V. Conclusion

The introduction of [1] claims that the "generalized" error, i.e., the $y-\hat{y}$ of this note, which is defined in [1, eq. (2.5)], converges to zero despite any initial parameter or state error regardless of input sequence or the magnitude of the constant adaptation gains. The counterexample of the preceding section disproves this claim. Furthermore, the PRBS used in the simulated failure to force the plant and identifier should provide the necessary richness requirements evoked in [1] for consistent parameter estimate convergence, thereby also belying that claim.

Presently the stall failure discussed in this note of the parallel MRAS with adaptive error filtering of [1] appears to be infrequent, but unpredictable. The only apparent conclusion is that a plant with poles near the unit circle in combination with an erratically converging identifier is most likely to generate $\hat{A}(z,k)$ root migration outside the unit circle, possibly leading to matching $\hat{C}(z,k)$ roots and identifier stall. Clearly, further study is required to firmly delineate the applicability of output error identifiers such as in [1], [4], and [5] since, unfortunately, a claim of desirable behavior, i.e., $\hat{y}\to y$, apparently does not proceed solely from the proof of "processed generalized" error decay to zero, i.e., $s(k)\to 0$ as $k\to\infty$.

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Instability of Optimal-Aim Control

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***Abstract*—It is shown that the so-called *optimal-aim control strategy* [1] might destabilize a controllable linear time-invariant system. This raises a serious question about the efficacy of this strategy when applied to a more complicated nonlinear power system.**

I. Introduction

In [1], a rather novel approach is described which is intended for regulation of a class of nonlinear systems. At each instant of time, an admissible control is chosen which minimizes the state-space angle between the state derivative and the direction of the equilibrium state. On the surface, such an approach appears to be an attractive alternative to the usual difficulties which one would encounter when applying classical techniques.[1] Delving a little below the surface, however, we find a number of fundamental difficulties associated with the stability properties of these so-called optimal-aim controls. It was demonstrated in [2] that instability might occur when using control strategies (see [3]) which are similar in spirit to the optimal-aim strategy of [1]. Strictly speaking, however, the instability theorem of [2] does *not* apply to the controllers of [1][2] (see [4] and [5]). Nevertheless, we shall see in the sequel that the optimal-aim control strategy might indeed induce instability in the case of [1] as well.

We shall present two counterexamples to the conjectured stability. In our first counterexample, an asymptotically stable linear time-invariant system is rendered unstable via application of optimal-aim control. In our second counterexample, we examine an unstable linear time-invariant system which is stabilizable via linear feedback [6]. When optimal-aim control is applied in lieu of linear feedback, the system becomes unstable.

II. Brief Review of Optimal Aiming

This control procedure (see [1] for full details) is developed for nonlinear regulation systems having additive linear control structures, i.e., we consider a system of state equations of the form

$$\dot{x}(t)=f(x(t))+Bu(t), \qquad t\geq 0,\ x(t)\in R^n$$

$$u(t)\in U=\{\omega\in R^m:\ \omega_j\in[\omega_{j0},\omega_{jl}], \qquad j=1,2,\cdots,m\}$$

with $f(\sigma_c)=0$ and $B'B=\operatorname{diag}\{C_j\}$, $C_j>0$.[3] It is implicitly assumed that $f(\cdot)$ is "sufficiently regular" to guarantee the existence and uniqueness of solutions.

Following the notation in [1], we define the set $\Delta(x(t))\triangleq\{\delta\in R^n:\ \delta=f(x(t))+B\omega:\ \omega\in U\}$ of *achievable derivatives* at the state $x(t)$ and the *reference vector* $\rho(x(t))\triangleq\sigma_c-x(t)$. ($\sigma_c$ is the desired equilibrium.) The angle between $\delta\in\Delta$ and $\rho\in R^n$ is then given by

$$\theta(\delta,\rho)=\begin{cases}0 & \text{for}\ \delta=0\ \text{and}\ \rho\neq 0\\ \cos^{-1}\left[\dfrac{\delta'\rho}{\|\delta\|\ \|\rho\|}\right]\in[0,\pi] & \text{for}\ \delta\neq 0\ \text{and}\ \rho\neq 0\\ \dfrac{\pi}{2} & \text{for}\ \rho=0\end{cases}$$

where $\|\cdot\|$ is the usual Euclidean norm.

Within this framework, a control $u(\cdot)$ and trajectory mate $x(\cdot)$ are termed on *optimal aim pair* if, at each instant of time t, we have[4]

$$\theta(\dot{x},\rho)=\min_{\delta\in\Delta}\theta(\delta,\rho)$$

and either

$$\|\dot{x}\|=\max_{\substack{\delta\in\Delta\\ \theta(\delta,\rho)=\theta(\dot{x},\rho)}}\|\delta\| \qquad \text{for}\ \theta(\dot{x},\rho)<\frac{\pi}{2}$$

or

$$\|\dot{x}\|=\min_{\substack{\delta\in\Delta\\ \theta(\delta,\pi)=\theta(\dot{x},\rho)}}\|\delta\| \qquad \text{for}\ \theta(\dot{x},\rho)\geq\frac{\pi}{2}.$$

Finally, the control $u(\cdot)$ giving rise to the minimum of $\theta(\dot{x},\rho)$ is called an *optimal-aim control*.

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[1] Infinite-dimensional function space optimizations are replaced by "simpler" finite-dimensional state space optimizations. Furthermore, the approach easily accommodates state-dependent input constraints and model variations.

[2] The possible nondifferentiable character of the control law does not enable us to satisfy the preconditions of the theorem in [2].

[3] The analysis to follow will also be valid if B is a full rank matrix. We use the diagonal matrix above to be consistent with the notation in [1].

[4] To simplify notation, the dependence on t and $x(t)$ has been suppressed. To be precise, $\dot{x}(t)$ should be interpreted as a right-hand derivative (see [1]).

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III. Counterexamples to Conjectured Stability

For our first example, we consider a linear time-invariant system described by

$$\left.\begin{aligned}\dot{x}_1(t)&=x_1(t)-\xi x_2(t)\\ \dot{x}_2(t)&=2x_1(t)-2x_2(t)+u(t); \qquad t\geq 0\end{aligned}\right\} \qquad (S_1)$$

with control restraint $|u(t)|\leq M(M>0)$ (ξ is a constant parameter).

To begin our analysis, we first note that this system (with $u(t)\equiv 0$) has a characteristic polynomial $p(\lambda)=\lambda^2+\lambda+(2\xi-2)$. Hence, this system is stable (also controllable) for all $\xi>1$. Now, we have the following result.

Theorem: Consider the system (S_1) with control restraint $|u(t)|\leq M$ and parameter $\xi>1$ fixed. Then this system is unstable when subjected to an optimal-aim control strategy.

Proof: Our proof will be accomplished by constructing a number $r>0$ and a set Ω_r having the following properties:

1) $[0 \quad 0]'\in\Omega_r$ (note that here $\sigma_c=[0 \quad 0]'$) and

2) for every initial state $x(0)\in\Omega_r$, $x(0)\neq[0 \quad 0]'$, there exists a time $T>0$ such that in the resulting "optimal-aim trajectory," $x(t)$ satisfies

$$\|x(T)\|=r.$$

To accomplish 1) and 2), we first choose $r>0$ so that

$$r\leq\frac{M}{3}. \qquad (1)$$

Next we define

$$\Omega_r\triangleq\{[x_1 \quad x_2]'\in\mathbb{R}^2: \quad x_1\geq(\xi+\epsilon)x_2, x_2\geq 0, \|x\|<r\} \qquad (2)$$

where ε is fixed so that

$$0<\epsilon<1. \qquad (3)$$

Clearly, Ω_r as defined in (2) has property 1), namely, $[0 \quad 0]'\in\Omega_r$. To prove property 2), let us define two angle mappings $\theta_+(\cdot,\cdot)$ and $\theta_-(\cdot,\cdot)$ on the set Ω_r by

$$\theta_+(x_1,x_2)\triangleq\pi+\tan^{-1}\frac{x_2}{x_1}-\tan^{-1}\frac{M+2(x_1-x_2)}{x_1-\xi x_2} \qquad (4a)$$

$$\theta_-(x_1,x_2)\triangleq\pi-\tan^{-1}\frac{x_2}{x_1}-\tan^{-1}\frac{M-2(x_1-x_2)}{x_1-\xi x_2}. \qquad (4b)$$

In the sequel, the following facts and observation will be useful.

1) All angles above, by convention, take values in $[0,\pi]$.

2) By (1) and (2), if $[x_1 \quad x_2]'\in\Omega_r$

$$x_2<\frac{M}{3(\xi+\epsilon)}.$$

Hence, by (3)

$$x_2<\frac{M}{3\xi}. \qquad (5)$$

3) By (1) and (2), if $[x_1 \quad x_2]'\in\Omega_r$

$$2(x_1-x_2)\leq 2x_1\leq 2r<\frac{2}{3}M<M. \qquad (6)$$

4) If the current state is $x(t)=[x_1 \quad x_2]'\in\Omega_r$, then $\theta_+(x_1,x_2)$ is the state-space angle between the reference vector and $\dot{x}(t)$ which results upon application of full positive control $u(t)=+M$. Similarly, $\theta_-(x_1,x_2)$ summarizes the effect of full negative control $u(t)=-M$.

5) If $x(t)=[x_1 \quad x_2]'\in\Omega_r$, then the optimal-aim control strategy dictates that either $u(t)=+M$ or $u(t)=-M$, i.e., any intermediate control value $m\in[-M,M]$ will result in a state-space angle which exceeds either $\theta_+(x_1,x_2)$ or $\theta_-(x_1,x_2)$.

This is apparent from (4) with M replaced by m and (6). (See also Fig. 1.)

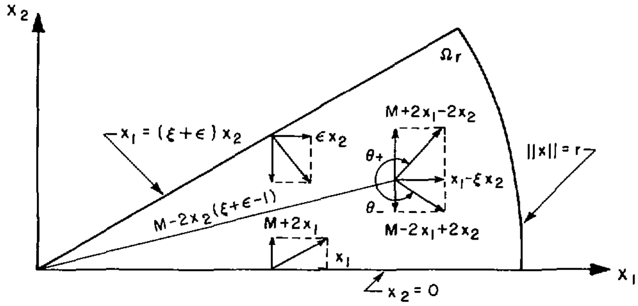

Fig. 1. The set Ω_r.

Facts 4) and 5) clearly imply that being in a state $x(t)=[x_1 \quad x_2]'\in\Omega_r$, the control will be

$$u=\begin{cases}+M & \text{if } \theta_+(x_1,x_2)-\theta_-(x_1,x_2)<0\\ -M & \text{if } \theta_+(x_1,x_2)-\theta_-(x_1,x_2)\geq 0.^5\end{cases} \qquad (7)$$

Next we are going to show that whenever on the boundaries $x_1=(\xi+\epsilon)x_2$ or $x_2=0$ of Ω_r, the optimal aim trajectory is directed into Ω_r.

On the boundary $x_1=(\xi+\epsilon)x_2$, we have

$$\theta_+(x_1,x_2)-\theta_-(x_1,x_2)=2\tan^{-1}\frac{1}{\xi+\epsilon}+\tan^{-1}\frac{M-2x_2(\xi+\epsilon-1)}{\epsilon x_2}-\tan^{-1}\frac{M+2x_2(\xi+\epsilon-1)}{\epsilon x_2}.$$

Using (3) and (5), we observe that

$$\frac{1}{\xi+\epsilon}>\frac{1}{\xi+1}$$

and

$$2x_2(\xi+\epsilon-1)<\frac{2}{3}M.$$

Using these two inequalities and the fact that $\tan^{-1}(\cdot)$ is a monotonically increasing function of its argument, we get

$$\theta_+(x_1,x_2)-\theta_-(x_1,x_2)>2\tan^{-1}\frac{1}{\xi+1}+\tan^{-1}\frac{M}{3\epsilon x_2}-\tan^{-1}\frac{5M}{3\epsilon x_2}$$

(using standard trigonometrical identities)

$$=\tan^{-1}\frac{2(\xi+1)}{\xi(\xi+2)}-\tan^{-1}\frac{12\epsilon x_2 M}{9\epsilon^2x_2^2+5M^2}$$

$$>\tan^{-1}\frac{2(\xi+1)}{\xi(\xi+2)}-\tan^{-1}\frac{4}{5\xi}\left(\text{recall } x_2<\frac{M}{3\xi}\right).$$

Again we use the monotonicity of $\tan^{-1}(\cdot)$ and note that for $\epsilon>1$

$$\frac{2(\xi+1)}{\xi(\xi+2)}>\frac{4}{5\xi}.$$

Hence, we conclude that

$$\theta_+(x_1,x_2)-\theta_-(x_1,x_2)>0$$

for $x_1=(\xi+\epsilon)x_2$. Hence, by (7), $u=-M$. Now, by (6), $\dot{x}_2(t)=2(x_1-x_2)-M<0$, which means that the trajectory is directed back into Ω_r from the boundary $x_1=(\xi+\epsilon)x_2$ (note that $\dot{x}_1(t)=x_1-\xi x_2>0$).

On the other boundary $x_2=0$, we have

$$\theta_+(x_1,x_2)-\theta_-(x_1,x_2)=\tan^{-1}\frac{M-2x_1}{x_1}-\tan^{-1}\frac{M+2x_1}{x_1}<0.$$

[5]Equality implies $u=-M$. This can be seen by noting that for $[x_1 \quad x_2]'\in\Omega_r$, $\theta(\dot{x},\rho)>\pi/2$. Hence, this choice of u will achieve the required minimum as defined by the "optimal-aim control."

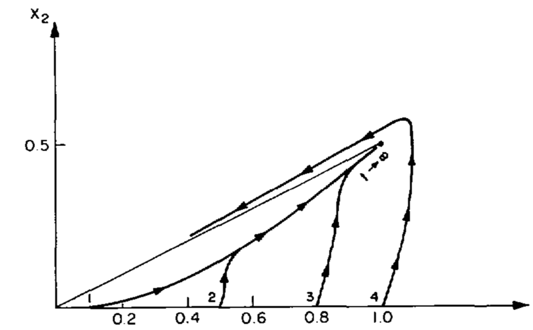

Fig. 2. Sample trajectories for Example 1 ($M=1$ and $\xi=2$).

Then by (7), $u=M$ and $\dot{x}_2(t)=2(x_1-x_2)+M>0$. Once again, the trajectory is directed from the boundary $x_2=0$ back into Ω_r.

Up to this point, we have shown that the only way an "optimal-aim trajectory" can escape from Ω_r is through its boundary $\|x\|=r$. Next, we are going to show that every such trajectory does, in fact, cross this boundary in finite time.

We have from (S_1)

$$\dot{x}_1(t)=x_1(t)-\xi x_2(t).$$

Hence, if $x(t)\in\Omega_r$, then by (2), it follows that $x_1(t)\geq(\xi+\epsilon)x_2(t)$ or, equivalently,

$$x_2(t)\leq\frac{1}{\xi+\epsilon}x_1(t).$$

Then

$$\dot{x}_1(t)\geq x_1(t)-\frac{1}{\xi+\epsilon}\xi x_1(t)=\frac{\epsilon}{\xi+\epsilon}x_1(t)$$

or

$$x_1(t)\geq x_1(0)e^{(\epsilon/(\xi+\epsilon))t}$$

which means that $\|x(T)\|=r$ for some $0<T\leq(\xi+\epsilon/\epsilon)\ln(r/x_1(0))$. So we see that for any initial condition, even if arbitrarily close to the origin, we cannot avoid future states having norm $r=M/3$ or larger. This enables us to conclude that the system is unstable and the theorem is proved. □

Remark: It is of interest to note that, contrary to our expectation, one is *not* able to do better with more available control effort. With larger M, we do, in fact, worse. The relation between r and M (from (1) it follows that we may choose $r=M/3$) and the proof of the theorem reveals the following. The larger M, the further away from the origin certain trajectories will be forced to go by the optimal aim control.

In Fig. 2, sample trajectories are indicated corresponding to $M=1$ and $\xi=2$. Clearly, these trajectories are approaching a point and the distance of this point from the origin grows with M.

For our second counterexample, we consider a linear time-invariant system described by

$$\left.\begin{aligned}\dot{x}_1(t)&=x_1(t)+x_2(t)\\ \dot{x}_2(t)&=x_2(t)+u(t); \qquad t\geq 0\end{aligned}\right\}\qquad (S_2)$$

with control restraint $-M\leq u(t)\leq 2M(M>0)$.

To begin, we note that this system can easily be stabilized using linear feedback.[6] Using optimal-aim control instead, we encounter difficulties.

Using the angles $\theta_+(x_1,x_2)$ and $\theta_-(x_1,x_2)$ as in the proof of Theorem 1, the following fact is readily established. Any optimal-aim trajectory $x(\cdot)$ which begins in the set

$$\Omega_+\triangleq\{(x_1,x_2)\in R^2:\quad x_1\geq 0, x_2\geq 0\}$$

remains *forever* within this set. Consequently, if $x(0)\in\Omega_+$, it follows that $\dot{x}_1(t)\geq x_1(t)$ (since $x_2(t)\geq 0$). Hence, $x_1(t)\geq e^t x_1(0)$ and we see this system is unstable. Hence, for this system, the optimal-aim control would not be desirable.

[6] It is controllable; its poles can be assigned arbitrarily. To accommodate the bound $-M\leq u(t)\leq 2M$, we simply use a so-called *saturation linear feedback*.

IV. Conclusion

Given the fact that comprehensive global stability criteria do not exist for arbitrary complex nonlinear systems, it is unreasonable (unfair) to judge the efficacy of a given control law in terms of its ability to provide an *a priori* guarantee of stability. Nevertheless, it is felt that any *sound* control strategy for nonlinear stabilization should satisfy the following necessary condition. Namely, when the strategy is specialized to a controllable linear time-invariant system, asymptotic stability should be assured. The optimal-aim control does not satisfy this necessary condition.

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Further Comments on "On the Numerical Solution of the Discrete Time Algebraic Riccati Equation"

ALAN J. LAUB

The purpose of this note is to make a few brief comments on [1] which, in turn, comments on [2]. It is unfortunate that the authors of [1] were unaware of the existence of [2], which is based on the Bachelor's degree thesis of Pappas [3]. Moreover, to set the historical record straight, the basic idea of the generalized eigenvalue problem formulation upon which [2] and [3] were based appeared as Appendix 1 of [4] (pp. 47–48), a report which achieved wide (although apparently not wide enough) circulation.

The authors of [1] claim that [2] does not address the issue of degeneracy. That claim is, of course, false as the proof (actually, it is almost a parenthetical remark) appears very clearly in the first paragraph of the proof of Theorem 4 in [2].

Several interesting points are raised in [1], particularly the formulation of the problem which avoids G_2^{-1}. This can be potentially a very important reformulation if G_2 is badly conditioned with respect to inversion. A thorough discussion of this and related questions appears in a fine paper of Van Dooren [5].

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