Bin Packing with Item Fragmentation

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Abstract. We investigate a variant of the bin packing problem in which items may be fragmented into smaller size pieces called fragments. While there are a few applications to bin packing with item fragmentation, our model of the problem is derived from a scheduling problem present in data over CATV networks. Fragmenting an item is associated with a cost which renders the problem NP-hard. We study two possible cost functions and as a result get two variants of bin packing with item fragmentation. In the first variant, called *bin packing with size-increasing fragmentation*, each item may be fragmented in which case overhead units are added to the size of every fragment. In the second variant each item has a size and a cost and fragmenting an item increases its cost but does not change its size. We call this variant *bin packing with size-preserving fragmentation.*
We develop several algorithms for the problem and investigate their performance. The algorithms we present are based on well known bin packing algorithms such as Next-Fit and First-Fit Decreasing, as well as of other algorithms. ...

1 Introduction

Because of its applicability to a large number of applications and because of its theoretical interest bin packing has been widely researched and investigated (see, e.g., [4], [6], [9] and [2] for a comprehensive survey). In the classical one-dimensional bin packing problem, we are given a list of items $L = (a_1, a_2, ..., a_n)$, each with a size $s(a_i) \in (0, 1]$ and are asked to pack them into a minimum number of unit capacity bins. Since the problem, as many of its derivatives, is NP-hard many approximation algorithms have been developed for it (see, e.g., [3], [7], [8] and [1] for a survey). The common assumption in bin packing problems is that an item may not be fragmented into smaller pieces. There are several applications, however, in which this assumption does not hold. The subject of item fragmentation in bin packing problems received almost no attention so far. This paper concentrates on aspects that were heretofore never researched, such as developing algorithms for the problem and investigating their performance.

The variant of bin packing presented in this paper is derived from a scheduling problem present in data over CATV (community antenna television) networks. In particular we refer to Data-Over-Cable Service Interface Specification

(DOCSIS), standard of the Multimedia Cable Network System (MCNS) standard committee, for a detailed description see [12]. When using CATV networks for data communication the data subscribers are connected via a cable modem to the headend. The headend is responsible for the scheduling of all transmissions in the upstream direction (from the cable modem to the headend). Scheduling is done by dividing the upstream, in time, into a stream of numbered mini-slots. The headend receives requests from the modems for allocation of datagram transmission. The length of each datagram can vary and may require a different number of mini-slots. From time to time, the headend publishes a *MAP* in which it allocates mini-slots to one modem or a group of modems. The scheduling problem is that of allocating the mini-slots to be published in the MAP, or in other words, how to order the datagrams transmission in the best possible way.

The headend must consider two kinds of datagram allocations:

1. *Fixed Location* - Allocations for connections with timing demands, such as a CBR (constant bit rate) connection. These connections must be scheduled so as to ensure delivering the guaranteed service. Fixed location datagrams are therefore scheduled in fixed, periodically located mini-slots.
2. *Free Location* - Allocations for connections without timing demands, such as a best effort connection. Free location datagrams can use any of the mini-slots.

The headend therefore performs the allocation in two stages: in the first stage it schedules, or allocates, all fixed location datagrams. We assume that after the fixed allocations have been made, a gap of U mini-slots is left between successive fixed allocations. In the second stage all free location datagrams are scheduled. The free allocations must fit into the gaps left by the fixed allocations.

The relation to the bin packing problem should now be clear. The items are the free location datagrams that should be scheduled, each of which may require a different number of mini-slots. The bins are defined by the gaps between every two successive fixed allocations in the MAP. The goal is to use the available mini-slots in the MAP in the best way. We point out that the practical scheduling problem may actually be somewhat more complicated since the bins (gaps between fixed allocation) may not be of uniform size. Dealing with variable size bins is beyond the scope of this paper (we present some results in [11]).

One of the capabilities of the system is the ability to break a datagram into smaller pieces called *fragments*. When a datagram is fragmented, i.e., transmitted in non successive mini-slots, extra bits are added to the original datagram to enable the reassembly of all the fragments at the headend. In a typical CATV network one mini-slot is added to every fragment.

To make the problem of bin packing with item fragmentation nontrivial a cost must be associated with fragmentation. We study two possible cost functions and as a result get two variants of bin packing with item fragmentation. In the first variant, called *bin packing with size-increasing fragmentation*, the cost function adds one (or more) overhead unit to the size of every fragment. In the second variant the cost function increases the cost of an item upon each fragmentation, but does not change its size. We call this variant, *bin packing*

with size preserving fragmentation. The scheduling problem, where the cost is due to the extra overhead bits that are added to each fragment, serves as a model to the first variant. The second variant is suitable for problems where the cost associated with fragmentation is a result of extra processing time or reassembly delay. The two cost functions we present here are not the only possible cost functions. In other applications, for example, the cost may be related to the size of the item, a combination of the two cost functions is also possible. It is interesting to note that when the cost associated with fragmentation is ignored the packing problem becomes trivial, and when the cost is very high it does not pay to fragment items and we face the classical problem. Hence, the problem is interesting with the middle-range costs. It has been shown in [10] that for non zero cost the ability to fragment items does not reduce the complexity of the problem, that is, the problem of bin packing with item fragmentation is NP-hard.

We present worst case analysis of both variants of bin packing with item fragmentation. We begin by showing that the two variants are NP-hard in the strong sense. We then devise approximation algorithms for the problem and investigate their performance. We restrict our attention to practical bin packing algorithms, i.e., of low order polynomial running time, and examine both *online* and *offline* algorithms. Online algorithms are applicable to cases where the items arrive in some order and must be assigned to the bins as soon as they arrive. Offline algorithms assume the entire list of items is known before the packing begins. We devise algorithms which are based on well known bin packing algorithms but include the capability of fragmenting items. We investigate the performance of algorithms such as Next-Fit (NF) and First-Fit Decreasing (FFD), as well as of other algorithms.

The reminder of the paper is organized as follows. In Section 2 we address the problem of bin packing with size-increasing fragmentation. Section 3 is devoted to the problem of bin packing with size-preserving fragmentation.

2 Bin Packing with Size-Increasing Fragmentation

In this section we study the variant of bin packing with item fragmentation where fragmentation increases the size of an item. We define the problem similar to the classical bin packing problem. The classical bin packing problem deals with equal-sized (unit capacity) bins and a list of items each of which can fit in every bin. To handle fragmentation we use a discrete version of the problem and add a fragmentation cost function that adds overhead units to each fragment. We proceed to formally define the problem.

Bin Packing with Size-Increasing Fragmentation (BP-SIF): We are given a list of items $L = (a_1, a_2, ..., a_n)$, each with a size $s(a_i) \in \{1, 2, ..., U\}$. The items must be packed into a minimum number of bins, which are all the size of U units. When packing a fragment of an item, one unit of overhead is added to the size of *every* fragment.

Performance Ratio: We use the same definition as is typically used in analyzing the classical problem. For a given list L and algorithm A, let $A(L)$ be the

number of bins used when algorithm A is applied to list L, let $OPT(L)$ denote the optimum number of bins for a packing of L, and let $R_A(L) \equiv A(L)/OPT(L)$. The ***asymptotic worst case performance ratio*** R_A^∞ is defined to be:

$$R_A^\infty \equiv \inf\{r \geq 1 : for some N > 0, R_A(L) \leq r for all L with OPT(L) \geq N\} \quad (1)$$

The bin packing problem is know to be NP-hard in the strong sense [13]. We show that the complexity of BP-SIF is the same.

Claim. BP-SIF is NP-hard in the strong sense.

Proof. We denote by D(BP-SIF) the decision version of BP-SIF and show that it is NP-complete in the strong sense. We do so by reducing the 3-PARTITION problem to a restricted instance of D(BP-SIF). The 3-PARTITION problem (defined formally below) is known to be NP-complete in the strong sense [5].

3-PARTITION: given a list L of $n = 3m$ integers: $w_1, w_2, ..., w_n$ and a bound $B \in Z^+$ such that $B/4 < w_j < B/2$ for $j = 1, ..., n$ and $\sum_{j=1}^{n} w_j = m\,B$, can L be partitioned into m disjoint subsets $S_1, ..., S_m$ such that $\sum_{j \in S_i} w_j = B$ for $i = 1, ..., m$?

We define D(BP-SIF) as follows: given a list of items L, a size $s(a) \in Z^+$ for each item $a \in L$, a positive integer bin capacity U and a positive integer K, is there a feasible packing of L in K bins of size U? Any instance I of 3-PARTITION can be polynomially transformed into an equivalent instance I' of D(BP-SIF) by setting $U = B$ and $K = m$. To realize the two decision problems are equivalent note that, since the total size of all items is equivalent to the total capacity of all bins, in any "yes" instance of D(BP-SIF) all n items are packed without fragmentation and the packing is therefore also valid for 3-PARTITION. Clearly the packing of any "yes" instance of 3-PARTITION is also valid for D(BP-SIF). It follows that the "yes" and "no" instances of the two problems are equivalent. □

When packing a list of n items into m bins, the maximum number of fragmentations possible is $n \cdot m$ (each item is fragmented over all bins). From the definition of the problem it is obvious that a good algorithm should try to perform the minimum number of fragmentations. Therefore we would only like to consider algorithms that do not fragment items unnecessarily.

Definition 1: An algorithm A is said to *prevent unnecessary fragmentation* if it follows the following two rules:

1. No unnecessary fragmentation: An item (or fragment of an item) is fragmented only if it is to be packed into a bin that cannot contain it. In case of fragmentation, the item (or fragment) is divided into two fragments. The first fragment must fill one of the bins. The second fragment is packed according to the packing rules of the algorithm.
2. No unnecessary bins: An item is packed in a new bin only if it cannot fit in any of the open bins used by A.

Algorithms that prevent unnecessary fragmentation have the following property.

Lemma 1. *For any algorithm A that prevents unnecessary fragmentations - $R_A^\infty \leq \frac{U}{U-2}$, for every $U > 2$.*

Proof. Assume that the number of bins used by the algorithm is $A(L) = m$. Since A prevents unnecessary fragmentation it can perform at most $m-1$ fragmentations while packing m bins (no fragmentation in the last bin), regardless of the size of the list. Each fragmentation adds two units of overhead at the most. Therefore, in the worst case, $2m-1$ units of overhead are added to the total size of all items. Note that in this case only the last bin may be left unfilled. Assuming the optimal packing does not fragment any item, the number of bins used by it satisfies: $OPT(L) \geq \frac{(m-1)U-2(m-1)+1}{U}$

The asymptotic performance ratio follows:

$$R_A(L) = \frac{A(L)}{OPT(L)} \leq \frac{mU}{(m-1)(U-2)+1} \overset{m\to\infty}{\longrightarrow} R_A^\infty \leq \frac{U-2}{U}$$

□

Remark: For the more general case, where r units of overhead (instead of one) are added to the size of every fragment, it can be shown by similar arguments, that: $R_A^\infty \leq \frac{U}{U-2r}, \quad U > 2r$.

We now have an upper bound on the performance ratio of any algorithm. In the remainder of this section we investigate specific algorithms to find their actual performance ratio. For a given algorithm A we define a version of the algorithm that allows item fragmentation and denote it by A_f. We investigate the worst case performance ratio of the following algorithms: NF_f, NFD_f (NFI_f) and FFD_f (BFD_f).

2.1 Next-Fit with Item Fragmentation - NF_f

The NF_f algorithm is defined similar to the NF algorithm.

Algorithm NF_f - In each stage there is only one open bin. The items are packed, according to their order in the list L, into the open bin. When an item does not fit in the open bin, it is fragmented into two parts. The first part fills the open bin and the bin is closed. The second part is packed into a new bin which becomes the open bin. Offline version of the algorithm sorts the items in decreasing (NFD_f) or increasing (NFI_f) order before packing the list.

The NF_f algorithm is very simple, can be implemented to run in linear time and requires only one open bin (bounded space). However, as we show next, similar to the classical problem, the performance ratio the algorithm achieves is the worst possible.

Theorem 1. *For algorithm NF_f - $R_{NF_f}^\infty = \frac{U}{U-2}, \quad \forall\, U \geq 6$.*

Proof. Lemma 1 provides an upper bound on the performance ratio of the algorithm. We present an example that proves the lower bound. Let us first consider the case where the bin size U is an even number. As a worst case example we

choose the following list L: The first item is of size $U/2$, the next $\left(\frac{U}{2}-2\right)$ items are of size 1. The rest of the list repeats this pattern kU times. The optimal packing avoids fragmentations by packing bins with two items of size $U/2$ or U items of size 1. The total number of bins used is: $OPT(L)=\frac{U}{2}k+\left(\frac{U}{2}-2\right)k=(U-2)\,k$. On the other hand, algorithm NF_f fragments each item of size $U/2$ (except for the first item). Overall $2(kU-1)$ units of overhead are added to the packing and therefore the number of bins used by NF_f is:

$$NF_f(L)=\left\lceil\frac{U}{2}k+2\frac{kU-1}{U}+\left(\frac{U}{2}-2\right)k\right\rceil=\left\lceil Uk-\frac{2}{U}\right\rceil=Uk. \tag{2}$$

A worst case example for the case where U is an odd number is similar. The first item in L is of size $(U-1)/2$, the next $\left(\frac{U-1}{2}-1\right)$ items are of size 1. The rest of the list repeats this pattern kU times. It is easy to verify that, as in the previous example, $OPT(L)=(U-2)k$, while $NF_f(L)=kU$. □

When the bin size is very small the above proof does not hold. For the values $3<U\leq 5$ we show in [11] that the worst case asymptotic performance ratio is: $R^{\infty}_{NF_f}=\frac{3}{2}$.

It is interesting to compare the NF_f algorithm to the classic NF algorithm for which the asymptotic worst case performance ratio is: $R^{\infty}_{NF}=\frac{2U}{U+1},\quad \forall U\geq 1$. While the performance ratio of NF is increasing with U the performance ratio of NF_f is decreasing with U. This is intuitive since as the bin size gets larger the effective cost of fragmentation gets smaller.

***NFD$_f$* and *NFI$_f$* Algorithms** Given the poor performance of the NF_f algorithm, one may ask whether sorting the items before applying the NF_f packing, would yield better results. It turns out that algorithms NFD_f and NFI_f are very similar to the NF_f algorithm in their worst case performance. The actual performance ratio of these algorithms depends on the bin size U, but in all cases it is not far from the ratio of NF_f. To avoid dealing with each value of U separately, we first present an example for a general value. Our list of items is made of k items of size $U-2$ and k items of size 2. The optimal packing uses k bins where each bin contains two items, one of each kind. The NFD_f algorithm first packs k items, of size U-2, into k bins. Then k-1 items of size 2 are packed into $\lceil 2\,(k-1)\,/U\rceil$ bins if U is even, or $\lceil 2\,(k-1)\,/\,(U-1)\rceil$ bins if it is odd. The NFI_f algorithm will use the same number of bins, since the only difference is that the items are packed in reverse order. This simple example gives us the following lower bounds:

$$R^{\infty}_{NFD_f}=R^{\infty}_{NFI_f}\geq\frac{U+2}{U},\quad \text{for any even } U\geq 6 \tag{3}$$

$$R^{\infty}_{NFD_f}=R^{\infty}_{NFI_f}\geq\frac{U+1}{U-1},\quad \text{for any even } U\geq 5 \tag{4}$$

The above example provides a lower bound. We now demonstrate that in some cases the NFD_f and NFI_f algorithms can perform just as bad as NF_f.

The first example is for U=5 and a list $L = \{3, ..., 3, 2, ..., 2\}$, for which $R^{\infty}_{NFD_f} = R^{\infty}_{NFI_f} = R^{\infty}_{NF_f} = \frac{3}{2}$. To see that such examples are not restricted to low values of bin size, consider $U = 32$ and choose a list of $15k$ items of size 10 and $15k$ items of size 6. In the optimal packing the content of each bin is {10, 10, 6, 6}, therefore $OPT(L) = 7.5k$. The total number of bins used by the algorithms is: $NFD_f(L) = NFI_f(L) = 8k$, since all bins (save two) contain two units of overhead. The performance ratio of all three algorithms in this case is $\frac{32}{30}$ which is the worst possible ratio. On the other hand for some values of bin size, NFI_f and NFD_f have a better performance ratio than NF_f. For example when $U = 6$, $R^{\infty}_{NFD_f} = R^{\infty}_{NFI_f} = \frac{4}{3}$, while $R^{\infty}_{NF_f} = \frac{3}{2}$.

2.2 First-Fit Decreasing with Item Fragmentation - FFD_f

We now develop an algorithm based on the FFD heuristic. Let us first describe algorithm FFD_f which packs items from a list L, into a *fixed* number of m bins.
Algorithm FFD_f - First-Fit Decreasing with item fragmentation: The algorithm packs the items in decreasing order. An item is packed into the lowest indexed bin into which it fits. If an item does not fit into any bin it is fragmented. When fragmenting an item the first fragment fills the lowest indexed bin that is as yet not full. If the second fragment can be packed without fragmentation, it is packed into the lowest indexed bin into which it fits, otherwise another fragmentation is performed according to the above rule.
Remark: Other definitions are possible. For example, upon fragmentation we may choose to insert the second fragment back to the list. Another possibility is to first go over the whole list and pack items without fragmentation and only then pack the remaining items into the available free space.

Note that FFD_f may not be able to pack all the items in L. To ensure all items are packed we use the following iterative algorithm:
Algorithm FFD_f **Iterative** $(FFD_f - I)$ - The $FFD_f - I$ algorithm tries to pack the list L into a fixed number of m bins. If it fails it increases m by one and tries again. Let $s(L)$ be the sum of all items in L, the first value of m is: $m_1 = \lceil s(L)/U \rceil$, which is the minimum number of bins possible.

The algorithm performs the following steps:

1. Set $m = m_1 = \lceil s\,(L)/U \rceil$.
2. Try to pack the list L into m bins using the FFD_f algorithm.
3. If all items were packed stop.
4. Otherwise set $m = m + 1$ and go to step 2.

It is interesting to see if $FFD_f - I$ improves the performance ratio of NF_f. As we shell see the improvement is significant for small values of bin size.

Theorem 2. *The asymptotic worst case performance ratio of the* $FFD_f - I$ *algorithm satisfies:*

(i) $R^{\infty}_{FFD_f - I} \leq \frac{U}{U-1}$ *when* $U \leq 15$

(ii) $\frac{U}{U-1} < R^{\infty}_{FFD_f - I} < \frac{U}{U-2}$ *when* $U \geq 16$

Proof. To prove the theorem we use a property which we call the *border bin property*. We assume $FFD_f - I$ has m bins to pack and number the bins $B_1, ..., B_m$ according to the order they are opened by the algorithm. Looking at the level (used space) of the bins at a certain time during algorithm execution, we say that bin B_j, $j < m$, is a *border bin* at that time if its level satisfies: $L(B_j) \neq L(B_{j+1})$.

Claim. At any time before the first item is fragmented, the packing of $FFD_f - I$ contains at most $2U$ border bins.

Proof. The $FFD_f - I$ algorithm packs the items in decreasing size order. We consider the number of border bins before the first item is fragmented. Denote by $BR(k)$ the number of border bins, after k different sizes of items have been packed by $FFD_f - I$. Clearly $BR(1) \leq 2$ since items of the same size are packed in a similar way. Whenever each additional size is packed the number of border bins may increase by at most two, therefore $BR(k) \leq BR(k-1)+2$. Since there are at most U different sizes, $BR(U) \leq 2U$, and the claim follows. □

We now proceed to prove the theorem for each range separately.

(i) $U \leq 15$: We establish $\frac{U}{U-1}$ as an upper bound on the asymptotic performance ratio. It is clear that as long as the number of bins with two units of overhead is small, i.e., O(1), the asymptotic performance ratio cannot exceed $\frac{U}{U-1}$ (the proof is similar to that of Lemma 1). We make the following observation:

Claim. The final packing of the $FFD_f - I$ algorithm **cannot** contain more than O(1) bins with 2 units of overhead, if one of the following conditions is met:

1. Before the first fragmentation occurs, the free space in the bins is 2 or less.
2. The fragmented items are of size 4 or less.

Proof. Omitted.

It is easy to verify that the conditions set by the above claim cannot be extended, since fragmenting items of size 5 over bins of size 3 results in one third of the bins containing 2 units of overhead. Therefore, in order to get a significant number of bins with 2 units of overhead, items of size 5 or more must be fragmented. This means that the list should contain items of size 6 or more. Note that if $FFD_f - I$ fragments items of sizes $s(a_i) \geq \frac{U}{2}$, they are also fragmented by the optimal packing. We conclude that in order to create a difference of more than one overhead unit, between the optimal packing and the packing of $FFD_f - I$, two items of size $s(a_i) \geq 6$ must be packed in a bin and leave a free space of at least 3. To do so, the bin size must satisfy $U \geq 15$. However, in the case of U=15, items of size 5 are packed without fragmentation and therefore the value $\frac{U}{U-1}$ is an upper bound on the asymptotic performance ratio for $U \leq 15$.

(ii) $U \geq 16$: We first prove the lower bound and then the upper bound.

Claim. For the $FFD_f - I$ algorithm - $\frac{U}{U-1} < R^{\infty}_{FFD_f-I}$.

Proof. For each value $U \geq 16$, there is a list for which the performance ratio exceeds $\frac{U}{U-1}$. We present here an example only for the case where U is even, an example for odd values of U is similar. Assume $U = 2u$ and choose a list L with k items of size $u-2$, then k items of size $u-3$ and finally k items of size 5. The optimal packing fills k bins each with one item of each size, $OPT(L) = k$. When $FFD_f - I$ is applied to L the first k bins contain all items except for the last $k/4$ items of size 5. In the best case (U is a multiple of 5) no more overhead is produced and $\frac{5k}{4U}$ more bins are used. The total number of bins required by the algorithm is: $FFD_f - I(L) \geq (k + 5k/4U)$. The performance ratio in this case is:

$$R^{\infty}_{FFD_f-I} \geq \frac{4U+5}{4U} > \frac{U}{U-1}, \quad \forall\, even\ U \geq 16. \tag{5}$$

□

We now turn to the upper bound on the performance ratio.

Claim. For the $FFD_f - I$ algorithm - $R^{\infty}_{FFD_f-I} < \frac{U}{U-2}$.

Proof. In order to prove the claim we show that the algorithm cannot produce a packing where each bin (maybe except for a negligible number) contains two units of overhead. The *border bins property* implies that just before the first item is fragmented the bins are arranged in long sequences where the free space in all bins is equal. The items are also packed in long sequences. Let us assume the free space in a sequence is x and the size of the items packed is y. Obviously $x < y$ otherwise the items are not fragmented. Note that when an item is fragmented over more than two bins, only the first and last bins can contain two units of overhead. Therefore it is enough to consider only the case where an item is fragmented over two bins. Assume that item number k is fragmented over two bins such that bin number 1 contains a fragment of size α and bin number 2 contains a fragment of size $y-\alpha$. The next item (number $k+1$) is also fragmented and a fragment of size $x-y+\alpha-2$ is packed in bin number 2. In order to create a repeated cycle of fragmentations the size of the first fragment of each item must be equal, that is $\alpha = x-y+\alpha-2$. This is true for $y = x-2$, but for that value the items are not fragmented in the first place. Since we can not create a repeated cycle of equal size fragmentations, the size of the first fragment packed in a bin increases, until it reaches the size of the bin, in which case only one unit of overhead is packed in the bin. This means that at least one of every y bins contains only one unit of overhead. Since $y \leq U/2$ we can establish that:

$$R^{\infty}_{FFD_f-I} \leq \frac{U}{U-2+\frac{2}{U}} < \frac{U}{U-2} \tag{6}$$

This concludes the proof of Theorem 2. □

The improvement of $FFD_f - I$ over NF_f is significant for low values of bin size ($U \leq 15$), which are the most meaningful values (since the performance ratio is decreasing with the bin size). Moreover, $FFD_f - I$ is superior for any value of U. We make the following remarks:

- For each of the following values: $U \in \{7, 9, 10, 11, 13, 14, 15\}$, it is possible to find an example where the ratio is $\frac{U}{U-1}$, hence $R^{\infty}_{FFD_f-I} = \frac{U}{U-1}$, [11].
- When $U \leq 5$, $R^{\infty}_{FFD_f-I} = 1$, however when $U \in \{6, 8\}$ $R^{\infty}_{FFD_f-I} > 1$. This is interesting, since for the classical problem the performance ratio of FFD is: $R^{\infty}_{FFD} = 1$.
- We may define algorithm $BFD_f - I$ in a similar way to $FFD_f - I$, only this algorithm is based on the Best-Fit Decreasing heuristic. We can show that the $BFD_f - I$ algorithm has the same performance ratio as $FFD_f - I$ [11].

3 Bin Packing with Size-Preserving Fragmentation

In this section we study a different fragmentation cost function in which fragmenting an item does not increase its size. Instead, we assume that packing an item is associated with a cost and fragmentation increases this cost. The cost of packing an item (or fragmenting it) depends on the application. As an example, take the scheduling problem but assume that fragmenting a datagram does not increase its size, because the format of the datagram already includes the fragmentation fields. On the other hand fragmentation requires additional resources from the system (CPU, memory) and takes longer to process. In other applications, such as in stock-cutting problems, it may simply cost to fragment an item (cut a piece of pipe for example) or put it back together. We proceed to formally define the problem.

Bin Packing with Size-Preserving Fragmentation (BP-SPF): We are given a list of n items $L = (a_1, a_2, ..., a_n)$, each with a size $s(a_i) \in \{1, 2, ..., U\}$ and a cost $c(a_i) \in Z^+$. The items must be packed into m identical bins, of size U. It is possible to fragment any item, in which case one unit is added to its cost but does not change its size. The goal is to minimize the total cost.

Denote by $s(L)$ and $c(L)$ the total size and cost of all items, respectively. To ensure all items can be packed, we assume $s(L) \leq mU$.

Performance: There are several ways to evaluate the performance of an algorithm for the problem. We observe that since the cost of fragmentation is not related to the size or cost of an item, the additional cost of an algorithm depends only on the number of fragmentations it performs. We therefore chose to evaluate the performance of an algorithm by its *overhead*. For a given list L and algorithm A, let $c(A, L)$ be the total cost of algorithm A, let $c(OPT, L)$ denote the optimal (minimal) cost and define the overhead of A as: $OH_A(L) \equiv c\,(A, L) - c\,(OPT, L)$. For the case of $OPT(L) = m$, we define the worst case overhead of algorithm A, OH^m_A, as:

$$OH^m_A \equiv \inf\{h : \ OH_A(L) \leq h \quad \text{for all } L \text{ with } OPT(L) = m\}. \tag{7}$$

We first show that the complexity of BP-SPF is similar to that of BP-SIF.

Claim. BP-SPF is NP-hard in the strong sense.

Proof. The proof is similar to that of BP-SIF and is omitted from this version.

We consider only algorithms that prevent unnecessary fragmentation (see Definition 1). Such algorithms have the following property.

Lemma 2. *For any algorithm A, that prevents unnecessary fragmentations* - $OH_A^m \leq m-1$.

Proof. Since A prevents unnecessary fragmentations, it may perform at most $m-1$ fragmentations when packing m bins. The maximum cost of algorithm A is therefore: $c(A,L) = c(L) + (m-1)$. Clearly $c(OPT,L) \geq c(L)$, which means that for any list L: $OH_A(L) \leq m-1$. □

We now examine the performance of the NF_f and FFD_f algorithms (defined in subsections 2.1 and 2.2, respectively). We show that the performance of the NF_f algorithm is the worst possible while FFD_f performs better.

Theorem 3. *The overhead of algorithm NF_f for every $m \geq 2$ is* - $OH_{NF_f}^m = m-1, \quad \forall U \geq 2$.

Proof. Lemma 2 provides an upper bound. As a worst case example choose a list of items with one item of size $U-1$ followed by $m-1$ items of size U. □

We now turn to the FFD_f algorithm. We expect FFD_f to perform better than NF_f and this is indeed true when the bin size U is small. However, we show that if the bin size is not bounded, the worst case overhead of FFD_f is the maximum possible, that is, there exist a list L for which, for any value of m, $c(FFD_f,L) - c(OPT,L) = m-1$.

Claim. For the FFD_f algorithm, for every $m \geq 2$ - $OH_{FFD_f}^m = m-1$.

Proof. We choose a bin size satisfying: $U > 2m+16$. The list, L, is made of k repetitions of the following set: $L' = \{U/2+2,\ U/2+1,\ U/4+2,\ U/4+1,\ U/4-3,\ U/4-3\}$. Two bins of size U are needed to pack L', therefore $m = 2k$ bins are needed to pack L. The optimal packing causes no fragmentations. The FFD_f algorithm packs the items in the following way: The first $5k$ items are packed without fragmentations. The free space in the first k bins is $U/4-4$, in the remaining k bins the free space is 1. Since the free space in all bins is smaller than the size of the items, the items are fragmented. Each item except the last is fragmented over two bins. When packing the last item, bins B_1,...B_{k-1} are full, bin B_k has free space of $\frac{U}{4}-k-3$ and each bin B_{k+1},...,B_{2k} has free space of 1. As a result, the last item is fragmented over the remaining free space, causing k more fragmentations. The total number of fragmentations is $m-1$ and the overhead is therefore $OH_{FFD_f}^m = m-1$. □

To hold for every value of m the above claim requires an unbounded bin size. Since we are mainly interested in asymptotic behavior, i.e., $U \ll m$, this is clearly not a practical assumption. For the more reasonable cases where $m > U$ the overhead of algorithm FFD_f is less than $m-1$.

Claim. For any bin size U there is $N > 0$ for which the overhead of FFD_f satisfies - $OH_{FFD_f}^m < m-1, \quad \forall N < m$.

Proof. We show that if $U \ll m$ then $c(A,L) < c(L) + (m-1)$. Recall that when analyzing the $FFD_f - I$ algorithm (subsection 2.2) we proved the border bin property. The property tells us that before the first item is fragmented the bins are ordered in long sequences of equal content bins. Note that by increasing m while keeping U constant, we can create sequences of any length. Consider the moment before FFD_f fragments the first item. Let us assume the free space in a sequence is x and the size of the items packed is y, where $x < y$. Clearly at least one out of every $x \cdot y$ bins is closed without fragmentation. The overhead is therefore always smaller than $m - 1$. □

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