Reconstruction of a fiber Bragg grating from noisy reflection data

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We develop a novel method that enables one to reconstruct the structure of highly reflecting fiber Bragg gratings from noisy reflection spectra. When the reflection spectrum is noisy and the grating reflectivity is high, noise in the Bragg zone of the reflection spectrum is amplified by the inverse scattering algorithms and prevents the reconstruction of the grating. Our method is based on regularizing the reflection spectrum in frequencies inside the Bragg zone by using the data on the grating spectrum outside the Bragg zone. The regularized reflection spectrum is used to reconstruct the grating structure by means of inverse scattering. Our method enables one to analyze gratings with a high reflectivity from a spectrum that contains a high level of noise. Such gratings could not be analyzed by using methods described in previous work [IEEE J. Quantum Electron. **39**, 1238 (2003)]. © 2005 Optical Society of America

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1. INTRODUCTION

The inverse scattering problem in fiber Bragg gratings has been studied intensively in recent years. In such a problem, the profile of the grating is extracted from its complex reflection spectrum.[1–5] Several inverse scattering algorithms were used for synthesizing the grating profile from a desired spectral response[1–4] and for reconstructing the grating profile from the measured complex reflection spectrum.[6]

In previous studies it has been shown that the inverse scattering problem becomes very sensitive to noise in the reflection spectrum when the grating reflectivity is high.[6–8] In strongly reflecting gratings, a forward-propagating wave with a frequency inside the Bragg zone of the grating attenuates strongly along the grating. Therefore the reflection from the region located close to the output end of the grating becomes very low and does not significantly affect the grating spectrum. Thus a small error in the amplitude of the spectrum inside the Bragg frequency region of the grating may yield a very large change in the reconstructed grating profile. In a previous study we showed that even when no noise is added to the reflection spectrum, most inverse scattering algorithms are unstable in analyzing very strong gratings because of numerical inaccuracies.[5] When the reflection spectrum is accurate, the integral layer-peeling (ILP) algorithm[5] can overcome the numerical instability that limits previous inverse scattering algorithms. However, when the reflection spectrum is inaccurate and the grating reflectivity is high, the reconstruction of the grating is inherently unstable regardless of the inverse scattering algorithm in use. When the reflection spectrum of the grating is measured, there is always an error added to the grating reflection spectrum. Because of the instability of the inverse scattering problem, this error often prevents the use of inverse scattering algorithms for characterizing highly reflecting fiber Bragg gratings, as are used in optical communication systems.[6]

The instability of the inverse scattering algorithms to errors in the reflection spectrum can be significantly reduced by measuring the complex reflection spectrum from both sides of the grating.[6] For the case in which the reflection spectrum is measured from one side of the grating, several regularization methods have been developed[8,9] to reduce the instability of the inverse scattering algorithms. In the work described in Ref. 8 a regularization of the problem was obtained by slightly decreasing the amplitude of the reflection spectrum. A small amount of *a priori* information on the grating profile such as, for example, knowledge of its length was needed to improve the result. The main disadvantage of this method was its inherent reduction in the accuracy of the result even when a noise-free spectrum was used. Moreover, such a method cannot solve the instability problem of the inverse scattering algorithm when the noise in the reflection spectrum is not small enough.[8] In Ref. 9, conditions on the complex reflection spectrum of a grating with a finite length were used to find an approximation to the noisy reflection spectrum. When the inverse scattering algorithm was used on the approximated reflection spectrum, the result did not diverge. This method requires an *a priori* knowledge of the grating length. However, the calculation of the approximated reflection spectrum requires mathematical operations, such as the Hilbert transform, that are very sensitive to noise when the grating reflectivity is high.[10] Therefore, although the algorithm regularizes the solution, the error between the original and the reconstructed profiles becomes very large for highly reflecting gratings. The two methods given in Refs. 8 and 9 were demonstrated only when the noise in the spectrum was weak and the signal to noise inside the Bragg zone of the reflection spectrum was on the order of 10^5.[9]

We demonstrate in this paper a new approach for reconstructing highly reflecting gratings from a noisy reflection spectrum. In contrast to previous methods, we sepa-

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rate data from the reflection spectrum inside and outside the Bragg zone. Noise in the Bragg zone of a highly reflecting grating is amplified in the reconstruction of the grating. Therefore the Bragg zone of a noisy reflection spectrum does not contain a significant amount of information on the output end of the grating. However, information on the output end of the grating may still be obtained from the reflection spectrum in frequencies outside the Bragg zone. In this frequency region, the signal-to-noise ratio may be lower than in the Bragg zone of the grating; however, the error in the data located outside the Bragg zone of the grating is not significantly enhanced in the reconstruction of the grating. Our method is based on regularizing the reflection spectrum inside the Bragg zone of the spectrum by using the reflection spectrum given outside the Bragg zone. After the reflection spectrum is regularized, the ILP inverse scattering algorithm can be used to accurately reconstruct the grating profile. The method is based on prior knowledge of the grating length. We note that the use of the ILP algorithm enables one to reduce numerical errors in the reconstruction that may limit other inverse scattering algorithms in an analysis of highly reflecting fiber Bragg gratings.[5]

The performance of our method depends on the accuracy of the data outside the Bragg zone of the reflection spectrum. The required signal-to-noise ratio of the reflection spectrum outside the Bragg frequency region depends on the product of the grating length and the bandwidth of the Bragg zone and on the grating reflectivity. We find that the stability of the method to noise is higher when the product of the grating length and the bandwidth of the Bragg zone of the grating is small. Therefore we obtained the best reconstruction results for a uniform grating, since the bandwidth of the Bragg zone in such a grating is narrow compared with that of apodized or chirped gratings with the same length and reflectivity. When the grating is uniform, the new method can be used to reconstruct gratings with a very high reflectivity, measured with a very high level of noise.

2. THEORETICAL BACKGROUND

In this section we derive the mathematical properties of the reflection spectrum of fiber Bragg gratings that are needed for our regularization method. We describe the propagation of waves inside a fiber Bragg grating without loss by using the coupled-mode equations[11]:

$$\frac{\mathrm{d}u_1(k,z)}{\mathrm{d}z} + iku_1(k,z) = q(z)u_2(k,z),$$

$$\frac{\mathrm{d}u_2(k,z)}{\mathrm{d}z} - iku_2(k,z) = q^*(z)u_1(k,z), \tag{1}$$

where $k = \beta - \beta_B$ is the wave-number detuning from the Bragg wavelength; β_B is the Bragg wave number of the grating; $u_1(k, z)$ and $u_2(k, z)$ are the complex amplitudes of the backward- and the forward-propagating waves respectively; and $q(z)$ is the complex coupling coefficient of the grating.[5] We assume that the grating is written in the region $[0, L]$ and consider a solution $U(k, z) = [u_1(k, z), u_2(k, z)]$ with the boundary conditions $U(k, z = L) = (0, 1)$. We define the functions $b(k)$ and $a(k)$ as $[b(k), a(k)] = U(k, z = 0)$. The reflection spectrum $r(k)$ and the transmission spectrum $t(k)$ can be expressed by using the functions $a(k)$ and $b(k)$:

$$r(k) = \frac{b(k)}{a(k)},$$

$$t(k) = \frac{1}{a(k)}. \tag{2}$$

In Appendix A we show that the functions $a(k)$ and $b(k)$ can be represented by using two time domain functions $\alpha(\tau)$ and $\beta(\tau)$ in the following integral form:

$$a(k) = \exp(-ikL) + \int_{-L}^{L} \alpha(\tau)\exp(ik\tau)\mathrm{d}\tau,$$

$$b(k) = \int_{-L}^{L} \beta(\tau)\exp(ik\tau)\mathrm{d}\tau. \tag{3}$$

Equation (3) shows that the Fourier transforms of the functions $a(k)$ and $b(k)$ are nonzero only for $z \in [-L, L]$. The boundary values of the functions $\alpha(\tau)$ and $\beta(\tau)$, derived in Appendix A, are given by

$$\alpha(\tau = -L) = -\frac{1}{2}\int_0^L |q(z)|^2 \mathrm{d}z, \qquad \alpha(\tau = L) = 0,$$

$$\beta(\tau = -L) = -\frac{q(z=0)}{2}, \quad \beta(\tau = L) = -\frac{q(z=L)}{2}. \tag{4}$$

When the energy is conserved, the amplitudes of the reflection and the transmission spectra are connected by the conservation-of-energy relation[12]:

$$|r(k)|^2 + |t(k)|^2 = 1. \tag{5}$$

Using Eqs. (2) and (5) we obtain

$$|a(k)|^2 = 1 + |b(k)|^2 = \frac{1}{1 - |r(k)|^2}. \tag{6}$$

Thus the intensity of the functions $a(k)$ and $b(k)$ can be calculated from the intensity of the reflection spectrum, $r(k)$. The function $a(k)$ is a minimum-phase-shift function, and therefore the phase of $a(k)$ can be calculated from the intensity $|a(k)|^2$ by using the Hilbert transform.[12] Then the phase of the function $b(k)$ can be found from the phase of the function $a(k)$ and from the phase of the complex reflection spectrum $r(k)$ by using Eq. (2). Therefore the functions $a(k)$ and $b(k)$ can be uniquely calculated from the reflection spectrum, $r(k)$. Alternatively, if the function $b(k)$ is known, it can be used to calculate the function $a(k)$ and the reflection spectrum $r(k)$.[10]

Since the function $b(k)$ contains the same information as the complex reflection spectrum of the grating, $r(k)$, we can uniquely reconstruct the grating profile from the function $b(k)$. In a previous study it was shown[8] that when the grating reflectivity is high, even a small error in the reflection spectrum can prevent the reconstruction of the grating. The reconstruction error is largest near the output end of the grating ($z = L$). On the other hand, when the function $b(k)$ is known, we can calculate directly the coupling coefficient at the output end of the grating ($z = L$) by using Eq. (4). Therefore it can be expected that the reconstruction of the grating from the function $b(k)$ is more stable than the reconstruction of the grating from the reflection spectrum. This conclusion is proved analytically for a uniform grating in Appendix B.

The inverse scattering problem can be stabilized by calculating accurately the function $b(k)$ from the reflection spectrum. However, the calculation of the function $b(k)$ cannot be performed directly when the grating reflectivity is strong and the reflection spectrum contains noise. In Section 3 we show how to regularize the function $b(k)$. When the function $b(k)$ is regularized, it can be used to regularize the complex reflection spectrum and to accurately reconstruct the grating profile by using the ILP algorithm.

3. REGULARIZATION METHOD

When the grating is reconstructed from its reflection spectrum, noise located in the frequencies inside the Bragg zone of the spectrum causes a very large error in the reconstruction of the grating. In Section 2 we showed that the function $b(k)$ uniquely defines the grating structure. When the grating is reconstructed from the function $b(k)$, noise in the function $b(k)$, located at frequencies inside the Bragg zone of the spectrum, is not significantly enhanced by the inverse scattering algorithm. Therefore to stabilize the inverse scattering algorithm it may be useful to reconstruct the grating structure from the function $b(k)$. We show in this section how the properties of the function $b(k)$ can be used to regularize the inverse scattering problem and develop a method for reconstructing the function $b(k)$ from a noisy reflection spectrum. Our method requires an approximated prior knowledge of the grating length.

When the reflection spectrum does not contain noise, we can use Eq. (6) to accurately calculate the magnitude of the functions $b(k)$ and $a(k)$ from the reflection spectrum $r(k)$. The magnitude of the function $a(k)$ can be used to calculate the phase of $a(k)$ by using the Hilbert transform. Then, with Eq. (2), the phase of $b(k)$ can be found from the phase of the function $a(k)$ and from the phase of the complex reflection spectrum, $r(k)$. When the reflection spectrum is noisy, the noise in the Bragg zone of the reflection spectrum is enhanced in the calculation of the functions $|b(k)|^2$ and $|a(k)|^2$ owing to the denominator part in Eq. (6). Thus, when the grating reflectivity is high, we cannot use Eq. (6) to calculate the functions $|a(k)|^2$ and $|b(k)|^2$ for frequencies within the Bragg zone of a noisy reflection spectrum. We develop a method that allows us to reconstruct both the function $|b(k)|^2$ and the function $|a(k)|^2$ inside the Bragg frequency region using the data of the reflection spectrum outside the Bragg region. Then the phases of the functions $a(k)$ and $b(k)$ can be calculated similarly to the case of a noiseless reflection spectrum. Once the functions $|b(k)|^2$ and $|a(k)|^2$ are recovered, we are able to calculate the function $b(k)$, which contains all the information on the grating structure. Therefore the calculation of the functions $|b(k)|^2$ and $|a(k)|^2$ allows us to regularize the inverse scattering problem. The regularization is performed by replacing the amplitude of the noisy reflection spectrum with the amplitude calculated from the function $|b(k)|^2$. Then the regularized reflection spectrum is used to reconstruct the grating profile.

The calculation of the functions $|a(k)|^2$ and $|b(k)|^2$ inside the Bragg zone from the data outside the Bragg zone is possible because of the analytical properties of these functions. Equation (3) indicates that the functions $a(k)$ and $b(k)$ are analytical functions for complex values of the wave number k, since the integrals in the equation are defined for all complex values of k.[13] Therefore the functions $a(k)a(k^*)^*$ and $b(k)b(k^*)^*$ are also analytical in the complex plane of the wave number k. Analytical functions are uniquely determined by their values on any arc in the complex plane.[14] Specifically, the values of the functions $|a(k)|^2$ and $|b(k)|^2$ outside the Bragg zone are theoretically sufficient to calculate the functions inside the Bragg zone. However, in practice, the functions $|a(k)|^2$ and $|b(k)|^2$ are not known continuously and exactly as required by Ref. 14 but are sampled with a finite resolution and also contain numerical and experimental errors. Therefore some additional information should be used in order to increase the accuracy of the extraction of the functions $|a(k)|^2$ and $|b(k)|^2$ inside the Bragg zone. In our work, we use an estimated knowledge of the grating length in order to calculate the functions $|a(k)|^2$ and $|b(k)|^2$ inside the Bragg zone.

We denote the error in the reflection spectrum by $\Delta r(k) = \bar{r}(k) - r(k)$, where $r(k)$ and $\bar{r}(k)$ are the accurate and the noisy reflection spectra, respectively. The functions $|b(k)|^2$ and $|\bar{b}(k)|^2$ can be calculated from the reflection functions $r(k)$ and $\bar{r}(k)$ by using Eq. (6). Because of the denominator part in Eq. (6), the function $|\bar{b}(k)|^2$ may have a large error with respect to the noise-free function $|b(k)|^2$ for frequencies inside the Bragg zone of the reflection spectrum. We denote the Fourier transforms of the functions $|b(k)|^2$ and $|\bar{b}(k)|^2$ by $B(\tau)$ and $\bar{B}(\tau)$, respectively.

The theoretical analysis given up to this point in the paper was developed by using a continuous model for the wave propagation inside the grating. In a practical problem the reflection spectrum is sampled with a finite resolution. Therefore we assume that the reflection spectrum is sampled by using a wave-number sampling period Δk. Since the error in the function $|\bar{b}(k)|^2$ receives its maximum values inside the Bragg frequency region, we assume that the error function $\Delta B(\tau) = \bar{B}(\tau) - B(\tau)$ can be approximated by

$$\Delta B(\tau) \simeq \sum_{n=1}^{N} c_n \exp(ik_n\tau), \tag{7}$$

where N is the number of sampled points inside the Bragg frequency zone and $\{k_n\}$ are the corresponding wave-number components inside the Bragg zone.

Equation (3) shows that the function $B(\tau)$ is confined to the time interval $[-2L, 2L]$. However, since the function $\bar{B}(\tau)$ contains a large error, it is not confined to the interval $[-2L, 2L]$. The function $\bar{B}(\tau)$ in the region outside the interval $[-2L, 2L]$ is equal to the error function $\Delta B(\tau)$ in that region. Therefore we can use the function $\bar{B}(\tau)$ outside the interval $[-2L, 2L]$ to calculate the coefficients $\{c_n\}$, given in relation (7). After the coefficients $\{c_n\}$ are calculated, they can be used to correct the function $|b(k)|^2$ for frequencies inside the Bragg zone and to reconstruct the grating. This procedure is summarized in more detail below.

The time region in which we can calculate the function $\bar{B}(\tau)$ is determined by the sampling period of the reflection spectrum Δk and is equal to $[-\pi/\Delta k, \pi/\Delta k]$. For our method to be used, the sampling period should be small enough that the interval $[-\pi/\Delta k, \pi/\Delta k]$ is significantly larger than the interval $[-2L, 2L]$, as explained below. We calculate the coefficients $\{c_n\}$ by minimizing the square error between the function $\bar{B}(\tau)$ and the function $\Sigma_{n=1}^{N} c_n \exp(ik_n\tau)$ in the time interval $I = [-\pi/\Delta k, -2L] \cup [2L, \pi/\Delta k]$. Since the functions $\{\exp(ik_n\tau)\}$ are not orthonormal in the Hilbert space $L^2(I)$, we cannot find directly the coefficients $\{c_n\}$ by using Fourier analysis. Instead, we need to find an orthonormal basis in $L^2(I)$ for the space spanned by the functions $\{\exp(ik_n\tau)\}$. The new basis functions $\{f_n(\tau)\}$ are found by using the Gram–Schmidt orthonormalization procedure.[15] The new basis functions $\{f_n(\tau)\}$ are a superposition of exponential functions $\{\exp(ik_n\tau)\}$.[15] Therefore the functions $\{f_n(\tau)\}$ are defined in the whole interval $[-\pi/\Delta k, \pi/\Delta k]$, although the orthonormality condition is fulfilled only in the Hilbert space $L^2(I)$. The function $\Delta B(\tau)$ can then be approximated accurately in the time interval $[-2L, 2L]$ by use of

$$\Delta B(\tau) \simeq \sum_{n=1}^{N} d_n f_n(\tau), \tag{8}$$

where the coefficients of the new expansion $\{d_n\}$ can be found from the connection $d_n = (\Delta B, f_n)$ and the operation $(\cdot, \cdot)$ denotes the inner product in the Hilbert space of $L^2(I)$.[15] Once the coefficients $\{d_n\}$ are found, they can be used to calculate the error function $\Delta B(\tau)$ over the interval $[-\pi/\Delta k, \pi/\Delta k]$ by use of relation (8). The function $\Delta B(\tau)$ is then used to correct the error in the function $|\bar{b}(k)|^2$ for frequencies inside the Bragg zone.

The separation in our algorithm between the data of the function $|b(k)|^2$ inside and outside the Bragg zone can be performed owing to the denominator in Eq. (6). When the reflectivity inside the Bragg zone is close to unity, the value of the denominator in Eq. (6) becomes close to zero. Therefore noise in the Bragg zone of the reflection spectrum is amplified in the calculation of the function $|\bar{b}(k)|^2$. In this case, the estimation of the function $|b(k)|^2$ inside the Bragg zone by our method is more accurate than the function $|\bar{b}(k)|^2$, calculated directly from the Bragg zone of the noisy reflection spectrum. Therefore, when the grating reflectivity is high (maximum reflectivity $\gtrsim 0.95$), the accuracy of our method is determined mainly by the transformation given in Eq. (6) and is less affected by the specific statistics of the noise in the reflection spectrum. As a result, our method gives accurate results for a wide variety of noise statistics, as is demonstrated in Section 4. When the grating reflectivity is low (maximum reflectivity $\lesssim 0.95$), the estimation of the function $|b(k)|^2$ in the Bragg zone according to the data outside the Bragg zone may be less accurate than the original data. In this case, our method does not reduce the noise in the reconstruction of the grating but may create an additional error. However, when the grating reflectivity is low, the reconstruction of the grating is stable and there is no need for a regularization of the solution.

Our method is based on recovering the coefficients $\{d_n\}$ from the function $\bar{B}(\tau)$ on the interval I. Thus, for the function $\bar{B}(\tau)$ to be defined in the region I, the sampling period Δk should be greater than $\pi/(2L)$. In a previous study we showed that the grating can be reconstructed from a noiseless reflection spectrum sampled with a sampling period of $\pi/(2L)$.[12] When the sampling period Δk is only slightly larger than $\pi/(2L)$, the method presented in this paper may become unstable since the size of the interval I is very small compared with the size of the interval $[-2L, 2L]$. Therefore our method requires increasing the spectral resolution of the reflection spectrum in order to reconstruct the grating from noisy data. Our numerical simulations show that as the spectral sampling period Δk is reduced, the stability of our method increases. However, our numerical simulations also indicate that there is no significant improvement in the stability of our method when the sampling period Δk is decreased below $\sim\pi/(10L)$.

The performance of our reconstruction technique is determined by the accuracy to which we are able to estimate the function $\Delta B(\tau)$ in the interval $[-2L, 2L]$ from the data of the function given outside that interval. Our numerical simulation shows that the energy of the basis functions $\{f_n(\tau)\}$ inside the interval $[-2L, 2L]$ increases exponentially as the function index n increases. Therefore, for a given grating length and a sampling resolution, the calculation of the coefficients $\{d_n\}$ will contain more error as the number of basis functions N increases. Thus for a given resolution it is desired that the bandwidth of the Bragg frequency region of the grating be as narrow as possible. Denoting the wave-number bandwidth of the Bragg zone by BW_B, the number of sampled points in the Bragg zone is equal to $N = \mathrm{BW}_B/\Delta k$. Since we have found from our numerical simulations that there is no significant improvement in the stability of our method when the sampling period Δk is reduced below $\pi/(10L)$, the accuracy of our method is better when the product $\mathrm{BW}_B L$ is as small as possible.

Numerical simulations show that for a grating with a given product $\mathrm{BW}_B L$, our reconstruction technique gives the lowest error for gratings with a high reflectivity. When the reflectivity of the grating increases, the error in the Bragg frequency region of the function $B(k)$ is also significantly enhanced, as indicated by Eq. (6). In this case the energy of the function $\bar{B}(\tau)$ outside the time interval $[-2L, 2L]$ also increases significantly, and the error function $\Delta B(\tau)$ can be more accurately found by using relation (8).

The two limitations of our method described above indicate that our reconstruction method is most accurate when the grating reflectivity is high and the Bragg frequency region is narrow. Since the bandwidth of the Bragg zone of a uniform grating is narrow compared with that of an apodized or a chirped grating with the same length and reflectivity, our method gives the lowest error for quasi-uniform gratings. Using this method we were able to reconstruct theoretically a uniform grating with a maximum reflectivity of $1-10^{-8}$ from a reflection spectrum that contained a high level of noise, as shown in Section 4. Our method also works for apodized gratings; however, for accurate results to be obtained, the noise level in the reflection spectrum of apodized gratings should be significantly lower than that in quasi-uniform gratings, as shown in the next section.

4. NUMERICAL RESULTS

In this section we demonstrate our method for reconstructing highly reflecting fiber Bragg gratings from noisy reflection spectra and compare its performance with a previous method given in Ref. 9. In all the examples given below, we added to each calculated point of the reflection spectrum a complex random variable that represented the noise in the experiment. In the first four examples, the random variables were independent and had a Gaussian distribution with zero mean and a uniformly distributed phase in the region $[-\pi, \pi]$. In the last example, the random variables were correlated Gaussian random variables. The complex reflection spectra of the gratings were calculated by using the method shown in Ref. 10. The basis functions $\{f_n(\tau)\}$, given in relation (8), were calculated by using a modified Gram–Schmidt procedure[16] with double-precision accuracy (64 bits) to avoid numerical errors in our calculations.

In the first example, we compared our method with the method given in Ref. 9. A uniform grating with a coupling coefficient $q = 500\,\mathrm{m}^{-1}$ and a length $L = 1$ cm, which was analyzed in Ref. 9, was reconstructed. The complex reflection spectrum of the grating was sampled with a bandwidth of 10 nm and resolution of 0.002 nm as performed in Ref. 9. When the standard deviation of the noise in the amplitude of the complex reflection spectrum was 10^{-5}, our method as well as the method used in Ref. 9 could accurately reconstruct the grating. However, when the noise level was increased, only our reconstruction technique resulted in an accurate reconstruction of the grating. Figure 1 shows the reconstruction of the grating by use of the method developed in Ref. 9 (solid curve) compared with a direct reconstruction of the grating by using the ILP algorithm (dashed curve) and a reconstruction of the grating using the method developed in this paper (dotted curve) when the standard deviation of the noise variables was 5×10^{-5}, i.e., five times larger than the noise level used in Ref. 9. The figure shows that the method developed in Ref. 9 gives a large error in the reconstruction of the grating even for a relatively low noise level, which does not affect the stability of the ILP algorithm. In contrast, the reconstruction technique given in this paper, as well as the ILP algorithm performed without any regularization, could accurately reconstruct the grating profile.

In the second example, we reconstructed a uniform grating with a coupling coefficient $q = 2.5 \times 10^3\,\mathrm{m}^{-1}$ and length $L = 4$ mm. The maximum reflectivity of the grating was approximately $1-10^{-8}$. The reflection spectrum of the grating was sampled with a bandwidth of 40 nm and a resolution of 0.01 nm. The standard deviation of the noise-variable amplitude, added to the complex reflection spectrum, was equal to 0.02. The Bragg zone was defined as the frequency region where the grating reflectivity was higher than 75%. We used the ILP algorithm to reconstruct the grating profile from the regularized reflection spectrum obtained by using the technique described in this paper. Figure 2 compares the amplitude of the coupling coefficient reconstructed from the regularized reflection spectrum (solid curve) with the original coupling coefficient (dashed curve). The results in Fig. 2

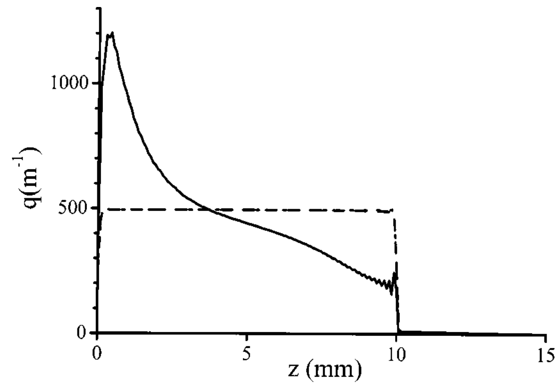

Fig. 1. Reconstruction of a uniform grating with coupling coefficient $q = 500\,\mathrm{m}^{-1}$ and length $L = 1$ cm from a noisy reflection spectrum. The figure compares the reconstruction obtained with the method developed in Ref. 9 (solid curve) with a direct reconstruction by the ILP algorithm (dashed curve) and with a reconstruction by the method presented in this paper (dotted curve). The reflection spectrum of the grating was sampled with a bandwidth of 10 nm and resolution of 0.002 nm. The standard deviation of the noise variables, added to the complex reflection spectrum, was equal to 5×10^{-5}. The ILP algorithm as well as the algorithm presented in this paper have accurately reconstructed the grating profile.

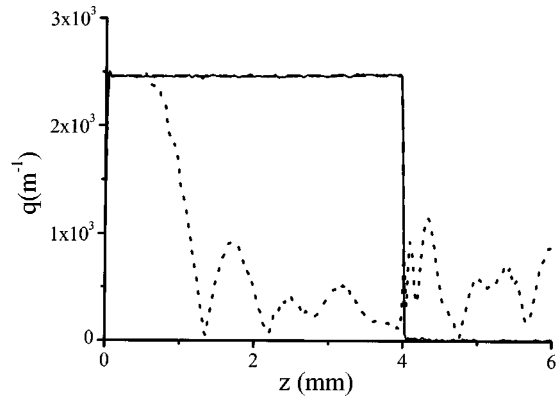

Fig. 2. Reconstruction of a uniform grating with length of $L = 4$ mm, coupling coefficient $q = 2.5 \times 10^3\,(\mathrm{m}^{-1})$, and maximum reflectivity $1-10^{-8}$ from a noisy reflection spectrum. The reconstruction was performed by using the method described in this paper (solid curve) and was compared with the original profile (dashed curve) and with a direct reconstruction of the grating from the noisy reflection spectrum (dotted curve). The standard deviation of the noise-variable amplitude added to the complex reflection spectrum was equal to 0.02.

were also compared with a direct reconstruction of the coupling coefficient from the noisy reflection spectrum by using the ILP method (dotted curve). The figure clearly shows an excellent reconstruction of the grating by our method, whereas the grating could not be reconstructed directly.

In the third example, a grating with length $L = 4$ mm and a Gaussian profile given by $q(z) = 1900 \exp[-6.2 \times 10^5 (z - L/2)^2]$ was reconstructed. The maximum reflectivity of the grating was approximately 0.999. The reflection spectrum of the grating was sampled with a bandwidth of 20 nm and a resolution of 0.02 nm. The standard deviation of the amplitude of the noise variables in the complex reflection spectrum was equal to 10^{-3}. The Bragg zone was defined as the frequency region where the grating reflectivity was higher than 75%. We used the ILP algorithm to reconstruct the coupling coefficient from the regularized spectrum obtained by using the technique shown in this paper. Figure 3 shows the coupling coefficients reconstructed from the regularized (solid curve) and noisy (dotted curve) reflection spectra, compared with the original coupling coefficient (dashed curve). An excellent reconstruction is again obtained when the reflection spectrum is regularized, whereas the grating could not be reconstructed directly.

In the fourth example, we reconstructed a uniform grating from a reflection spectrum that contained a very high level of noise. The grating had a coupling coefficient of $q = 1320\,\mathrm{m}^{-1}$ and a maximum reflectivity of ~0.9999. The reflection spectrum of the grating was sampled with a bandwidth of 20 nm and a resolution of 0.005 nm. The standard deviation of the noise-variable amplitude in the complex reflection spectrum was equal to 0.1. The noisy reflection spectrum of the grating is shown in Fig. 4. The Bragg zone was defined as the frequency region where the grating reflectivity was higher than 50%. Figure 5 compares the amplitude of the coupling coefficient reconstructed from the regularized reflection spectrum (solid curve) with the original grating structure (dashed curve). The results in Fig. 5 were also compared with a direct reconstruction of the coupling coefficient from the noisy reflection spectrum (dotted curve). The figure shows clearly that a very good reconstruction of the grating can be obtained with our method in spite of the high level of noise that was added to the reflection spectrum.

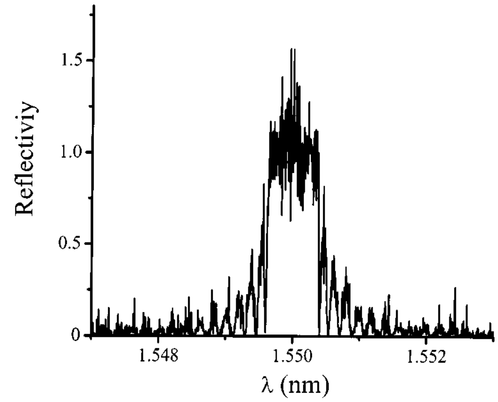

Fig. 4. Noisy reflectivity of a uniform grating with length 4 mm, coupling coefficient $q = 1320\,(\mathrm{m}^{-1})$, and maximum reflectivity 0.9999. The standard deviation of the noise variable amplitude added to the complex reflection spectrum was equal to 0.1.

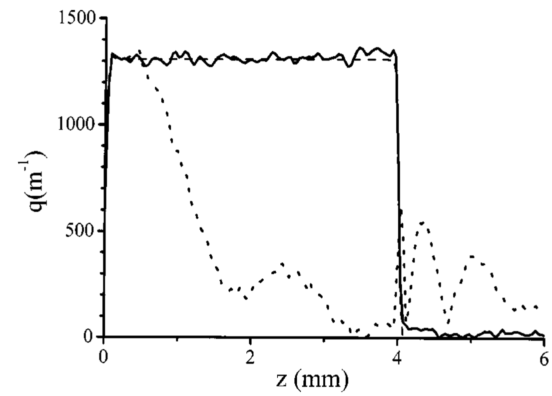

Fig. 5. Reconstruction of the uniform grating with the reflectivity shown in Fig. 4. Curve definitions as in Fig. 2.

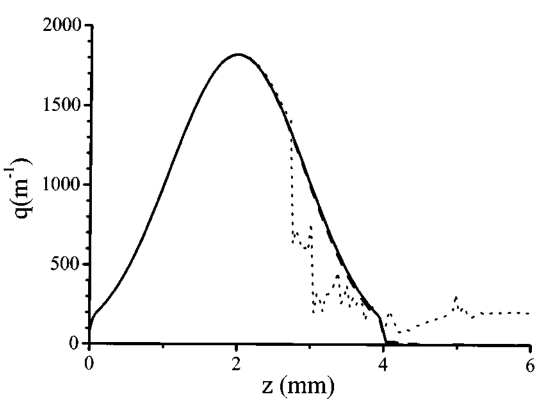

Fig. 3. Reconstruction of a Gaussian grating with length $L = 4$ mm, coupling coefficient $q(z) = \{1900 \exp[-6.2 \times 10^5 (z - L/2)^2]\}\,\mathrm{m}^{-1}$, and maximum reflectivity 0.999. The reflection spectrum of the grating was sampled with a bandwidth of 20 nm and a resolution of 0.02 nm. The standard deviation of the noise-variable amplitude added to the complex reflection spectrum was equal to 10^{-3}. Curve definitions as in Fig. 2.

In the last example, we reconstructed a uniform grating with a coupling coefficient of $q = 1320\,\mathrm{m}^{-1}$ and a maximum reflectivity of ~0.9999. We added to each point in the reflection spectrum a Gaussian random variable with zero mean. Unlike in previous examples, the noise variables added to the reflection spectrum were correlated; they were generated by a Gaussian ARMA(1, 0) process.[17] The covariance matrix of the random variables is given by

$$\mathrm{cov}[n(k_i),\, n(k_j)] = 2 \times 10^{-2} \rho^{|i-j|} \tag{9}$$

where $n(k_i)$ and $n(k_j)$ are the noise variables in the wave numbers k_i and k_j, respectively, and $\rho = 0.9$. The reflection spectrum of the grating was sampled with a bandwidth of 20 nm and a resolution of 0.02 nm. The Bragg zone was defined as the frequency region where the grating reflectivity was higher than 75%. Figure 6 compares the amplitude of the coupling coefficient reconstructed from the regularized reflection spectrum (solid curve) with the original grating structure (dashed curve). The results in Fig. 6 were also compared with a direct reconstruction of the coupling coefficient from the noisy reflection spectrum (dotted curve). The figure shows that a

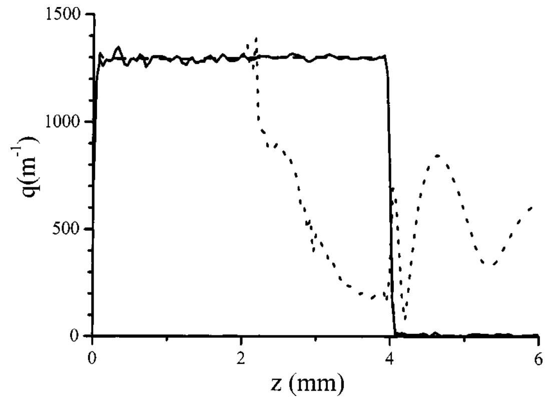

Fig. 6. Reconstruction of a uniform grating with length 4 mm, coupling coefficient of $q = 1320\,(\mathrm{m}^{-1})$, and maximum reflectivity 0.9999. We added to each point in the reflection spectrum a Gaussian random variable with zero mean. The noise variables were generated by a Gaussian ARMA(1, 0) process with a covariance matrix given in Eq. (9) with a parameter $\rho = 0.9$. Curve definitions as in Fig. 2.

very good reconstruction of the grating can be obtained by our method even in the case where the noise variables added to the reflection spectrum are correlated. We have also verified that an accurate reconstruction is obtained for other values of the parameter ρ that were tested: $\rho = 0.2, 0.5, 0.8, 0.99, 0.999$.

5. CONCLUSION

We have demonstrated a new method for reconstructing highly reflecting fiber Bragg gratings from noisy reflection spectra. In a case in which the grating reflectivity is high, noise in the Bragg zone of the reflection spectrum is enhanced by the inverse scattering algorithms. Our method is based on regularizing the reflection spectrum by using the data obtained outside the Bragg zone of the reflection spectrum. The mathematical properties of the reflection spectrum of a grating with a known length enable us to accurately regularize the data in the Bragg frequency region of the reflection spectrum. Using our method, we were able to accurately reconstruct the structure of uniform and apodized gratings from noisy reflection spectra. The performance of our method was optimal for uniform gratings. Our algorithm can be used to accurately reconstruct, to our knowledge for the first time, the structure of highly reflecting gratings from a reflection spectrum that contains a high level of noise. Such gratings cannot be reconstructed with methods described in previous studies.

APPENDIX A

In this appendix we give the derivation of the results presented in Eqs. (3) and (4). We assume that the solution $U(k, z)$ with the boundary conditions $U(k, z = L) = (0, 1)$ can be represented in the following integral form:

$$U(k, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp[ik(z - L)] + \int_{z-L}^{L-z} F(\tau, z)\exp(ik\tau)d\tau, \qquad 0 \leq z \leq L, \tag{A1}$$

where $F(\tau, z) = [f_1(\tau, z), f_2(\tau, z)]$ is the time-domain kernel function of the solution $U(k, z)$. Substituting Eq. (A1) into Eq. (1) we obtain the following equations,

$$\frac{\mathrm{d}f_1(\tau, z)}{\mathrm{d}z} - \frac{\mathrm{d}f_1(\tau, z)}{\mathrm{d}\tau} = q(z)f_2(\tau, z),$$
$$\frac{\mathrm{d}f_2(\tau, z)}{\mathrm{d}z} + \frac{\mathrm{d}f_2(\tau, z)}{\mathrm{d}\tau} = q^*(z)f_1(\tau, z), \tag{A2}$$

with the boundary conditions

$$f_1(z - L, z) = -\frac{q(z)}{2},$$
$$f_2(L - z, z) = 0. \tag{A3}$$

Using the theory of characteristics,[18] as performed in Ref. 19 for an infinite grating, we obtain that Eq. (A2) with the boundary conditions given in Eq. (A3) has a unique solution. Therefore the solution $U(k, z)$ can be represented in the integral form given in Eq. (A1). Integrating Eq. (A2) we obtain

$$f_1(L - z, z) = -\frac{q(z = L)}{2},$$
$$f_2(z - L, z) = -\frac{1}{2}\int_z^L |q(z)|^2 \mathrm{d}z. \tag{A4}$$

Equations (3) and (4) are obtained directly by using Eqs. (A1), (A3), and (A4) and the definition of the functions $a(k)$ and $b(k)$.

APPENDIX B

In this appendix we calculate the effect of a small perturbation in the coupling coefficient of a highly reflecting grating on the functions $a(k)$ and $b(k)$ and on the reflection spectrum $r(k)$. We show that a perturbation in the grating structure has a small effect on the reflection spectrum of the grating for frequencies inside the Bragg zone. Thus any small inaccuracy or a noise in the reflection spectrum in frequencies within the Bragg zone may cause a large error in the reconstruction of the grating. We also show that the spectral functions $a(k)$ and $b(k)$ are significantly affected by a perturbation in the grating structure. Thus the reconstruction of the grating structure is not sensitive to noise in the functions $a(k)$ and $b(k)$. The analysis is performed for a uniform grating since such gratings have a closed-form solution for the coupled-mode equations. The results can be generalized for gratings with a more general profile using approximated solutions such as the WKB method.[20]

We consider a uniform grating with a coupling coefficient $q(z) = q$, where q is a real constant. The fields that propagate inside the grating $U(k, z)$ can be found by using the following relation[10]:

$$U(k, z) = \begin{bmatrix} \cosh(\gamma\Delta) + \frac{ik}{\gamma}\sinh(\gamma\Delta) & -\frac{q}{\gamma}\sinh(\gamma\Delta) \\ -\frac{q}{\gamma}\sinh(\gamma\Delta) & \cosh(\gamma\Delta) - \frac{ik}{\gamma}\sinh(\gamma\Delta) \end{bmatrix} \times U(k, z = L) \tag{B1}$$

where $\Delta = L - z$ and $\gamma = (q^2 - k^2)^{1/2}$.

We add a small perturbation to the coupling coefficient $q(z) = q + \epsilon\Delta q(z)$, where $\Delta q(z)$ is a complex coupling coefficient, and calculate the perturbation in the fields inside the grating up to the first order in ϵ, $U(k, z) + \epsilon\Delta U(k, z)$. The propagation equation for the fields $\Delta U(k, z) = [\Delta u_1(k, z), \Delta u_2(k, z)]$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\Delta u_1(k, z) + ik\Delta u_1(k, z) = q\Delta u_2(k, z) + \Delta q(z)u_2(z),$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\Delta u_2(k, z) - ik\Delta u_2(k, z) = q\Delta u_1(k, z) + \{\Delta q(z)\}^* u_1(z). \tag{B2}$$

Equation (B2) is an inhomogeneous equation for the variable $\Delta U(k, z)$, with a null boundary condition, $\Delta U(k, z = L) = 0$.

Since Eq. (B2) is a linear equation, we solve it using the superposition method.[21] We first consider the solution to Eq. (B2), $\Delta V(k, z)$, with the coupling coefficient $\Delta q(z) = \delta(z - z_0)$ and the boundary conditions $\Delta V(k, z = L) = 0$. The solution to Eq. (B2) for $z < z_0$ is the same as the solution to the homogeneous part of the equation [with $\Delta q(z) = 0$] and with the boundary conditions $\Delta V(k, z = z_0^-) = [-u_2(k, z = z_0^-), -u_1(k, z = z_0^-)]$. The unperturbed waves at $z = z_0^-$, $u_1(k, z = z_0^-)$, and $u_2(k, z = z_0^-)$ are calculated by using the transmission matrix given in Eq. (B1). Then the homogeneous equation with the boundary conditions at $z = z_0^-$ should be solved again by using the transmission matrix given in Eq. (B1). After some calculations we obtain

$$\Delta v_1(k, z = 0) = -\frac{q^2}{\gamma^2}\cosh(\gamma L) + \frac{k^2}{\gamma^2}\cosh[\gamma(L - 2z_0)] - i\frac{k}{\gamma}\sinh[\gamma(L - 2z_0)],$$

$$\Delta v_2(k, z = 0) = \frac{q}{\gamma}\sinh(\gamma L) - i\frac{k}{\gamma}\cosh(\gamma L) + i\frac{k}{\gamma}\cosh[\gamma(L - 2z_0)]. \tag{B3}$$

Since the system is linear, the solution for an arbitrary perturbation, $\Delta q(z)$, can be calculated by using the connection $\Delta q(z) = \int_{-\infty}^{\infty}\Delta q(z_0)\delta(z - z_0)\mathrm{d}z_0$. The solution in this case is given by the superposition integral,[21]

$$\Delta U(k, z = 0) = \int_0^L [\Delta q(z_0)\Delta v_1(k, z_0), \{\Delta q(z_0)\}^*\Delta v_2(k, z_0)]\mathrm{d}z_0. \tag{B4}$$

The perturbation to the functions $a(k)$ and $b(k)$, $\epsilon\Delta b(k)$ and $\epsilon\Delta a(k)$, respectively, can be obtained from Eq. (B4) by using the relation $[\Delta b(k), \Delta a(k)] = \Delta U(k, z = 0)$. The perturbation to the amplitude of the reflection spectrum $\epsilon\Delta|r|(k)$ can be calculated from the perturbation to the functions $a(k)$ and $b(k)$, $\epsilon\Delta b(k)$ and $\epsilon\Delta a(k)$, and Eq. (2).

To find the effect of the perturbation in the coupling coefficient on the functions $a(k)$, $b(k)$, and $|r|(k)$, we calculate the relative change in these functions that is due to the perturbation in the grating profile. We define the functions $q(k)$ and $\Delta q(k)$ as the Fourier transforms of the coupling coefficient $q(z)$ and the perturbation to the coupling coefficient $\Delta q(z)$, respectively. When the grating is highly reflecting, $\exp(qL) \gg 1$, we obtain that for frequencies inside the Bragg zone that fulfill $k \ll 1/L$, the relative change in the functions $a(k)$, $b(k)$, and $|r|(k)$ is given by

$$\left|\frac{\Delta a(k)/a(k)}{\Delta q(k)/q(k)}\right| \simeq qL,$$

$$\left|\frac{\Delta b(k)/b(k)}{\Delta q(k)/q(k)}\right| \simeq qL,$$

$$\left|\frac{\Delta|r|(k)/|r(k)|}{\mathrm{Re}\{\Delta q(k)\}/q(k)}\right| \simeq t(k)^2 qL. \tag{B5}$$

Equation (B5) shows that the perturbation in the grating profile causes a relative change in the reflection spectrum inside the Bragg zone that is proportional to the square of the transmission function of the grating, $t(k)^2$. When the grating reflectivity is high, the transmission of the grating is very small for frequencies located inside the Bragg zone of the grating. Therefore a large change in the grating profile causes only a small change in the amplitude of the reflection spectrum for frequencies located inside the Bragg zone of the grating. Thus in the inverse scattering problem a perturbation in the amplitude of the reflection spectrum may be amplified by a factor of $1/|t(k)|^2$ in the calculation of the coupling coefficient. This conclusion is in agreement with the results given in Ref. 8. In contrast, Eq. (B5) shows that a perturbation in $q(z)$ causes a significant change in the functions $a(k)$ and $b(k)$ for frequencies inside the Bragg zone. Thus perturbations in the functions $a(k)$ and $b(k)$ are not amplified in the reconstruction of the grating structure.

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