$$d_1^2 = \min\{k : \tilde{r}_1^1 A^k B \neq 0\}, \; d_1^2 = 2$$

$$B_2^* = \hat{B}_2^* = [-1 \quad 0], \; \hat{P}_2 = I_1, \; \bar{P}_2 = 0, \; P_2 = \hat{P}_2, \; \tilde{P}_2 = I_3, \; R_2 = \tilde{P}_2 \bar{R}_1 = \bar{R}_1$$

$$\text{rank}\begin{bmatrix}\hat{B}_0^* \\ \hat{B}_1^* \\ \hat{B}_2^*\end{bmatrix} = \text{rank}\begin{bmatrix}1 & 1 \\ 1 & 0 \\ -1 & 0\end{bmatrix} = 2 = \min(p, m)$$

$$R = \tilde{P}_2 \tilde{P}_1 \tilde{P}_0 = \begin{bmatrix}1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1\end{bmatrix}$$

$$Q_1 = [\tilde{c}_1]$$

$$RC = \begin{bmatrix}\tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3\end{bmatrix} = \begin{bmatrix}0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0\end{bmatrix} \quad Q_2 = \begin{bmatrix}\tilde{c}_2 \\ \tilde{c}_2 A\end{bmatrix} \quad Q_3 = \begin{bmatrix}\tilde{c}_3 \\ \tilde{c}_3 A \\ \tilde{c}_3 A^2\end{bmatrix}$$

$$Q = \begin{bmatrix}0 & -1 & 1 & 0 & 1 \\ \hdashline 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0\end{bmatrix} \quad \hat{Q}_e = \begin{bmatrix}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$V_{\max} = \ker \hat{Q}_e = \text{span}\{[0 \quad 0 \quad 0 \quad 1 \quad 0]^T\}.$$

Note that although

$$\text{rank}\begin{bmatrix}\hat{B}_0^* \\ \hat{B}_1^*\end{bmatrix} = 2 \quad V_{\max} \neq \ker\begin{bmatrix}\tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_2 A\end{bmatrix}.$$

V. Computational Aspects of the Presented Methods

The approach presented seems to be useful for computer aided computations because only elementary operations (available in any standard computer programs library) on matrices are used, i.e., reduction of a matrix into echelon form and verification of rank of a triangular matrix. The computer program can be written in a form of a loop. The loop stops when (21) is satisfied. Then the rank test of (25) should be performed. If (25) is not satisfied, then the whole loop should be repeated for the system $(\hat{Q}_e, A, B)$. Computation of $\hat{Q}_e$ demands only reduction of Q into the echelon form. The program demands only one computation of a linear equation of the form $\hat{Q}_e X = 0$. Such a program can be found in a standard computer programs library.

The considered types of matrices significantly diminish the amount of computer calculations because rank tests can be reduced in many cases to evaluation of positions of zero elements in tested matrices in the echelon form of low dimensions.

The approach presented seems to be more effective computationally than Wonham's method [7] because Wonham's method demands at any stage of a sequential program computation of two linear equations of the form $X_i R_i = 0$ and $P_i Y_i = 0$ [7; p. 101]. Moreover, at any stage of Wonham's algorithm we have to compute

$$\text{rank}\,[Y_{i-1}, \; Y_i]$$

which is complicated in the case of large scale systems.

The Vardulakis approach [6] is very simple from a computational point of view, provided that the system (C, A, B) is in the Luenberger controllable canonical form (LCCF). In the general case the reduction of (C, A, B) into LCCF is necessary and it demands computation of controllability indexes of (A, B) and inversion of a certain nonsingular matrix (cf. [5]).

Conclusions

In this paper a simple method for computation, useful in computer aided calculations, of a maximal (A, B)-invariant subspace contained in ker C was presented. The method showed how to unify the approach of Bhattacharyya, Wonham, and Vardulakis. A simple example was worked out.

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On Minimum Spanning Blocks in Discrete Linear Systems

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Abstract—**The question of spanning blocks in linear discrete systems arises in many adaptive identification and control problems and is related to the convergence of these algorithms. Specifically, in block-invariant adaptive control, it is of importance to know the minimum length of these spanning blocks. Establishing this minimum length is the topic of the present note. It is found to be equal to the sum of the dimensions of the system's state space and its controllable subspace.**

I. Introduction

A problem that has received a great deal of attention in the literature on adaptive control has been that of establishing conditions for persistency of excitation required for global convergence of parameters. One cause of difficulty in establishing such conditions is that the excitation condition must be satisfied inside a time-varying loop around the unknown plant. When employed in algorithms for adaptive control of nonminimum phase plants, an added difficulty has been that the maintenance of loop stability depended on parameter convergence.

In several recent papers [1]–[3], approaches were proposed for ensuring persistency of excitation in adaptive control of nonminimum phase plants via the technique of *block-invariant* feedback. With this technique, the feedback gain is held constant for periods of sufficient length so as to circumvent the difficulties caused by the time variation of the closed-loop plant.

Manuscript received June 25, 1985. This work was supported in part by the Technion fund for promotion of research.

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IEEE Log Number 8607614.

We illustrate the approach by using the terminology and notation of [3] where it was shown that a large class of adaptive control problems can be transformed into the following standard state-space formulation.

Consider the discrete-time linear system described by the equations

$$\left.\begin{aligned}\psi_{k+1}&=A\psi_k+bv_k\\ \varphi_k&=C\psi_k\end{aligned}\right\}\tag{1.1}$$

where $\psi_k \in \mathbb{R}^{\hat{n}}$, $\varphi_k \in \mathbb{R}^m$, $u_k \in \mathbb{R}^1$ are the state, output, and input at time k with A, b, and C real matrices of suitable dimensions but with unknown parameters. The control to be employed is given by the state-feedback equation

$$u_k=f^*\psi_k+u_k\tag{1.2}$$

where $v_k \in \mathbb{R}^1$ is a new command input and where $f^* \in \mathbb{R}^{1\times\hat{n}}$ is a feedback gain matrix (row vector) which depends on the (unknown) plant parameters and on the desired closed-loop performance. The vector f^* is given by a formula of the form

$$f^*=\vartheta^{*T}G+H\tag{1.3}$$

where $\vartheta^* \in \mathbb{R}^m$ is a vector of (also unknown) parameters and where G and H are given constant matrices. (Here and below $(\cdot)^T$ denotes the transpose.) Finally, the parameter vector ϑ^* is related to the output of the system (1.1) by the *regression equation*

$$\eta_k=\varphi_k^T\vartheta^*\tag{1.4}$$

where $\{\eta_k\}$, $\eta_k \in \mathbb{R}^1$ is a sequence of (known) signals that is generated from the sequences $\{\psi_k\}$ and $\{u_k\}$.

In an adaptive control implementation, (1.4) serves to obtain an estimate ϑ_j of ϑ^*, say by the recursive least squares (or RLS) algorithm [4]. This estimate is then used in (1.3) instead of ϑ^* to obtain an estimate f_j for the feedback gain vector that is used in (1.2) in place of the unknown f^*.

To ensure convergence of the algorithm, it is clearly necessary that the estimates ϑ_j converge to the true value of the parameter vector ϑ^* and to this end it is necessary that a persistency of excitation condition be satisfied. With the RLS algorithm, the persistency of excitation condition is that [4]

$$\lim_{k\to\infty}\lambda_{min}\left(\sum_{j=1}^{k}\varphi_j\varphi_j^T\right)=\infty\tag{1.5}$$

where $\lambda_{\min}(\ )$ denotes the minimal eigenvalue. To satisfy (1.5), it is obviously sufficient that there exists a real number $\epsilon > 0$ and an integer $N > 0$ such that for all integers $l \geq 0$

$$\sum_{j=lN+1}^{(l+1)N}\varphi_j\varphi_j^T\geq\epsilon I.\tag{1.6}$$

The block-invariant feedback approach that was used in [1]–[3] rests on satisfying the persistency of excitation via condition (1.6). Thus, the integer N is considered as the *block length* and to avoid time variation of the plant dynamics during the spanning process, the feedback gain vector is held constant during spanning blocks (of length N) and is changed only between them. During each invariant block, the closed-loop dynamics is then given by

$$\left.\begin{aligned}\psi_{k+1}&=A_j\psi_k+bv_k\\ \varphi_k&=C\psi_k\end{aligned}\right\}\tag{1.7}$$

where $A_j = A + bf_j$. In order to be able to generate a spanning output sequence $\{\varphi_k\}$ of (1.7) from the input, it is obviously necessary that (1.7) be output reachable. Assuming that the output reachability property holds, the problem arises of what is the least block size N for which an input sequence $\{v_k\}$ can be found such that for all initial conditions ψ_0 of (1.7) and independently of the (unknown) parameter values of (A_j, b, C), the spanning condition (1.6) is satisfied for all $l \geq 0$. This will be the central topic of the present note.

We refer the reader to a related problem concerning input sequence properties for reachable systems which has been studied recently in [5].

[1] Clearly, if $\mathbb{R}^m$ is not spanned in $N = \hat{n} + \nu - 1$ steps it will not be spanned in $N < \hat{n} + \nu - 1$ steps.

II. Minimal Spanning Blocks

We consider the class $L := L(m, \hat{n})$ of all systems of the form (1.7) or, equivalently, (1.1) with fixed dimensions m and $\hat{n}$ ($m \leq \hat{n}$). For an integer ν satisfying $m \leq \nu \leq \hat{n}$ we let C_ν denote the class of all elements in L that are output reachable and whose reachable subspace (in $\mathbb{R}^{\hat{n}}$) is of dimension $n_r \leq \nu$. Clearly, then $m \leq n_r \leq \nu$. We now make the following.

Definition 2.1: An input sequence $\{u_k\}$ to (1.1) is said to be N-spanning for class C_ν if for every initial state $\psi_0 \in \mathbb{R}^{\hat{n}}$ and every system in C_ν the condition is satisfied that

$$\operatorname{rank}\,[\varphi_1,\ \cdots,\ \varphi_N]=m.\tag{2.1}$$

The main result of the present note is the following.

Theorem 2.2: Let $\nu \geq m$ be a given integer. Then an N-spanning input sequence for class C_ν exists if and only if $N \geq \hat{n} + \nu$.

Before proving the theorem, it is of interest to relate the present result to some cases dealt with in the literature. In [2], the convergence of a block-invariant pole placement adaptive controller for nth-order SISO plants has been investigated. That controller is also examined in [3] where a spanning block of length $10n$ is proposed. Comparing to the present result, we note that this length is actually minimal (with $\hat{n} = 6n$ and $n_r = 4n$). In [2], however, it has been suggested that a block length of $6n$ ($=\hat{n}$) suffices for spanning, raising the question of whether spanning can actually be guaranteed there independently of the initial conditions. In [4], on the other hand, for a case where $\hat{n} = n_r = 2n$, an input sequence is employed for which a block of length $10n$ is suggested. Our present result indicates that an input sequence could be used there for which spanning occurs in blocks of length $4n$ only.

Proof of Theorem 2.2: For any vector $\omega \in \mathbb{R}^m$, let

$$\alpha_k := \omega^T\varphi_k\tag{2.2}$$

so that condition (2.1) is equivalent to the statement that $\alpha_k = 0$ for $k = 1, 2, \cdots, N$ only if $\omega = 0$.

The z-transform of the output $\{\varphi_k\}$ of system (1.1) is given by

$$\varphi(z)=C[zI-A]^{-1}bU(z)+zC[zI-A]^{-1}\psi_0\tag{2.3}$$

where $U(z)$ $[= \sum_{i=0}^{\infty} u_i t^{-i}]$ is the z-transform of the input sequence $\{u_k\}$. For a vector $\omega \in \mathbb{R}^m$, denote

$$\omega^T\varphi(z)=\alpha(z)=\alpha_0+\alpha_1z^{-1}+\alpha_2z^{-2}\cdots\tag{2.4}$$

(so that $\alpha_k = \omega^T\varphi_k$, $k = 0, 1, 2, \cdots$) and let

$$\omega^TC[zI-A]^{-1}b=\frac{r_1z^{-1}+r_2z^{-2}+\cdots+r_{\hat{n}}z^{-\hat{n}}}{1+p_1z^{-1}+\cdots+p_{\hat{n}}z^{-\hat{n}}}=\frac{r(z^{-1})}{p(z^{-1})}\tag{2.5}$$

and

$$\omega^TC[zI-A]^{-1}\psi_0=\frac{q_1z^{-1}+\cdots+q_{\hat{n}}z^{-\hat{n}}}{1+p_1z^{-1}+\cdots+p_{\hat{n}}z^{-\hat{n}}}=\frac{q(z^{-1})}{p(z^{-1})}.\tag{2.6}$$

After multiplication of both sides of (2.3) by ω^T, we obtain with the notation of (2.4)–(2.6)

$$\alpha(z)=\frac{r(z^{-1})U(z)+q(z^{-1})}{p(z^{-1})}\tag{2.7}$$

and upon multiplying (2.7) through by $p(z^{-1})$ and equating coefficients of

z^{-1}, the following $N + 1$ equations are obtained:

$$\begin{bmatrix} 1 & & & & & & & \\ p_1 & 1 & & & & & & \\ p_2 p_1 & & 1 & & & & 0 & \\ \vdots & & & \ddots & & & & \\ p_{\hat{n}-1} & & & & \ddots & & & \\ p_{\hat{n}} & \cdot & & & & \ddots & & \\ 0 & \cdot & \cdot & & & & \ddots & \\ \vdots & & \ddots & & & & & \ddots \\ 0 & \cdots & 0 & p_{\hat{n}} p_{\hat{n}-1} & \cdots & p_2 p_1 & & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \vdots \\ \alpha_N \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & & \\ u_0 & \cdot & & 0 & \\ u_1 & \cdot & \cdot & & \\ \vdots & & \ddots & & \\ u_{\hat{n}-2} & \cdots & u_1 & u_0 & 0 \\ u_{\hat{n}-1} & \cdots & & u_1 & u_0 \\ u_{\hat{n}} & \cdots & & & u_1 \\ \vdots & & & & \vdots \\ u_{N-1} & \cdots & & & u_{N-\hat{n}} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{\hat{n}} \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\hat{n}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.8}$$

To prove the "if" part of the theorem, we set $N = \hat{n} + \nu$ and choose the input sequence

$$u_k = \begin{cases} 0 \text{ for } k = 0, 1, \cdots, \hat{n}-1 \\ 1 \text{ for } k = \hat{n} \\ \text{any value for } \hat{n} < k < N. \end{cases} \tag{2.9}$$

Suppose now that for some system in C_ν with initial state ψ_0, there exists $\omega \in \mathbb{R}^m$ so that when the above input is used $\alpha_i = 0$, $i = 1, 2, \cdots, \hat{n} + \nu$. Then the last ν equations of (2.8) become

$$0 = \begin{bmatrix} 1 & & & & 0 \\ u_{\hat{n}+1} & 1 & & & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ u_{\hat{n}-\nu-1} & \cdot & & u_{\hat{n}+1} & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{\hat{n}} \end{bmatrix}$$

so that

$$r_1 = r_2 = \cdots = r_\nu = 0.$$

By (2.5) the above implies that

$$\omega^T C A^{i-1} b = 0, \qquad i = 1, 2, \cdots, \nu$$

and since the dimension n_r of the system's controllable subspace satisfies $n_r \le \nu$, we get

$$\omega^T C A^{i-1} b = 0 \qquad \text{for all } i \ge 1$$

so that

$$\omega^T C [Iz - A]^{-1} b = 0. \tag{2.10}$$

From the output reachability of the system and (2.10), we conclude that $\omega = 0$. Hence, as pointed out earlier, (2.1) holds and $\{u_k\}$ as defined in (2.9) is $(\hat{n} + \nu)$-spanning for C_ν.

To prove the "only if" part let $N = \hat{n} + \nu - 1$.[1] We will show that for every given input sequence, a system can be found in C_ν for which this input sequence is not N-spanning.

[1] Clearly, if $\mathbb{R}^m$ is not spanned in $N = \hat{n} + \nu - 1$ steps, it will not be spanned in $N < \hat{n} + \nu - 1$ steps.

Let $\{u_k\}$ be a given input sequence and define the following.

Case 1: $\nu = \hat{n}$

Let $r_1, r_2, \cdots, r_{\hat{n}}$ be a nontrivial solution of

$$\begin{bmatrix} u_{\hat{n}} & u_{\hat{n}-1} & \cdots & u_1 \\ u_{\hat{n}+1} & u_{\hat{n}} & & u_2 \\ \vdots & \vdots & & \vdots \\ u_{2\hat{n}-2} & u_{2\hat{n}-3} & \cdots & u_{\hat{n}-1} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{\hat{n}} \end{bmatrix} = 0 \tag{2.11}$$

Case 2: $\nu < \hat{n}$

Let $r_1, r_2, \cdots, r_{\hat{n}}$ be the nontrivial solution of

$$\begin{bmatrix} u_{\hat{n}-1} & u_{\hat{n}-2} & \cdots & u_0 \\ u_{\hat{n}} & u_{\hat{n}-1} & & u_1 \\ \vdots & \vdots & & \vdots \\ u_{\hat{n}+\nu-2} & u_{\hat{n}+\nu-3} & \cdots & u_{\nu-1} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{\hat{n}} \end{bmatrix} = 0. \tag{2.12}$$

For either case choose $p_1, p_2, \cdots, p_{\hat{n}}$ so that $p(z^{-1})$ and $r(z^{-1})$ have at least $\hat{n} - \nu$ and at most $\hat{n} - m$ roots in common. Then choose $\alpha_0, q_1, q_2, \cdots, q_{\hat{n}}$ so as to satisfy

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\hat{n}} \\ 0 \end{bmatrix} = \alpha_0 \begin{bmatrix} 1 \\ p_1 \\ \vdots \\ p_{\hat{n}-1} \\ p_{\hat{n}} \end{bmatrix} - \begin{bmatrix} 0 & & & 0 \\ u_0 & \ddots & & \\ \vdots & \ddots & \ddots & \\ u_{\hat{n}-2} & \cdots & u_0 & 0 \\ u_{\hat{n}-1} & \cdots & & u_0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{\hat{n}} \end{bmatrix}. \tag{2.13}$$

Define now

$$b = [r_1, r_2, \cdots, r_{\hat{n}}]^T \tag{2.14}$$

and

$$A = \begin{bmatrix} 0 & 0 \cdots 0 & -p_{\hat{n}} \\ 1 & 0 \cdots 0 & -p_{\hat{n}-1} \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots 1 & -p_1 \end{bmatrix} \tag{2.15}$$

then clearly

$$e_{\hat{n}}^T [Iz - A]^{-1} b = \frac{r(z^{-1})}{p(z^{-1})}$$

where $e_{\hat{n}}^T = [0, 0, \cdots, 1]$.

Since $p(z^{-1})$ was chosen so that it has no more than $\hat{n} - m$ (and no less than $\hat{n} - \nu$) roots in common with $r(z^{-1})$ and since $(e_{\hat{n}}^T, A)$ is an observable pair, the controllable subspace of (A, b) has dimension n_r satisfying $m \le n_r \le \nu$. This implies that there exist indexes $i_1, i_2, \cdots, i_{m-1} < \hat{n}$ so that $\{e_{\hat{n}}^T [Iz - A]^{-1} b, e_{i_j}^T [Iz - A]^{-1} b, j = 1, 2, \cdots, m - 1\}$ are linearly independent over the reals.

Hence, if

$$C = [e_{\hat{n}}, e_{i_1}, \cdots, e_{i_{m-1}}]^T, \tag{2.16}$$

the system represented by (C, A, b) as defined, is output reachable and hence belongs to class C_ν. This system, with initial state

$$\psi_0 = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\hat{n}} \end{bmatrix},$$

and $\omega = e_1$ satisfies (2.5) and (2.6) with the coefficients defined in (2.11)-(2.13). However, substituting (2.11) [or (2.12)] and (2.13) in (2.8) and solving for α_i we get

$$\alpha_i = 0 \qquad i = 1, 2, \cdots, N.$$

Hence, $\{u_k\}$ is not $N = \hat{n} + \nu - 1$ spanning for C_ν, and this completes the proof of the theorem. □

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Some Discrete-Time Counterparts to Kharitonov's Stability Criterion for Uncertain Systems

C. V. HOLLOT AND A. C. BARTLETT

Abstract—**In [1], Kharitonov gave an elegant and simple stability criterion for continuous-time systems. This note reports on similar results for discrete-time systems.**

I. INTRODUCTION

In [1], Kharitonov stated a most surprising result concerning the stability of continuous-time uncertain systems. The goal of this work is to give similar results for discrete-time systems. To do this, we shall first state the major results contained in [1]. To this end, consider the uncertain system described by the state equations

$$\dot{x}(t) = A(r)x(t) \tag{1}$$

where $x(t) \in R^n$ is the system state and $r \in \mathcal{R} \subset R^p$ is the system uncertainty. This uncertainty is a vector of fixed but otherwise unknown parameters. The uncertainty bounding set $\mathcal{R}$ is compact and the entries of $A(r)$ depend continuously on r. The asymptotic stability of the origin of (1) requires that the family of characteristic polynomials

$$\Lambda_s \triangleq \left\{ |sI - A(r)| = \sum_{i=0}^{n} a_i(r)s^{n-i} : r \in \mathcal{R} \right\}$$

be contained in the set of nth degree polynomials whose zeros lie in the left half plane; i.e., the set G_s. Now, for $i = 0, 1, \cdots, n$, define

$$\underline{\alpha}_i \triangleq \min_{r \in \mathcal{R}} a_i(r); \quad \bar{\alpha}_i \triangleq \max_{r \in \mathcal{R}} a_i(r)$$

and construct the family of polynomials

$$S_\alpha \triangleq \left\{ \sum_{i=0}^{n} \bar{a}_i s^{n-i} : \bar{a}_i \in [\underline{\alpha}_i, \bar{\alpha}_i], \qquad i = 0, 1, \cdots, n \right\}.$$

Manuscript received August 2, 1985; revised November 13, 1985.

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IEEE Log Number 8607613.

Clearly, $\Lambda_s \subset S_\alpha$. The next theorem gives Kharitonov's main result.

Theorem 1: (See [1] for proof.) *$S_\alpha \subset G_S$ if and only if the four polynomials*

$$f_1(s) = \underline{\alpha}_0 s^n + \bar{\alpha}_1 s^{n-1} + \bar{\alpha}_2 s^{n-2} + \underline{\alpha}_3 s^{n-3} + \underline{\alpha}_4 s^{n-4} + \cdots;$$

$$f_2(s) = \underline{\alpha}_0 s^n + \underline{\alpha}_1 s^{n-1} + \bar{\alpha}_2 s^{n-2} + \bar{\alpha}_3 s^{n-3} + \underline{\alpha}_4 s^{n-4} + \cdots;$$

$$f_3(s) = \bar{\alpha}_0 s^n + \bar{\alpha}_1 s^{n-1} + \underline{\alpha}_2 s^{n-2} + \underline{\alpha}_3 s^{n-3} + \bar{\alpha}_4 s^{n-4} + \cdots;$$

$$f_4(s) = \bar{\alpha}_0 s^n + \underline{\alpha}_1 s^{n-1} + \underline{\alpha}_2 s^{n-2} + \bar{\alpha}_3 s^{n-3} + \bar{\alpha}_4 s^{n-4} + \cdots$$

are contained in G_s. Consequently, if $f_1(s)$ through $f_4(s)$ are contained in G_s, then $\Lambda_s \subset G_s$.

Compared to a direct application of the Routh-Hurwitz criterion, this theorem offers a simple sufficient condition for $\Lambda_s \subset G_s$. Moreover, if $\Lambda_s = S_\alpha$, then these conditions are necessary as well. An interpretation to Theorem 1 can be given by first considering a polynomial's representation in coefficient space. In this way we can think of Λ_s as some set in $(n+1)$-dimensional space, S_α as an $(n+1)$-dimensional rectangle, and the polynomials $f_i(s)$ as four "special" vertices of S_α. By construction, the rectangle S_α contains Λ_s and has edges which are parallel to the coordinate axes. Theorem 1 states that all the polynomials associated with the rectangle S_α are stable if and only if the four vertices $f_1(s)$ through $f_4(s)$ are stable.

This note reports on efforts to extend these results to discrete-time systems. Clearly, one can begin by simply "translating" Kharitonov's result to the z-domain via the bilinear transformation $s = (z+1)/(z-1)$. Indeed, we shall do this in the next section and state these almost trivial results as a corollary to Theorem 1. A more challanging line of research is concerned with the validity of a direct restatement of Kharitonov's result in the z-domain. That is, suppose we have a rectangle in coefficient space with edges parallel to the axes. Each point in the rectangle corresponds to an nth degree polynomial in the variable z. Prompted by Theorem 1 we ask: Do there exist four vertices whose stability implies stability of the complete rectangle? In general, the answer is "no." In fact, even if all the vertices are stable, then the rectangle may be unstable. However, we will be able to prove the following. Consider an $n/2$-dimensional (assume n even) rectangle lying in the first $n/2$ dimensions of $(n+1)$-dimensional coefficient space. If all the vertices of this rectangle are stable, then the rectangle is stable. This result is stated in Theorem 2 and is applicable to uncertain systems whose uncertainties enter only in the coefficients of $z^0, z^1, \cdots, z^{n/2}$ of the system's characteristic polynomial.

II. DISCRETE-TIME SYSTEMS

Consider the discrete-time system described by the state equations

$$x(k+1) = \Phi(q)x(k) \tag{2}$$

where $x(k) \in R^n$ is the system state and $q \in Q \subset R^p$ is the system uncertainty. The vector q is fixed but unknown, the uncertainty bounding set Q is compact, and the entries of $\Phi(q)$ depend continuously on q. Analogous to the continuous-time case, we shall form the set of characteristic polynomials Λ_z generated by (2). Presently, we shall write this set as

$$\Lambda_z \triangleq \left\{ |zI - \Phi(q)| = \sum_{i=0}^{n} b_i(q)(z+1)^{n-i}(z-1)^i : q \in Q \right\} \tag{3}$$

and take G_z to be the set of polynomials of degree n whose zeros lie in the unit circle. For $i = 0, 1, \cdots, n$, define

$$\underline{\beta}_i \triangleq \min_{q \in Q} b_i(q); \quad \bar{\beta}_i \triangleq \max_{q \in Q} b_i(q)$$

and let

$$S_\beta \triangleq \left\{ \sum_{i=0}^{n} \bar{b}_i (z+1)^{n-i}(z-1)^i : \bar{b}_i \in [\underline{\beta}_i, \bar{\beta}_i], \qquad i = 0, 1, \cdots, n \right\}.$$