Fundamentals of Stochastic Processes 048868

Home Assignment 5

Submit Part I by May 25, Part II by June 8.

Ito calculus

Part I: Stochastic integrals

1. For the stochastic integral, prove properties 1–6 (from the lectures) for processes in $\mathcal{E}$.

2. For processes in $\mathcal{L}^2$: If $Y_n \to Y$ in L^2, show that $E_n \to EY$. Now prove property 1.

3. Show that if $X_n \to X$ in L^2 and $Y_n \to Y$ in L^2 then $X_n + Y_n \to X + Y$ in L^2. Now prove property 2 in $\mathcal{L}^2$.

4. Prove proerty 3 in $\mathcal{L}^2$.

Part II: Stochastic Differential Equations

5. Let $X_t = W_t$ and $Y_t = e^{W_t^2}$. Show that (X_t, Y_t) solves the set of stochastic differential equations

$$dX_t = dW_t, \qquad X_0 = 0,$$

$$dY_t = 2X_tY_tdW_t + (Y_t + 2X_t^2Y_t)dt, \qquad Y_0 = 1.$$

6. Let $W^1, \dots, W^n$ be independent BMs and denote $W = (W^1, \dots, W^n)$ (W is called an n-dimensional BM). Let

$$R = |W| = \left(\sum_{i=1}^{n}(W^i)^2\right)^{1/2}.$$

Show that R solves the stochastic Bessel equation:

$$dR = \sum_i \frac{W^i}{R}dW^i + \frac{n-1}{2R}dt.$$

7. Let W be an n-dimensional BM, for $n \geq 3$. Write $X = W + x_0$ where the point x_0 lies in the region $U = \{0 < R_1 < |x| < R_2\}$. Calculate explicitly the probability that X will hit the outer sphere $\{|x| = R_2\}$ before hitting the inner sphere $\{|x| = R_1\}$. Hint: Check that $\Phi(x) = |x|^{2-n}$ satisfies $\Delta\Phi = 0$ for $x \neq 0$. Modify Φ to build a function u which equals 0 on the inner sphere and 1 on the outer sphere.