Greedy-Merge Degrading has Optimal Power-Law

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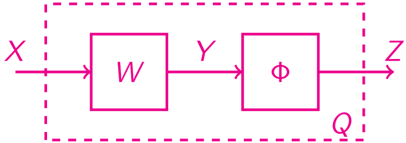
Motivation

- $W : \mathcal{X} \to \mathcal{Y}$, P_X, $|\mathcal{Y}|$ is very large
- Common problem in
  - Digital receiver design
  - Polar code construction

$\Rightarrow$ Quantize $\mathcal{Y}$ to L letters

Motivation



- $Q : \mathcal{X} \to \mathcal{Z}, \quad |\mathcal{Z}| = L$
- $\Delta I \triangleq I(X;Y) - I(X;Z) \geq 0$

Question

Given $|\mathcal{X}|$, what is

$$\Delta I^* \triangleq \min_Q \Delta I = O(?)$$

in terms of L?

Previous Results

- Binary input, $|\mathcal{X}| = 2$

Pedarsani et al. 2011	$O(L^{-1.5} \log L)$

- Finite $|\mathcal{X}|$ (constant)

Gulcu, Ye, and Barg 2016	$O(L^{-1/(|\mathcal{X}|-1)})$
Tal 2015	$\Omega(L^{-2/(|\mathcal{X}|-1)})$

- Related work

Kurkoski and Yagi 2014
Nazer, Ordentlich, and Polyanskiy 2017

Main Result

Theorem

$$\begin{cases} |\mathcal{Y}| > 2|\mathcal{X}| \\ L \geq 2|\mathcal{X}| \end{cases} \implies \Delta I^* = O(L^{-\frac{2}{|\mathcal{X}|-1}})$$

In particular,

$$\Delta I^* \leq \frac{\pi|\mathcal{X}|(|\mathcal{X}|-1)}{2\left(\sqrt{1+\frac{1}{2(|\mathcal{X}|-1)}}-1\right)^2}\left(\frac{2|\mathcal{X}|}{\Gamma\left(1+\frac{|\mathcal{X}|-1}{2}\right)}\right)^{\frac{2}{|\mathcal{X}|-1}} \cdot L^{-\frac{2}{|\mathcal{X}|-1}}$$

This bound is:

- *Attained by "greedy-merge" algorithm*
- *Tight in power-law sense*

Proof - Main Ideas

- Greedy-merge algorithm
- Simple upper bounds on ΔI
- “Sphere-packing”

Notation

- Channel, input and output probabilities:

$$W(y|x) \triangleq \mathbb{P}(Y = y|X = x) \qquad \pi_x \triangleq \mathbb{P}(X = x)$$
$$W(x|y) \triangleq \mathbb{P}(X = x|Y = y) \qquad \pi_y \triangleq \mathbb{P}(Y = y)$$

- Mutual information:

$$I(W, P_X) \triangleq I(X; Y) = \sum_{x \in \mathcal{X}} \eta(\pi_x) - \sum_{\substack{x \in \mathcal{X}, \\ y \in \mathcal{Y}}} \pi_y \eta(W(x|y))$$

$$\eta(p) \triangleq \begin{cases} -p \log p & p > 0 \\ 0 & p = 0 \end{cases}$$

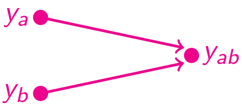
- Loss in mutual information:

$$\Delta I = I(W, P_X) - I(Q, P_X)$$

Merging a Pair of Letters

- For $y_a, y_b \in \mathcal{Y}$ define:

$$\alpha_x \triangleq W(x|y_a) \qquad \boldsymbol{\alpha} \triangleq (\alpha_x)_{x\in\mathcal{X}} \qquad \pi_a \triangleq \pi_{y_a}$$
$$\beta_x \triangleq W(x|y_b) \qquad \boldsymbol{\beta} \triangleq (\beta_x)_{x\in\mathcal{X}} \qquad \pi_b \triangleq \pi_{y_b}$$



- Merging y_a, y_b to y_{ab}:

$$W(x|y_{ab}) = \frac{\pi_a\alpha_x + \pi_b\beta_x}{\pi_a + \pi_b} \qquad \pi_{y_{ab}} = \pi_a + \pi_b$$

- Loss by a single merger:

$$\Delta I_x \triangleq (\pi_a + \pi_b)\eta\left(\frac{\pi_a\alpha_x + \pi_b\beta_x}{\pi_a + \pi_b}\right) - \pi_a\eta(\alpha_x) - \pi_b\eta(\beta_x)$$
$$\Delta I = \sum_{x\in\mathcal{X}} \Delta I_x$$

Greedy-Merge Algorithm

- Algorithm:
  - Merge y_a, y_b that minimize ΔI
  - Repeat $|\mathcal{Y}| - L$ times
- If $\min \Delta I = O(|\mathcal{Y}|^{-\frac{|\mathcal{X}|+1}{|\mathcal{X}|-1}}) \Rightarrow$ proof is finished

New Goal

Prove existence of $y_a, y_b \in \mathcal{Y}$ s.t.

$$\Delta I = O(|\mathcal{Y}|^{-\frac{|\mathcal{X}|+1}{|\mathcal{X}|-1}})$$

Simple upper bounds on ΔI

- ΔI is complicated
- Upper bound ΔI:

$$\Delta I_x \leq (\pi_a + \pi_b)|\alpha_x - \beta_x| \qquad \Delta I_x \leq (\pi_a + \pi_b)\frac{(\alpha_x - \beta_x)^2}{\min(\alpha_x, \beta_x)}$$

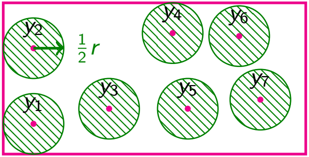
$$\Rightarrow \quad \Delta I \leq (\pi_a + \pi_b)|\mathcal{X}| \cdot \underbrace{\max_{x \in \mathcal{X}} \min\left(|\alpha_x - \beta_x|\ ,\ \frac{(\alpha_x - \beta_x)^2}{\min(\alpha_x, \beta_x)}\right)}_{\triangleq d(\boldsymbol{\alpha}, \boldsymbol{\beta})}$$

- Limit search to:

$$\mathcal{Y}_{\text{small}} \triangleq \left\{y \in \mathcal{Y} : \pi_y \leq \frac{2}{|\mathcal{Y}|}\right\} \qquad |\mathcal{Y}_{\text{small}}| \geq \frac{|\mathcal{Y}|}{2}$$

$$\Rightarrow \min_{y_a, y_b \in \mathcal{Y}_{\text{small}}} \Delta I \leq \frac{4|\mathcal{X}|}{|\mathcal{Y}|} \cdot \min_{y_a, y_b \in \mathcal{Y}_{\text{small}}} d(\boldsymbol{\alpha}, \boldsymbol{\beta})$$

Sphere-Packing Essentials



- A metric $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}_0^+$
- $r/2$-radius volumed spheres

$$\mathcal{B}\left(\boldsymbol{\alpha}, \frac{r}{2}\right) \triangleq \left\{\zeta \in \mathbb{M} : d(\boldsymbol{\alpha}, \zeta) \leq \frac{r}{2}\right\}$$

- Find $r = r_{\text{critical}} > 0$ s.t.:

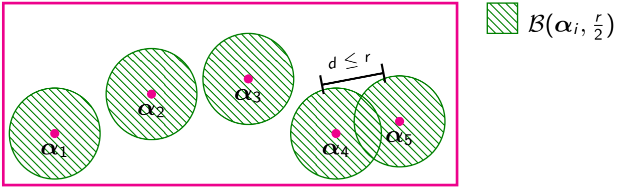
$$\sum_{\boldsymbol{\alpha} \in S} \mathrm{Vol}\left[\mathcal{B}\left(\boldsymbol{\alpha}, \frac{r}{2}\right)\right] = \mathrm{Vol}\Big[\text{whole space}\Big]$$

$\Rightarrow d(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq r$ for some $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{M}$

Sphere-Packing Reasoning

- $\mathcal{B}\left(\boldsymbol{\alpha}, \frac{r}{2}\right) \cap \mathcal{B}\left(\boldsymbol{\beta}, \frac{r}{2}\right) \neq \emptyset \;\Rightarrow\; d(\boldsymbol{\alpha}, \boldsymbol{\zeta}), d(\boldsymbol{\beta}, \boldsymbol{\zeta}) \leq \frac{r}{2}$
- Triangle inequality:

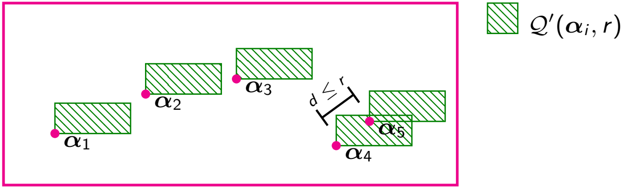
$$\Rightarrow\; d(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq d(\boldsymbol{\alpha}, \boldsymbol{\zeta}) + d(\boldsymbol{\zeta}, \boldsymbol{\beta}) \leq r$$



Sphere-Packing Reasoning

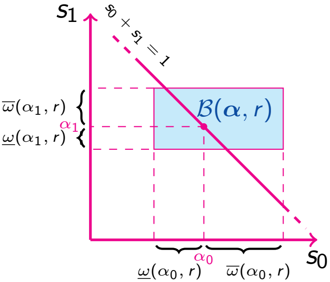
- d is a semimetric
- Find $\mathcal{Q}'(\cdot, r)$ s.t.:

$$\mathcal{Q}'(\boldsymbol{\alpha}, r) \cap \mathcal{Q}'(\boldsymbol{\beta}, r) \neq \emptyset \ \Rightarrow \ d(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq r$$



Towards a "Sphere"

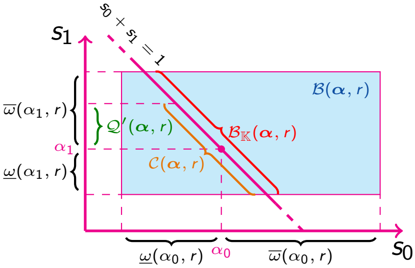
- Our $d(\cdot,\cdot)$ is a semimetric
- $\mathcal{B}(\boldsymbol{\alpha}, r)$ is a box



$$\underline{\omega}(\alpha_x, r) \triangleq \max\left(\sqrt{\frac{r^2}{4} + \alpha_x r} - \frac{r}{2}, r\right) \qquad \overline{\omega}(\alpha_x, r) \triangleq \max\left(\sqrt{\alpha_x r}, r\right)$$

- $\sum_{x\in\mathcal{X}} \alpha_x = 1 \;\Rightarrow\;$ dimension reduction is preferable

Towards a “Sphere”



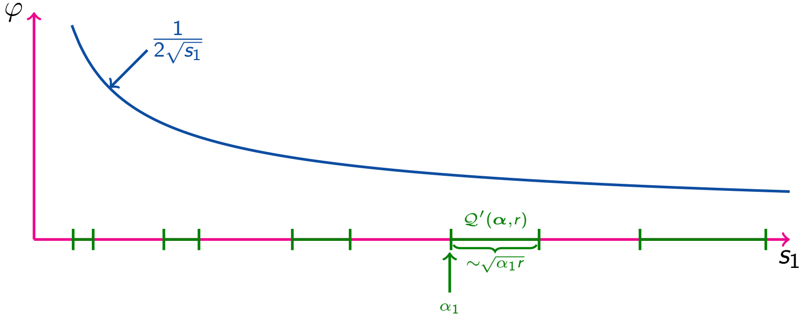
- $\mathcal{B}_{\mathbb{K}}(\boldsymbol{\alpha}, r) \triangleq \mathcal{B}(\boldsymbol{\alpha}, r) \cap \left\{\boldsymbol{\zeta} \in \mathbb{R}^{|\mathcal{X}|} : \sum_{x\in\mathcal{X}} \zeta_x = 1\right\}$ - complicated
- $\mathcal{C}(\boldsymbol{\alpha}, r) \subseteq \mathcal{B}_{\mathbb{K}}(\boldsymbol{\alpha}, r)$ - a box in $\mathbb{R}^{|\mathcal{X}|-1}$:

$$\mathcal{C}(\boldsymbol{\alpha}, r) \cap \mathcal{C}(\boldsymbol{\beta}, r) \neq \emptyset \qquad \not\Rightarrow \qquad d(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq r$$

- $\mathcal{Q}'(\boldsymbol{\alpha}, r) \subseteq \mathbb{R}^{|\mathcal{X}|-1}$ - suitable

Weighted "Sphere"-Packing

- Variable volume "spheres"



- $|\mathcal{X}| - 1$ dimensional density:

$$\varphi(\zeta') \triangleq \prod_{x \in \mathcal{X}'} \frac{1}{2\sqrt{\zeta_x}}$$

- Volume $\rightarrow$ Weight

Conclusion and Further Results

Conclusion

- $\Delta I^* = O(L^{-\frac{2}{|\mathcal{X}|-1}})$
- Tight in power-law
- Attained by "greedy-merge" algorithm

Further results (full paper)

For the upgrading setting:

- $\Delta I^* = \Omega(L^{-\frac{2}{|\mathcal{X}|-1}})$, same sequence of channels
- $\Delta I^* = O(L^{-\frac{2}{|\mathcal{X}|-1}})$ for $|\mathcal{X}| = 2$
- Optimal upgrading algorithm for $|\mathcal{X}| = 2$