Fundamentals of stochastic processes 048868

Home assignment 1—Probability and random variables, solution to Exercise 3.(b).

3. Probability spaces and random variables.

   (b) Let $(\Omega, \mathcal{F})$ be a measuragble space and let X_n be a sequence of random variables. Assume that for each $\omega \in \Omega$ the limit $\lim_{n\to\infty} X_n(\omega)$ exists, and denote it by $X(\omega)$. Prove that X is a random variable.

Solution: we need to show that for all α,

$$\{\omega : X(\omega) \leq \alpha\}$$

is a measurable set. Since complements of measurable sets are measurable, we may instead show that

$$\{\omega : X(\omega) > \alpha\}$$

is a measurable set. But since

$$X(\omega) = \lim_{n\to\infty} X_n(\omega)\,,$$

we have for each ω

$$\{\omega : X(\omega) > \alpha\} = \{\omega : X_n(\omega) > \alpha\} \quad \text{for all } n \geq m(\omega).$$

That is, since X_n converges, $X > \alpha$ if and only if $X_n > \alpha$ for all large enough n. Let us formalize this. Note that

$$\cap_{n=m}^{\infty} \{\omega : X_n(\omega) > \alpha\}$$

is exactly the set of ω such that $X_n(\omega)$ is larger than α for all $n \geq m$. Therefore

$$\cup_{m=1}^{\infty} \cap_{n=m}^{\infty} \{\omega : X_n(\omega) > \alpha\}$$

is exactly the set of ω such that $X_n(\omega)$ is larger than α for all n larger than some $m(\omega)$. So, finally,

$$\{\omega : X(\omega) > \alpha\} = \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} \{\omega : X_n(\omega) > \alpha\}\ .$$

Since each of the sets on the right hand side is measurable (as X_n are measurable), so are their countable intersections and unions.