Fundamentals of Stochastic Processes, May 2009 048868

Home Assignment 4

Submit part I by 25 May, part II by 1 June.

I. Brownian motion

1. Let W_t be a Brownian motion (BM). Show that $W_t^2 - t$ is a martingale on $\sigma\{W_s, s \le t\}$.

2. For the BM, show that if we choose a sequence of partitions so that $\delta_n = 2^{-n}$, then $S_n \to t$ a.s. (Use Borel-Cantelli and Chebychev).

3. Let W_t be a BM. Then so are $c^{1/2}W(t/c)$ and $tW(1/t)$.

4. $W_i(t)$ are independent BMs. Find a necessary and sufficient condition on α_i so that $\sum_{i=1}^{N} \alpha_i W_i(t)$ is a standard BM.

5. Let X be a Brownian motion with drift $-b$, i.e., $X_t = B_t - bt$. Let τ be the hitting time of level $a > 0$:

$$\tau = \inf\{t : X_t = a\}.$$

Prove that $P(\tau < \infty) = e^{-2ab}$, as follows. (i) Use the martingale (prove!) $\exp(\theta B_t - \theta^2 t/2)$ with $\theta = b + (b^2 + 2\lambda)^{1/2}$ to compute the moment generating function:

$$E\exp(-\lambda\tau) = \exp(-a\{b + (b^2 + 2\lambda)^{1/2}\}).$$

(ii) Send $\lambda \to 0$.

II. Stochastic integrals

1. For the stochastic integral, prove properties 1–6 (from the lectures) for processes in $\mathcal{E}$.

2. For processes in $\mathcal{L}^2$: If $Y_n \to Y$ in L^2, show that $E_n \to EY$. Now prove property 1.

3. Show that if $X_n \to X$ in L^2 and $Y_n \to Y$ in L^2 then $X_n + Y_n \to X + Y$ in L^2. Now prove property 2 in $\mathcal{L}^2$.

4. Prove proerty 3 in $\mathcal{L}^2$.