Noise distribution in the radio frequency spectrum of optoelectronic oscillators

Etgar C. Levy,[1,*] Moshe Horowitz,[1] and Curtis R. Menyuk[2]

[1]*Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa 32000 Israel*
[2]*Department of Electrical Engineering, University of Maryland, Baltimore, Maryland 21250, USA*
Corresponding author: etgarlevy@gmail.com

Received April 28, 2008; revised August 11, 2008; accepted October 12, 2008; posted November 4, 2008 (Doc. ID 95520); published December 3, 2008

We analyze the distribution of the rf spectrum in optoelectronic oscillators due to the finite duration of the spectrum measurement. The distribution of the periodogram or the rf spectrum at a given frequency is calculated using a reduced model and is compared to a comprehensive numerical simulation. The model shows that the rf spectrum at a given frequency fluctuates from measurement to measurement with an exponential distribution.

OCIS codes: 060.2320, 070.4790, 230.0250, 230.4910.

Since optoelectronic oscillators (OEOs) were invented [1,2], extensive work has been performed to minimize their phase noise and to improve their Q. Owing to the stochastic nature of the phase noise, the noise spectrum fluctuates from one measurement to another. Yao and Maleki [2] presented a reduced model to analyze the average rf spectrum in a single-cavity OEO. However, in experiments the rf spectrum is measured over a limited duration and changes from one measurement to another. This effect, which is observed in experiments, cannot be described using the Yao–Maleki model [2].

The rf spectrum or periodogram is an estimate of the average rf spectrum that is obtained by calculating or measuring the signal spectrum over a finite time duration. The variations or the fluctuations in the rf spectrum determine the shortest measurement duration that is required for applications based on the OEO, such as a Doppler radar system that measures the velocity of slow-moving targets [3].

In a previous work [4], we described a new comprehensive model for simulating phase noise and dynamic effects in OEOs. This model is based on Monte Carlo simulations and takes into account physical effects, such as the fast nonlinearity of the electro-optic modulator, gain saturation and the different noise sources that exist in OEOs. In this Letter, we calculate the fluctuations of the rf spectrum in OEOs when the duration of the spectral measurement is finite. The distribution of the rf spectrum at a given frequency is calculated using a reduced model and is compared to the results of our comprehensive numerical simulation. We show that the rf spectrum at every frequency fluctuates from one simulation run to another with an exponential distribution assuming that the resolution of the spectral measurement is equal to the inverse of the measurement duration. Given the spectral distribution, it is possible to calculate the spectral fluctuations when a fixed averaging time is used in an experiment [5] or when the measurement bandwidth is increased beyond the inverse of the measurement duration. The rf spectral distribution at a given frequency is needed to determine the expected fluctuation in the rf spectrum obtained in different measurements as a function of the measurement duration and bandwidth. This result, which cannot be calculated using the Yao–Maleki model, is important to determine the measurement duration that is required to assure a reliable measurement of the phase noise.

In OEOs the rf spectral density is nearly equal to the phase-noise spectral density [2,4], because the amplitude noise is negligible relative to the phase noise and because the phase fluctuation is significantly smaller than unity for frequency offsets that are smaller than about half the cavity mode spacing. The output signal of an OEO can be approximated by a sinusoidal wave with a radial carrier frequency ω_c, a time-dependent phase $\phi(t)$, and a time-dependent amplitude $|a(t)|$ [2,4] so that

$$V(t) = |a(t)|\cos[\omega_c t + \phi(t)] = a(t)\exp(-i\omega_c t)/2 + \text{c.c.} \tag{1}$$

We assume that the complex signal envelope, denoted by $a(t)=|a(t)|\exp[-i\phi(t)]$, is slowly varying relative to the carrier period $|\mathrm{d}\phi/\mathrm{d}t| \ll \omega_c$ and $|\mathrm{d}|a(t)|/\mathrm{d}t| \ll |a(t)|\omega_c$.

Assuming that the OEO signal is defined over a time T, where $T \gg 1/\omega_c$, we can expand $a(t)$ as a Fourier series,

$$a(t) = \sum_{k=-\infty}^{\infty} \tilde{a}_k \exp(-i\omega_k t), \tag{2}$$

where the $\tilde{a}_k$ denote the Fourier coefficients and $\omega_k = 2\pi k/T$. The rf spectrum of the normalized signal $a(t)/\sqrt{2RP_{\text{osc}}}$ is given by

$$S_{\text{rf}}(f_k) = \frac{|\tilde{a}_k|^2}{2RP_{\text{osc}}\delta f}, \tag{3}$$

where $f_k = \omega_k/2\pi$ is the frequency offset with respect to the carrier frequency, R is the impedance at the output of the detector, P_{osc} is the carrier oscillation power, and $\delta f = 1/T$ is the frequency resolution. The average rf spectrum $\langle S_{\text{rf}}(f_k)\rangle$ is equal to the power spectral density of the rf noise.

0146-9592/08/242883-3/$15.00

In our numerical model [4] we calculated the rf spectrum using a Monte Carlo simulation. We model in each round trip the effect of the electro-optic modulator, the fiber delay, the photodiode, the rf amplifier, and the rf filter. Noise is added to the OEO by the amplifier, the detector, and the laser. The injected noise can be modeled as white and complex Gaussian noise with zero mean that is added at each round trip of the signal in the OEO [2,4]. Since the injected noise changes in each round trip, each simulation run gives a different rf spectrum. The average rf spectrum is obtained by averaging the rf spectrum from a single simulation run over a sufficiently large number of simulation runs.

We study the noise in an OEO that operates in a stationary condition and that has an oscillating signal with a complex amplitude $a(t)$. White complex Gaussian noise is added to the signal in each round trip. The added noise can be decomposed into two components: a component that is in phase with the signal $a(t)$ at that round trip and a second component that is phase shifted by 90°, which we will refer to as quadrature noise. Owing to the fast response time of the electro-optic modulator, the in-phase noise at a frequency offset that is less than the cavity mode spacing is suppressed. By contrast, the quadrature noise accumulates in the OEO loop and results in phase noise. In OEOs the injected noise in each round trip is small compared to the oscillating signal. Therefore, the change of the signal phase $\phi(t)$ in a round trip is small compared to unity and is proportional to the quadrature noise component that is added to the signal in that round trip. Moreover, owing to the small noise power, the response of the system to the injected noise can be approximated by linearizing about its stationary behavior. Since a Gaussian distribution is maintained under a linear transformation, the phase deviations from one round trip to another are independent and identically distributed with a real Gaussian distribution. Therefore, the imaginary and the real parts of the Fourier coefficients $\tilde{a}_k$ $(k \neq 0)$ will also have a Gaussian distribution with the same distribution. This assumption was verified using Monte Carlo simulations, as is described shortly. In our model, we are only requiring that the phase change of the signal in a single round trip is small. Over time the phase change may become large.

The complex normal distribution of the Fourier coefficients $\tilde{a}_k$ is completely defined by its first two moments. The mean of the real and the imaginary parts of $\tilde{a}_k$ is equal to zero for $k \neq 0$, and the variance of both components is equal to σ_k^2. The complex normal distribution of $\tilde{a}_k$ implies that the coefficients $|\tilde{a}_k|$ $(k \neq 0)$ have a Rayleigh distribution with a mean $\sqrt{\pi/2}\sigma_k$. Therefore, the distribution of the squared norm of the Fourier coefficients $|\tilde{a}_k|^2$ $(k \neq 0)$ has an exponential distribution with a mean of $2\sigma_k^2$. It follows from Eq. (3) that

$$\sigma_k^2 = RP_{\mathrm{osc}}\langle S_{\mathrm{rf}}(f_k)\rangle \delta f, \tag{4}$$

so that the rf spectrum at f_k, $S_{\mathrm{rf}}(f_k)$ has an exponential distribution with a probability density function

$$p(x) = \frac{1}{\langle S_{\mathrm{rf}}(f_k)\rangle} \exp[-x/\langle S_{\mathrm{rf}}(f_k)\rangle], \tag{5}$$

where $x \geq 0$. The probability that $x < S_{\mathrm{rf}}(f_k) < x + \mathrm{d}x$ is equal to $p(x)\mathrm{d}x$. The power spectral density of the rf noise can be found either from the Yao–Maleki model [2] or more accurately from the model in [4].

To verify the results in Eq. (5) we have simulated a simple single-loop OEO using a Monte Carlo approach. The OEO contained a Lorentzian filter with a FWHM of $\Gamma = 20$ MHz, a Mach–Zehnder modulator with a half-wave voltage of $V_\pi = 3.14$ V, and a bias voltage of $V_B = 3.14$. The rf amplifier voltage gain was equal to $G_A = 7.5$, the small-signal open-loop gain was equal to $G_S = 1.5$, the noise power density was equal to $\rho_N = 10^{-17}$ mW/Hz, and the average oscillation power at the output of the amplifier was equal to 30 mW. The loop delay τ was set equal to 0.28 μs. Figure 1 shows both the rf spectrum and the average rf spectrum as a function of the frequency offset from the carrier frequency. The measurement duration T for calculating the rf spectrum $S_{\mathrm{rf}}(f)$ was set equal to 2.8 ms and the frequency resolution was set equal to 357 Hz. For comparison, the average phase-noise spectrum was added to Fig. 1. Figure 1 indicates that the average rf spectrum is approximately equal to the average phase noise spectrum in a wide frequency range up to about 100 kHz.

Equation (4) shows that the variance of the complex normal distributed Fourier coefficients, σ_k^2, depends on the frequency offset f_k, owing to the frequency dependence of the rf spectral density, $\langle S_{\mathrm{rf}}(f_k)\rangle$. The dependence of the Fourier coefficients on the frequency offset can be eliminated by defining a normalized Fourier coefficient, $\xi_{\mathrm{rf}}(f_k) = \tilde{a}_k/[2RP_{\mathrm{osc}}\langle S_{\mathrm{rf}}(f_k)\rangle \delta f]^{1/2}$. Therefore, all of the nor-

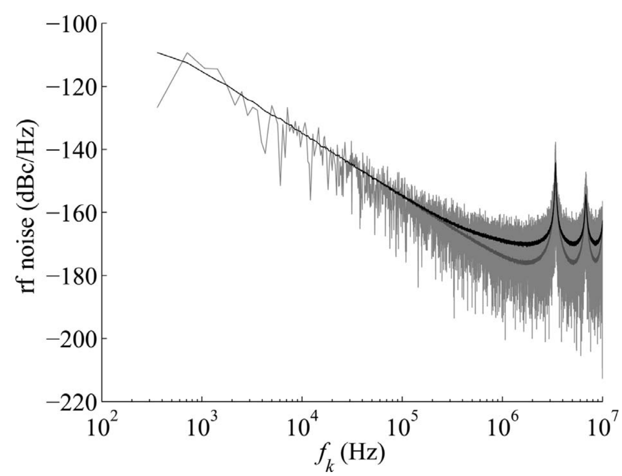

Fig. 1. Radio frequency spectrum $S_{\mathrm{rf}}(f_k)$ as a function of the frequency offset from the carrier frequency. The spectrum was calculated from a single simulation run with a measurement duration of $T = 2.8$ ms (light gray curve). The rf spectral density (black curve) and the phase spectral density (dark gray curve) were calculated by averaging the results the over 350 runs.

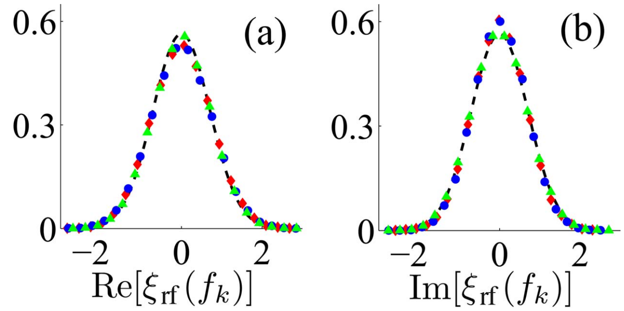

Fig. 2. (Color online) Distribution of the real and the imaginary normalized Fourier coefficients, $\xi_{\text{rf}}(f_k)$, calculated from 30,000 simulation runs at three frequencies: $f_k\tau=1/10$ (diamonds), $f_k\tau=1/20$ (circles), and $f_k\tau=1/50$ (triangles). The results are compared to a normal distribution with a variance $\sigma^2=0.5$ (dashed curve).

malized Fourier coefficients when $k\neq 0$ have a complex normal distribution with zero mean and a variance of 0.5.

Figure 2 shows the distribution of the real and the imaginary parts of the normalized Fourier coefficients, Re(ξ_{rf}) and Im(ξ_{rf}), calculated at three different frequencies: 357, 179, and 71 kHz. Figure 2 demonstrates that the distribution of each of the normalized Fourier coefficients converges to a complex normal distribution with a variance of 0.5. This result is in agreement with our model for calculating the phase noise distribution.

Figure 3 shows that the distribution of the normalized rf spectrum, $|\xi_{\text{rf}}(f_k)|^2$, at a fixed frequency $f_k\tau=1/10$, converges to an exponential distribution with a mean value of 1. Identical distributions were obtained for 20 different frequencies in the frequency range 70–350 kHz. The predicted experimental distribution of the rf spectrum, which is the main result of this Letter, can be verified by repeatedly measuring the spectrum with a fixed duration and bandwidth.

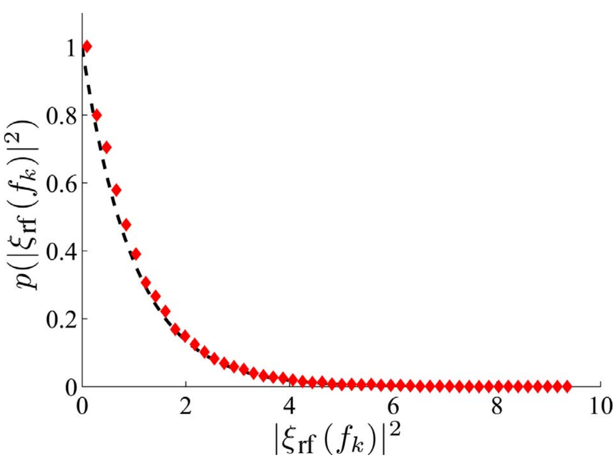

Fig. 3. (Color online) Distribution of the normalized rf spectrum, $|\xi_{\text{rf}}(f_k)|^2$, at a frequency $f_k\tau=1/10$ calculated from 30,000 runs. An exponential distribution with an average of 1 is added for comparison (dashed curve).

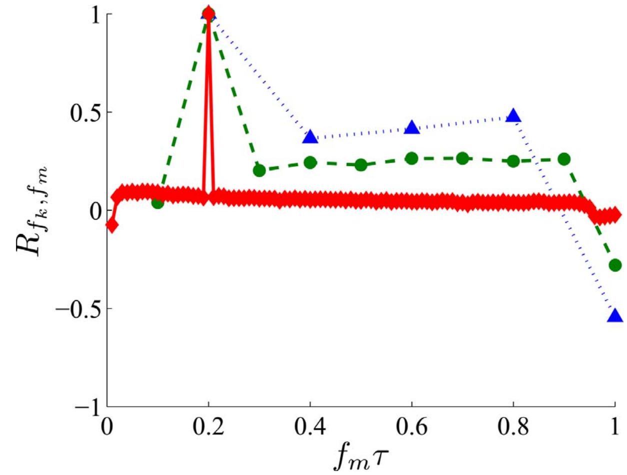

Fig. 4. (Color online) Cross correlation between the normalized Fourier coefficients $\xi_{\text{rf}}(f_k=1/5\tau)$ and $\xi_{\text{rf}}(f_m)$ calculated for three different measurement durations: $T=1/f_k$ (dotted triangles), $T=2/f_k$ (dashed circles), and $T=20/f_k$ (solid diamonds).

Our simulation enables us also to calculate the cross-correlation between the normalized Fourier coefficients and its dependence on the measurement duration:

$$R_{f_k,f_m}=\langle \xi_{\text{rf}}(f_k)\xi^*_{\text{rf}}(f_m)\rangle. \tag{6}$$

Figure 4 shows the cross-correlation value obtained for $f_k=1/5\tau$. The cross correlation was calculated for three different measurement durations: $T=1/f_k$, $T=2/f_k$, and $T=20/f_k$. From Fig. 4, we conclude that the cross-correlation between the components at different frequencies is less than 0.1 when the measurement duration is larger than $T>20/f_k$. When the measurement duration is small a correlation is found between the rf spectral components at nearby frequencies. This result indicates that the phase noise in OEOs is coherent for a limited duration.

To conclude, we have analyzed theoretically the distribution of the rf spectrum at a given frequency using a reduced model and a comprehensive numerical simulation. The rf spectral distribution at a fixed frequency has an exponential distribution.

This work is supported in part by the Defense Advanced Research Project Agency (DARPA).

References

1. X. S. Yao and L. Maleki, Opt. Lett. **21**, 483 (1996).
2. X. S. Yao and L. Maleki, J. Opt. Soc. Am. B **13**, 1725 (1996).
3. M. C. Li, IEEE Trans. Antennas Propag. **52**, 3319 (2004).
4. E. C. Levy, M. Horowitz, and C. R. Menyuk, "Modeling optoelectronic oscillators," J. Opt. Soc. Am. B (to be published).
5. K. S. Trivedi, in *Probability and Statistics with Reliability, Queueing and Computer Science Applications* (Wiley, 2002), p. 637.