

An Analytical Study of the Min-Sum Approximation for Polar Codes

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Abstract— The min-sum approximation is widely used in the decoding of polar codes. Although it is a numerical approximation, hardly any penalties are incurred in practice. We give a theoretical justification for this. We consider the common case of a binary-input, memoryless, and symmetric channel, decoded using successive cancellation and the min-sum approximation. Under mild assumptions, we show the following. For the finite length case, we show how to exactly calculate the error probabilities of all synthetic (bit) channels in time $\mathcal{O}(N^{1.585})$, where N is the codeword length. This implies a code construction algorithm with the above complexity. For the asymptotic case, we develop two rate thresholds, denoted $R_L = R_L(\lambda)$ and $R_U = R_U(\lambda)$, where $\lambda(\cdot)$ is the labeler of the channel outputs (essentially, a quantizer). For any $0 < \beta < \frac{1}{2}$ and any code rate $R < R_L$, there exists a family of polar codes with growing lengths such that their rates are at least R and their error probabilities are at most 2^{-N^β} . That is, strong polarization continues to hold under the min-sum approximation. Conversely, for code rates exceeding R_U , the error probability approaches 1 as the code-length increases, irrespective of which bits are frozen. We show that $0 < R_L \leq R_U \leq C$, where C is the channel capacity. The last inequality is often strict, in which case the ramification of using the min-sum approximation is that we can no longer achieve capacity.

I. INTRODUCTION

Polar codes are a family of capacity-achieving error correcting codes with efficient encoding and decoding algorithms, introduced by Arkan [1]. In this paper, we study the setting of a binary-input, memoryless and symmetric channel. Although many generalizations to this case exist [2]–[16], it is arguably the most basic and common one. Moreover, it affords a very efficient hardware implementation using the numerical min-sum approximation (MSA) in the decoder.

The seminal decoding algorithm of polar codes is called successive-cancellation (SC) decoding. It is a recursive algorithm that makes repeated use of the following two functions:

$$f(L_a, L_b) = 2 \tanh^{-1} \left(\tanh \left(\frac{L_a}{2} \right) \cdot \tanh \left(\frac{L_b}{2} \right) \right), \quad (1)$$

$$g_u(L_a, L_b) = (-1)^u \cdot L_a + L_b. \quad (2)$$

The functions g_0 and g_1 are simple to implement, since addition and subtraction are hardware-friendly operations. However, the f function is somewhat complicated, since hyperbolic functions are expensive in terms of calculation time and power consumption. Therefore, in many practical implementations the MSA is used [17]. That is, similar to what is done in LDPC

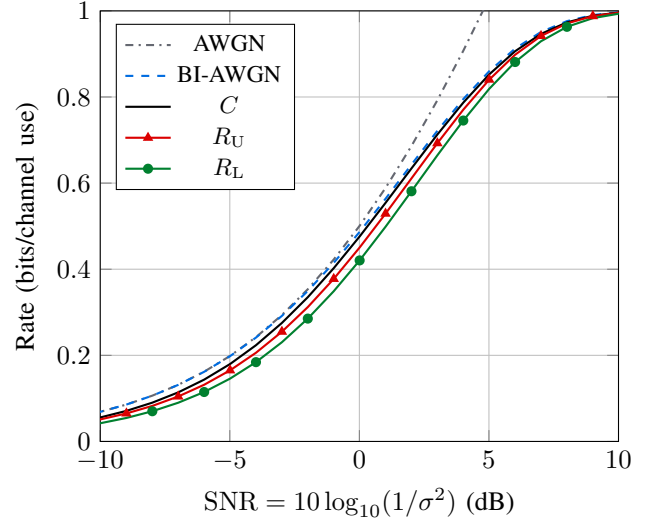


Fig. 1. The capacity C and the thresholds R_U and R_L of a BI-AWGN with 3-bit quantized output. For reference, the capacities of the corresponding non-quantized BI-AWGN and AWGN are also given.

decoder implementation [18], the f function is replaced with a simpler function $\tilde{f}$ given by

$$\tilde{f}(L_a, L_b) = \text{sgn}(L_a) \cdot \text{sgn}(L_b) \cdot \min \{|L_a|, |L_b|\}, \quad (3)$$

where $\text{sgn}(\cdot)$ is the sign function defined as

$$\text{sgn}(x) \triangleq \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For the non-approximated setting, L_a , L_b , and the outputs of f and g are log-likelihood ratios (LLRs) corresponding to certain channel outputs. For the approximated setting, we use the generalized term ‘labels’ for the corresponding quantities. At the base of the recursion the labels L_a and L_b are obtained by applying a labeling function $\lambda(\cdot)$ on the channel outputs. The full definition of $\lambda(\cdot)$ is given in Section II. Informally, $\lambda(y)$ is a quantized version of the LLR corresponding to the channel output y , up to a positive scaling constant.

The MSA is also used in decoders that are derivatives of the SC decoder, such as the SC list decoder [19] and the SC stack decoder [20]. Often, the MSA incurs only a small penalty in

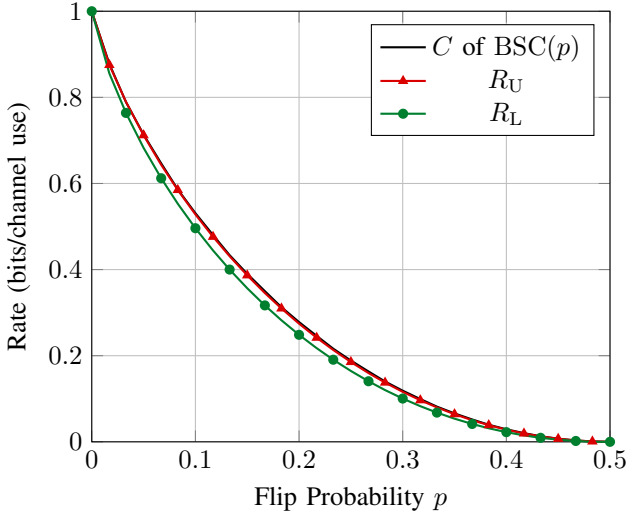


Fig. 2. The capacity C and the thresholds R_U and R_L for the BSC(p).

error rate [21, Figure 7], [22, Figures 7-8], and [23, Figure 4]. In this paper, we analyze this phenomenon.

The following theorem is our main result for the asymptotic case. The theorem promises two rate thresholds, R_L and R_U , when employing the MSA in SC decoding. Below R_L , strong polarization is guaranteed, while above R_U the error probability approaches 1. Figures 1 and 2 plot these thresholds and the channel capacity C for the binary-input additive white Gaussian noise (BI-AWGN) channel with quantized output, and for the binary symmetric channel (BSC), respectively. As can be seen in these figures, R_L , R_U , and C are all rather close. However, note that R_U is strictly smaller than C . That is, in these cases using the MSA means that we can no longer achieve capacity. The theorem assumes that a “fair labeler” is used, as defined in Definition 2 below.

Theorem 1. *Let W be a binary-input, memoryless and symmetric channel. Fix $0 < \beta < \frac{1}{2}$. Let $\lambda(\cdot)$ be a fair labeler. Then, there exist thresholds $R_L = R_L(\lambda)$ and $R_U = R_U(\lambda)$, such that $0 < R_L \leq R_U$. When using SC decoding and the MSA, the following holds. For any code rate $R < R_L$ there exists a family of polar codes with growing length N such that their rates are at least R and their word error probabilities are at most 2^{-N^β} . Conversely, for code rates exceeding R_U , the word error probability approaches 1 as the code-length increases, irrespective of which bits are frozen.*

If we only assume a fair labeler, R_L is weak but still positive, and R_U is trivial. For a significant subclass of fair labelers, “good labelers” (Definition 1), both bounds can be significantly strengthened. A good labeler is often the case in practice.

For the finite-length case and the good labeler setting, we develop an algorithm for calculating the exact error probability of each min-sum synthetic channel, defined in (10). The running time of our algorithm is $\mathcal{O}(N^{1.585})$. Note that in the non-approximated setting, no such algorithm exists, only a method to calculate bounds on the error probabilities [24].

II. NOTATION

Denote by $W : \mathcal{X} \rightarrow \mathcal{Y}$ a general binary-input, memoryless, and symmetric channel with input alphabet $\mathcal{X} = \{0, 1\}$ and output alphabet $\mathcal{Y}$. For each pair $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ the input probability is $p(x)$, and the transition probability is $W(y|x)$. Hence, the joint probability is given by $W(y; x) = p(x) \cdot W(y|x)$. We will assume that $p(x)$ is symmetric, i.e. $p(0) = p(1) = 1/2$.

For $n \in \mathbb{N}$ denote $N = 2^n$ and let $(X_i, Y_i)_{i=0}^{N-1}$ be N i.i.d. pairs, each distributed according to $W(y; x)$. Denote by U_0^{N-1} the polar transform of X_0^{N-1} . For $0 \leq i < N$, define the following synthetic joint distribution¹:

$$W_N^{(i)}(y_0^{N-1}, u_0^{i-1}; u_i) = \Pr(Y_0^{N-1} = y_0^{N-1}, U_0^{i-1} = u_0^{i-1}, U_i = u_i). \quad (4)$$

By [1, Proposition 3],

$$W_N^{(2j)}(y_0^{N-1}, u_0^{2j-1}; u_{2j}) = \sum_{u_{2j+1}} W_{N/2}^{(j)}(y_0^{\frac{N}{2}-1}, u_{0,e}^{2j-1} \oplus u_{0,o}^{2j-1}; u_{2j} \oplus u_{2j+1}) \cdot W_{N/2}^{(j)}(y_{\frac{N}{2}}^{N-1}, u_{0,o}^{2j-1}; u_{2j+1}), \quad (5)$$

and

$$W_N^{(2j+1)}(y_0^{N-1}, u_0^{2j}; u_{2j+1}) = W_{N/2}^{(j)}(y_0^{\frac{N}{2}-1}, u_{0,e}^{2j-1} \oplus u_{0,o}^{2j-1}; u_{2j} \oplus u_{2j+1}) \cdot W_{N/2}^{(j)}(y_{\frac{N}{2}}^{N-1}, u_{0,o}^{2j-1}; u_{2j+1}), \quad (6)$$

where $W_1^{(0)}(y; x) = W(y; x)$ and “ $\oplus$ ” is addition over GF(2). In the above, $u_{0,e}^{2j-1}$ and $u_{0,o}^{2j-1}$ are the even and odd entries of u_0^{2j-1} , respectively. As shown in [1], $W_N^{(2j)}$ and $W_N^{(2j+1)}$ are the result of applying the “−” and “+” transforms, respectively, on $W_{N/2}^{(j)}$, up to a relabeling of the output.

For each joint distribution $W_N^{(i)}$ we define the LLR $L_N^{(i)}$ as

$$L_N^{(i)}(y_0^{N-1}, u_0^{i-1}) \triangleq \log_2 \left(\frac{W_N^{(i)}(y_0^{N-1}, u_0^{i-1}; u_i = 0)}{W_N^{(i)}(y_0^{N-1}, u_0^{i-1}; u_i = 1)} \right). \quad (7)$$

Using the relations described in (5) and (6) we obtain recursive transforms for the LLRs. Namely, these are (8a) and (8b) below, once we remove all the tildes. The starting condition for this recursion is $L_1^{(0)}(y) = \log_2(W(y; 0)/W(y; 1))$.

The SC decoder uses f and g to recursively calculate the LLRs of all synthetic joint distributions, yielding a decoding algorithm with running time $\mathcal{O}(N \log N)$.

The min-sum SC decoder is a simplified version of the original SC decoder, as it uses $\tilde{f}$ (see (3)) instead of the computationally heavier f during the recursion. Unlike f , both $\tilde{f}$ and g are positive homogeneous (i.e. multiplying both inputs by a positive constant multiplies the output by the same constant). This implies that the min-sum decoder is not affected

¹We find it notationally easier to track joint distributions instead of channels. The latter is simply obtained from the former by multiplying by 2.

by scaling. Therefore, we further extend the approximation and allow the initial labels at the base of the recursion not to be LLRs, but some values obtained by applying a labeling function λ on the channel outputs. We now list 3 properties required of a labeler λ to be called a “good labeler”.

Definition 1 (Good labeler). A labeler $\lambda : \mathcal{Y} \rightarrow \mathbb{R}$ is a good labeler with respect to a binary-input memoryless symmetric channel $W : \mathcal{X} \rightarrow \mathcal{Y}$ if the following holds:

- 1) Symmetry preservation: since W is symmetric, there exists a permutation $\pi : \mathcal{Y} \rightarrow \mathcal{Y}$ such that for all $y \in \mathcal{Y}$, $W(y|1) = W(\pi(y)|0)$ and $\pi(\pi(y)) = y$. We require that $\lambda(\pi(y)) = -\lambda(y)$ for all $y \in \mathcal{Y}$.
- 2) Sign consistency: for all positive t we have $\alpha_t \geq \alpha_{-t}$, where $\alpha_t = \sum_{y: \lambda(y)=t} W(y|0)$, and the inequality is strict for at least one t .
- 3) Finite integer range: the range of λ is contained in $\{-\gamma, -\gamma+1, \dots, \gamma-1, \gamma\}$, for some positive integer γ .

We also define a “fair labeler” as follows.

Definition 2 (Fair labeler). A labeler λ is a fair labeler if the first two requirements of a good labeler are met.

Note that if we were to take $\lambda(y) = \text{LLR}(y)$, we would have a fair labeler, for any channel with positive capacity. The last property of the good labeler is required only for computational reasons, and is often the case due to quantization. The justification for it is by the homogeneous property of $\tilde{f}$ and g and its implications, as described above.

Under the MSA, labels are calculated recursively by

$$\tilde{L}_N^{(2j)}(y_0^{N-1}, u_0^{2j-1}) = \quad (8a)$$

$$\tilde{f}\left(\tilde{L}_{N/2}^{(j)}(y_0^{N/2-1}, u_{0,e}^{2j-1} \oplus u_{0,o}^{2j-1}), \tilde{L}_{N/2}^{(j)}(y_{N/2}^{N-1}, u_{0,o}^{2j-1})\right),$$

$$\tilde{L}_N^{(2j+1)}(y_0^{N-1}, u_0^{2j}) = \quad (8b)$$

$$g_{u_{2j}}\left(\tilde{L}_{N/2}^{(j)}(y_0^{N/2-1}, u_{0,e}^{2j-1} \oplus u_{0,o}^{2j-1}), \tilde{L}_{N/2}^{(j)}(y_{N/2}^{N-1}, u_{0,o}^{2j-1})\right),$$

with starting condition $\tilde{L}_1^{(0)}(y) = \lambda(y)$.

III. POSYNOMIAL REPRESENTATION

For a fair labeler, we now define the synthetic joint distributions (on the label t and input u_i) at stage i of the SC decoder and min-sum SC decoder. These are, respectively,

$$Q_N^{(i)}(t; u_i) \triangleq \sum_{\substack{y_0^{N-1}, u_0^{i-1}: \\ L_N^{(i)}(y_0^{N-1}, u_0^{i-1})=t}} W_N^{(i)}(y_0^{N-1}, u_0^{i-1}; u_i), \quad (9)$$

$$\tilde{Q}_N^{(i)}(t; u_i) \triangleq \sum_{\substack{y_0^{N-1}, u_0^{i-1}: \\ \tilde{L}_N^{(i)}(y_0^{N-1}, u_0^{i-1})=t}} W_N^{(i)}(y_0^{N-1}, u_0^{i-1}; u_i). \quad (10)$$

Denote by $\tilde{\mathcal{T}}_N^{(i)}$ the support of $\tilde{Q}_N^{(i)}(t; u_i)$ with respect to t . In the setting of a good labeler, by definition, $\tilde{\mathcal{T}}_1^{(0)} \subseteq \{-\gamma, \dots, \gamma\}$. As will soon become apparent, under this setting,

$$\tilde{\mathcal{T}}_N^{(i)} \subseteq \{-2^{\text{wt}(i)} \cdot \gamma, \dots, 2^{\text{wt}(i)} \cdot \gamma\}, \quad (11)$$

and $\text{wt}(i)$ is the Hamming weight of the vector whose entries are the binary representation of i . Indeed, this follows by (12a) and (12b) below, and the definitions of $\tilde{f}$ and g in (3) and (2).

Using the relations in (5)–(8), we obtain the following minus and plus transforms of synthetic min-sum joint distributions.

Lemma 2 (Transforms of synthetic joint distributions).

$$\tilde{Q}_N^{(2j)}(t; u_{2j}) = \quad (12a)$$

$$\sum_{\substack{t_a, t_b, u_{2j+1}: \\ \tilde{f}(t_a, t_b)=t}} \tilde{Q}_{N/2}^{(j)}(t_a; u_{2j} \oplus u_{2j+1}) \cdot \tilde{Q}_{N/2}^{(j)}(t_b; u_{2j+1}),$$

$$\tilde{Q}_N^{(2j+1)}(t; u_{2j+1}) = \quad (12b)$$

$$\sum_{\substack{t_a, t_b, u_{2j}: \\ g_{u_{2j}}(t_a, t_b)=t}} \tilde{Q}_{N/2}^{(j)}(t_a; u_{2j} \oplus u_{2j+1}) \cdot \tilde{Q}_{N/2}^{(j)}(t_b; u_{2j+1}).$$

The following lemma ensures that symmetry holds for the min-sum synthetic distributions.

Lemma 3 (Symmetry of synthetic joint distribution).

$$\tilde{Q}_N^{(i)}(t; u_i) = \tilde{Q}_N^{(i)}(-t; u_i \oplus 1). \quad (13)$$

The above symmetry implies that the probability of error at the i -th stage of the min-sum SC decoder, when aided by a genie that reveals the correct values of u_0^{i-1} , is given by

$$P_e(\tilde{Q}_N^{(i)}) = \tilde{Q}_N^{(i)}(0; 0) + 2 \cdot \sum_{t < 0} \tilde{Q}_N^{(i)}(t; 0). \quad (14)$$

To derive a Bhattacharyya-like upper bound on P_e , and to aid in notation in general, we abuse notation and define the following posynomial, in the indeterminate ξ :

$$\tilde{Q}_N^{(i)}(\xi) \triangleq \sum_t \tilde{Q}_N^{(i)}(t; 0) \cdot \xi^t. \quad (15)$$

The above is indeed a posynomial: all the coefficients are non-negative as they are probabilities, while t is not restricted to non-negative numbers. We further define the following:

$$Z(\tilde{Q}_N^{(i)}, \xi) \triangleq 2 \cdot \tilde{Q}_N^{(i)}(\xi). \quad (16)$$

Our upper bound on P_e is presented in the following lemma.

Lemma 4 (Bhattacharyya-like bound). For $0 < \xi_0 \leq 1$,

$$P_e(\tilde{Q}_N^{(i)}) \leq Z(\tilde{Q}_N^{(i)}, \xi_0). \quad (17)$$

We remark that setting $\xi_0 = 1/\sqrt{2}$ and removing the tildes yields the Bhattacharyya bound on the error probability at the i -th stage of the non-approximated genie-aided SC decoder. Also, we may optimize over ξ_0 to yield the tightest upper bound, denoted

$$Z^*(\tilde{Q}_N^{(i)}) \triangleq \min_{0 < \xi_0 \leq 1} Z(\tilde{Q}_N^{(i)}, \xi_0). \quad (18)$$

The above optimization is an instance of geometric programming, and can thus be efficiently computed [25, Section 4.5].

The following shows that the evolution of Z and Z^* is similar to the evolution of the Bhattacharyya parameter in the non-approximated setting.

Lemma 5 (Bhattacharyya-like evolutions). *For $0 < \xi_0 \leq 1$ and $0 \leq j < N/2$ we have*

$$Z\left(\tilde{Q}_N^{(2j)}, \xi_0\right) \leq 2 \cdot Z\left(\tilde{Q}_{N/2}^{(j)}, \xi_0\right), \quad (19a)$$

$$Z\left(\tilde{Q}_N^{(2j+1)}, \xi_0\right) = \left(Z\left(\tilde{Q}_{N/2}^{(j)}, \xi_0\right)\right)^2. \quad (19b)$$

Furthermore,

$$Z^*\left(\tilde{Q}_N^{(2j)}\right) \leq 2 \cdot Z^*\left(\tilde{Q}_{N/2}^{(j)}\right), \quad (20a)$$

$$Z^*\left(\tilde{Q}_N^{(2j+1)}\right) = \left(Z^*\left(\tilde{Q}_{N/2}^{(j)}\right)\right)^2. \quad (20b)$$

To prove the above, we state the following two lemmas.

Lemma 6 (Bound on posynomial minus transform). *For all $0 < \xi_0 \leq 1$ we have*

$$\tilde{Q}_N^{(2j)}(\xi_0) \leq 2 \cdot \tilde{Q}_{N/2}^{(j)}(\xi_0). \quad (21)$$

Lemma 7 (Posynomial plus transform).

$$\tilde{Q}_N^{(2j+1)}(\xi) = 2 \cdot \left(\tilde{Q}_{N/2}^{(j)}(\xi)\right)^2. \quad (22)$$

The previous lemma implies that the coefficients of $\tilde{Q}_N^{(2j+1)}(\xi)$ can be calculated efficiently from those of $\tilde{Q}_{N/2}^{(j)}(\xi)$. We now show an analogous result for $\tilde{Q}_N^{(2j)}(\xi)$. In aid of this, we define the “above” and “below” posynomials:

$$\tilde{A}_N^{(i)}(\xi) \triangleq \sum_{t \in \tilde{\mathcal{T}}_N^{(i)}} \left(\sum_{t' > t} \tilde{Q}_N^{(i)}(t'; 0) \right) \cdot \xi^t, \quad (23)$$

$$\tilde{B}_N^{(i)}(\xi) \triangleq \sum_{t \in \tilde{\mathcal{T}}_N^{(i)}} \left(\sum_{t' < t} \tilde{Q}_N^{(i)}(t'; 0) \right) \cdot \xi^t. \quad (24)$$

Namely, if we write out $\tilde{Q}_N^{(i)}(\xi)$ in ascending order of powers of ξ , then the coefficient of ξ^t in $\tilde{A}_N^{(i)}(\xi)$ (resp. $\tilde{B}_N^{(i)}(\xi)$) is the sum of the coefficients strictly above (resp. below) the monomial $\tilde{Q}_N^{(i)}(t; 0)\xi^t$.

Let $\Gamma(\xi)$ and $\Lambda(\xi)$ be two posynomials. Denote by $[\xi^t] \Gamma(\xi)$ the coefficient of ξ^t in $\Gamma(\xi)$. Define the “positive” and “negative” operators, and Hadamard (element-wise) product: these operators return posynomials, where for all t ,

$$[\xi^t] \text{pos} \langle \Gamma(\xi) \rangle = \begin{cases} [\xi^t] \Gamma(\xi) & t \geq 0, \\ [\xi^{-t}] \Gamma(\xi) & t < 0. \end{cases} \quad (25)$$

$$[\xi^t] \text{neg} \langle \Gamma(\xi) \rangle = \begin{cases} [\xi^t] \Gamma(\xi) & t \leq 0, \\ [\xi^{-t}] \Gamma(\xi) & t > 0. \end{cases} \quad (26)$$

$$[\xi^t] (\Gamma(\xi) \odot \Lambda(\xi)) = ([\xi^t] \Gamma(\xi)) \cdot ([\xi^t] \Lambda(\xi)). \quad (27)$$

Lemma 8 (Posynomial minus transform).

$$\begin{aligned} \tilde{Q}_N^{(2j)}(\xi) = & 2 \left(\tilde{Q}_{N/2}^{(j)}(\xi) \odot \left(2 \cdot \text{pos} \langle \tilde{A}_{N/2}^{(j)}(\xi) \rangle + \tilde{Q}_{N/2}^{(j)}(\xi) \right) \right) \\ & + 2 \left(\tilde{Q}_{N/2}^{(j)}(1/\xi) \odot \left(2 \cdot \text{neg} \langle \tilde{B}_{N/2}^{(j)}(\xi) \rangle + \tilde{Q}_{N/2}^{(j)}(1/\xi) \right) \right) \\ & - 2 \left([\xi^0] \tilde{Q}_{N/2}^{(j)}(\xi) \right)^2. \end{aligned} \quad (28)$$

IV. FINITE-LENGTH CASE

In this section, we assume a good labeler. For the finite length case, our aim is to calculate $P_e\left(\tilde{Q}_N^{(i)}\right)$ for all $0 \leq i < N$, where the codeword length is $N = 2^n$. The expression for this is given in (14), which we can recast using (26) as

$$P_e\left(\tilde{Q}_N^{(i)}\right) = \text{neg} \left\langle \tilde{Q}_N^{(i)}(\xi) \right\rangle \Big|_{\xi=1}. \quad (29)$$

We use (22) and (28) to calculate $\tilde{Q}_N^{(i)}(\xi)$ for all i , and then apply (29) to yield the error probability. The following two lemmas specify the complexity of calculating $\tilde{Q}_N^{(2j)}(\xi)$ and $\tilde{Q}_N^{(2j+1)}(\xi)$ from $\tilde{Q}_{N/2}^{(j)}(\xi)$. Namely, the complexity of calculating all the coefficients of the former, given all the coefficients of the latter. Recall that $\tilde{\mathcal{T}}_N^{(i)}$ is defined in (11).

Lemma 9. *The complexity of calculating $\tilde{Q}_N^{(2j)}(\xi)$ from $\tilde{Q}_{N/2}^{(j)}(\xi)$ is $\mathcal{O}\left(|\tilde{\mathcal{T}}_{N/2}^{(j)}|\right)$.*

Lemma 10. *The complexity of calculating $\tilde{Q}_N^{(2j+1)}(\xi)$ from $\tilde{Q}_{N/2}^{(j)}(\xi)$ is $\mathcal{O}\left(|\tilde{\mathcal{T}}_{N/2}^{(j)}| \cdot \log(|\tilde{\mathcal{T}}_{N/2}^{(j)}|)\right)$.*

The following theorem is our main result for this section. It shows that the complexity of calculating all the probabilities of error $P_e\left(\tilde{Q}_N^{(i)}\right)$ is polynomial in the codeword length N and in γ (recall Definition 1).

Theorem 11 (Total complexity of evaluating P_e). *When using a good labeler λ , the complexity of calculating $P_e\left(\tilde{Q}_N^{(i)}\right)$ for all $0 \leq i < N$ is $\mathcal{O}(N^{\log_2 3} \log N \cdot \gamma \log \gamma)$. We simplify this to $\mathcal{O}(N^{1.585} \cdot \gamma \log \gamma)$.*

V. ASYMPTOTIC CASE

In this section we prove Theorem 1. We first do so assuming a fair labeler, and then show how to significantly improve the thresholds R_L and R_U for the case of a good labeler. The following three results are required for deriving R_L .

Proposition 12. *Let $B_1, B_2, \dots$ be i.i.d. random variables such that $\Pr(B_i = 0) = \Pr(B_i = 1) = 1/2$. Let $S_0, S_1, \dots$ be a $[0, 1]$ -valued random process that satisfies*

$$S_{n+1} \leq \kappa \cdot \begin{cases} S_n, & B_{n+1} = 0, \\ S_n^2, & B_{n+1} = 1, \end{cases} \quad n \geq 0. \quad (30)$$

Then, for every $\epsilon' > 0$ and $\delta' > 0$ there exist $n' = n'(\epsilon', \delta', \kappa)$ and $\eta = \eta(\epsilon', \delta', \kappa) > 0$ such that if $S_0 \leq \eta$ then

$$\Pr(S_n \leq \epsilon' \text{ for all } n \geq n') \geq 1 - \delta'. \quad (31)$$

This is [26, Equation 171], and is the crux of proving [26, Proposition 49]. The expression for η is given in the penultimate displayed equation in [26, Appendix A], where r is defined slightly before as the largest positive solution of $\kappa^r + (2\kappa)^{-r} = 2$. In our setting, S_n will be related to Z^* . Thus, by (20), we specialize to $\kappa = 2$. Plugging $x = 2^r$ into $2^r + 4^{-r} = 2$ yields $x + 1/x^2 = 2$. The three roots of this equation are 1, φ , and $-\varphi^{-1}$, where $\varphi = \frac{1}{2} \cdot (1 + \sqrt{5})$ is the golden ratio. Thus, $r = \log_2(\varphi)$ and $\eta(\delta') = \frac{1}{8} \cdot (\delta'/2)^{1/\log_2 \varphi}$.

The following result is an immediate corollary.

Corollary 13. *Let $S_0, S_1, \dots$ be as in Proposition 12, with $\kappa = 2$. Fix $\epsilon' > 0$ and $\eta > 0$. Then there exists $n' = n'(\epsilon', \eta)$ such that if $S_0 \leq \eta$ then*

$$\Pr(S_n \leq \epsilon' \text{ for all } n \geq n') \geq 1 - \delta'(\eta), \quad (32)$$

where

$$\delta'(\eta) \triangleq 2 \cdot (8\eta)^{\log_2 \varphi} \quad \text{and} \quad \varphi = (1 + \sqrt{5})/2. \quad (33)$$

The following result is of primary importance and will be used directly to prove Theorem 1.

Proposition 14. *Let $S_0, S_1, \dots$ be as in Proposition 12, with $\kappa = 2$. Fix $0 < \beta < 1/2$, $\eta > 0$, and $\delta > \delta'(\eta)$, where $\delta'(\eta)$ is given in (33). Then, there exists $n_0 = n_0(\beta, \delta - \delta'(\eta))$ such that if $S_0 \leq \eta$ then*

$$\Pr(S_n \leq 2^{-2^{n\beta}} \text{ for all } n \geq n_0) \geq 1 - \delta. \quad (34)$$

A. Fair Labeler

Proof of Theorem 1: For R_U , we first recall that any decoder operates on the output of W , after it has been labeled by λ . Thus, it effectively sees the channel $\tilde{Q}(t|x) = 2 \cdot \tilde{Q}_1^{(0)}(t; x)$, as defined in (10). We take R_U as the capacity of this channel, which is valid by the strong converse to the coding theorem, see [27, Theorem 5.8.5].

We now work towards deriving R_L . Consider a polar code of length $N = 2^n$ with non-frozen index set $\mathcal{A} = \{0 \leq i < N : Z^*(\tilde{Q}_N^{(i)}) < 2^{-N^{\beta'}}\}$, where $\beta' = \frac{\beta+1/2}{2}$. By the “genie-aided decoder” argument in [1], the union bound, and Lemma 4, the error probability of such a code is at most $|\mathcal{A}| \cdot 2^{-N^{\beta'}} \leq N \cdot 2^{-N^{\beta'}} < 2^{-N^\beta}$, where the last inequality holds for N large enough. Thus, we must find an R_L such that for $R < R_L$ fixed and all N large enough, $|\mathcal{A}| \geq N \cdot R$. Consider the set $\mathcal{A}' = \{0 \leq i < N : \zeta_N^{(i)} < 2^{-N^{\beta'}}\}$, where $\zeta_1^{(0)} = Z^*(\tilde{Q}_1^{(0)})$ and

$$\zeta_N^{(i)} = \begin{cases} 2 \cdot \zeta_{N/2}^{(i/2)} & i \text{ is even,} \\ \left(\zeta_{N/2}^{((i-1)/2)}\right)^2 & i \text{ is odd.} \end{cases} \quad (35)$$

By (20), we have for all i that $\zeta_N^{(i)} \geq Z^*(\tilde{Q}_N^{(i)})$. Namely, $\mathcal{A}' \subseteq \mathcal{A}$. Thus, it suffices to find an R_L such that for $R < R_L$ fixed and all N large enough, $|\mathcal{A}'| \geq N \cdot R$. For any $M = 2^m$, we use the definition of $\delta'(\cdot)$ in (33) and define the following:

$$R_L(M) \triangleq \frac{1}{M} \sum_{j=0}^{M-1} \max \left\{ 1 - \delta'(\zeta_M^{(j)}), 0 \right\}. \quad (36)$$

Proving the following two items will complete the proof:

- 1) For a given M and $R < R_L(M)$ there exists n_0 such that for all $n \geq n_0$ we have $|\mathcal{A}'| \geq N \cdot R$.
- 2) There exists an M such that $R_L(M) > 0$.

To prove the first item, assume that R , and therefore $R_L(M)$, are positive, otherwise the claim is trivial. For each one of the M indices $0 \leq j < M$, we invoke Proposition 14 with $\delta = \delta'(\zeta_M^{(j)}) + (R_L(M) - R)$, $\eta = S_0 = \zeta_M^{(j)}$, and $\beta'' = \frac{\beta'+1/2}{2}$

in place of β . Denote the n_0 promised by the proposition as $n_0^{(j)}$. Now define $n_0^{\max} = \max_j n_0^{(j)}$. By (34), for $n \geq m + n_0^{\max}$ the fraction of indices $0 \leq i < N$ such that $\zeta_N^{(i)} \leq 2^{-2^{(n-m)\beta''}}$ is at least

$$\begin{aligned} & \frac{1}{M} \sum_{j=0}^M \max \left\{ 1 - \left(\delta'(\zeta_M^{(j)}) + R_L(M) - R \right), 0 \right\} \\ & \geq \frac{1}{M} \sum_{j=0}^M \max \left\{ 1 - \delta'(\zeta_M^{(j)}) + R - R_L(M), R - R_L(M) \right\} \\ & = \frac{1}{M} \sum_{j=0}^M \max \left\{ 1 - \delta'(\zeta_M^{(j)}), 0 \right\} + R - R_L(M) \\ & = R. \end{aligned}$$

For the first item to hold, we take $n_0 \geq m + n_0^{\max}$ large enough such that for all $n \geq n_0$ we have $2^{-2^{(n-m)\beta''}} \leq 2^{-2^{n\beta'}} = 2^{-N^{\beta'}}$ (ensuring $|\mathcal{A}'| \geq N \cdot R$) and $N \cdot 2^{-N^{\beta'}} < 2^{-N^\beta}$.

We now prove the second item. That is, it is always possible to find an M such that $R_L(M) > 0$. We first show that $Z^*(\tilde{Q}_1^{(0)}) < 1$. Indeed,

$$Z(\tilde{Q}_1^{(0)}, \xi) \Big|_{\xi=1} = 1 \quad \text{and} \quad \frac{d}{d\xi} Z(\tilde{Q}_1^{(0)}, \xi) \Big|_{\xi=1} > 1,$$

where the inequality follows by item 2 in Definition 1. Hence, for $\xi_0 < 1$ sufficiently close to 1 it must hold that $Z(\tilde{Q}_1^{(0)}, \xi_0) < 1$. Thus, $\zeta_1^{(0)} = Z^*(\tilde{Q}_1^{(0)}) < 1$. Next, note that $\zeta_M^{(M-1)} = \left(\zeta_1^{(0)}\right)^M$. Take M as the smallest power of 2 that is at least $\log_a b$ where $a = \zeta_1^{(0)}$ and $b = \eta(1/2) \approx 0.327254$. For this choice, $R_L(M) \geq \frac{1}{2M} > 0$, by considering the last term in (36). ■

B. Good Labeler

We now show how both thresholds R_L and R_U can be strengthened in the case of a good labeler. We give a simplified description here. We give a full and more nuanced description in the expanded version [28]. For R_L , we observe the following regarding the proof of Theorem 1. Any definition of $\zeta_N^{(i)}$ that satisfies $\zeta_N^{(i)} \geq Z^*(\tilde{Q}_N^{(i)})$ for all $0 \leq i < N$ is valid. Thus, for a parameter $V = 2^v \leq M$, and all indices $0 \leq k < V$, define $\zeta_V^{(k)} = Z^*(\tilde{Q}_V^{(k)})$. For $N > V$, define $\zeta_N^{(i)}$ recursively according to (35). This improves $R_L(M)$, which we now denote as $R_L(V, M)$, since for polarization stage v we are calculating the exact values of $Z^*(\tilde{Q}_V^{(k)})$, as opposed to bounds on them. By Lemmas 9 and 10, we can indeed calculate $\tilde{Q}_V^{(k)}$ efficiently.

To strengthen R_U , we now define $R_U(V)$ as the average capacity of the channels corresponding to $\tilde{Q}_V^{(k)}$ over $0 \leq k < V$. The proof of this threshold being valid is given in [28]. In essence, we employ a so called “block-genie” that corrects us after N/V decisions have been made. Each block of size N/V corresponds to N/V uses of one of the above channels, and hence we cannot code for this block at a rate exceeding the capacity of that channel.

The two figures in this paper were derived with $V = 12$ and $M = 36$, and pruning as described in [28].

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