Fundamental Bounds and Algorithms in Distributed Reliable Storage

Research Thesis

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List of Publications

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Abstract

In recent years we see an exponential increase in storage capacity demands, creating a need for big data storage solutions. Additionally, today's economy emphasizes consolidation, giving rise to massive data centers and clouds. In this era, distributed storage plays a key role. This is marked by two clear trends: First, the storage market gravitates towards distributed storage solutions within data centers as well as across multiple data centers; such systems are typically made up of many cheap, low-reliability storage nodes, and achieve durability and high availability via redundancy. Second, we see more and more users store data in clouds that are accessed remotely over the Internet. In this thesis, we study two fundamental aspects of reliable distributed storage: storage space cost and dynamic storage reconfiguration.

Storage space cost. Given the inherent failure-prone nature of storage and network components, a reliable distributed storage algorithm must store redundant information in order to allow data to be reconstructed when some subset of the system fails. The most common approach to achieve this is via replication, i.e., storing multiple copies of each data block. The well-known ABD result shows that an f-tolerant storage can be emulated using a collection of $2f + 1$ fault-prone servers, each storing a single read-modify-write object type, which is known to be optimal. We first generalize this bound by investigating the inherent space cost of emulating reliable storage as a function of the object type exposed by the underlying servers. Then, we focus on read-modify-write object type and investigate whether erasure codes can help mitigate the significant cost of replication that results from the immense size of the data. Some previous works attempted to do it, but a closer look at existing solutions reveals that they incur extra storage costs in other places. Specifically, the use of coding creates scenarios where old information cannot be over-written, leading the storage to grow without bound if concurrency is high. We shed light on the fundamental tradeoff between lower redundancy per block (as achieved by codes), concurrency, and allowing old data to be over-written (as achieved by replication).

Dynamic reconfiguration. A key challenge for distributed storage is the problem of reconfiguration. Clearly, any production storage system that provides data reliability and availability for long periods must be able to reconfigure in order to remove failed

or old nodes and add healthy or new ones. Reconfiguration is also essential in order to realize the greatest advantage of distributed storage over traditional monolithic enterprise storage solutions, namely, that it supports elastic incremental growth- the main motivation for the economic model of cloud computing. In this thesis we contribute to the understanding of reconfiguration of distributed storage by showing both negative (impossibility) and positive (algorithms) results.

List of Acronyms

RMW	read modify write
MWMR	multi-writer multi-reader
MWSR	multi-writer single-reader
SWSR	single-writer single-reader
SWMR	single-writer multi-reader
RR	ranked register
CoS	common set

List of Symbols

$\diamond P^D$ dynamic failure detector

Chapter 1

Introduction

Data collection and consumption is growing exponentially in recent years (see, e.g., [46]), and is expected to continue to grow even faster [76]. This creates a demand for larger and larger storage capacities and support for elastic growth thereof. The growth of individual disk capacities cannot match this phenomenal surge in demand, and so *distributed storage* has become the method of choice. Storage solutions nowadays consist of large interconnected collections of disks, controllers, and servers. In some cases they reside within a single data center, while in others they are geo-distributed so as to allow local access to geographically dispersed clients as well as for disaster recovery. In many cases the storage is provided as a cloud-based service. In this thesis, we deal with fundamental challenges that arise in today's massive distributed storage. In particular, we study reliable asynchronous storage space cost and its dynamic reconfiguration. Part of the results in this thesis were previously published in PODC [27, 69], DISC [73], and SIROCCO [71].

1.1 Storage cost

Reliable storage emulations seek to construct fault-tolerant shared objects, such as read/write registers, using a collection of *base objects* hosted on failure-prone servers. A reliable distributed storage emulation must therefore store redundant information in order to allow data to be reconstructed when some subset of the system fails. The most common approach to achieve this is via *replication*, i.e., storing multiple copies of each data block, and most existing storage emulation algorithms are constructed from storage services capable of supporting custom-built read-modify-write (RMW) primitives [10, 35, 41, 31, 39, 7, 64, 57], where perhaps the most famous one is ABD [10]. This algorithm emulates an f-tolerant atomic wait-free register, accessed by an unbounded number of processes (readers and writers), on top of $2f + 1$ servers, each of which stores a single RMW object.

The power of primitives. Since f-tolerant register emulation is impossible with less than $2f + 1$ servers [14, 55], the ABD algorithm's space complexity is optimal in terms of

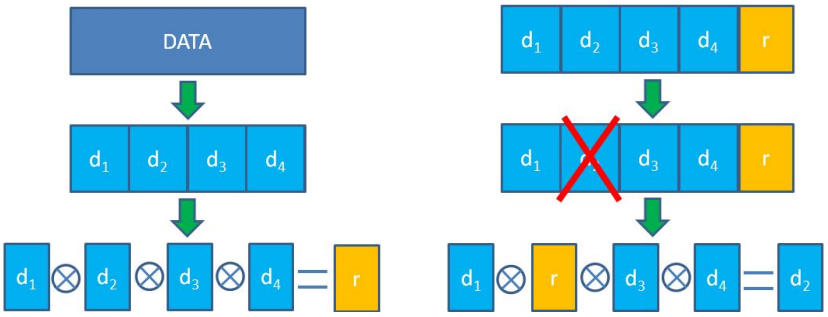


Figure 1.1: Encoding and decoding using xor.

number of base objects. However, support for atomic RMW is not always available: the operations exposed by network-attached disks are sometimes limited to basic read/write capabilities, and the interfaces exposed by cloud storage services sometimes augment this with simple conditional update primitives similar to Compare-and-Swap (CAS). A natural question that we study in Chapter 2 is therefore how the ABD results generalize when only weaker primitives (e.g., read/write registers) are available. More specifically, we are interested whether reliable storage emulations are possible with weaker primitives, and if so, what their space cost in terms of number of objects is, and in particular, does their cost depend on the number of writers and the number of servers. This question led us to define a new space hierarchy of fault tolerant storage emulations.

Erasure codes. Given the immense size of data, the storage cost of full replication becomes significant. A promising approach mitigates this cost via the use of *erasure codes* [6, 21, 42, 23, 77, 30]. In a nutshell, erasure codes allow redundant information to be shared among multiple data blocks. For example, it is possible to keep a single redundant block constructed as the xor of k data blocks instead of holding replicas of all k blocks. Either way, the data can be reconstructed despite the loss of any single (original or redundant) block as illustrated in Figure 1.1. Given data of size D bits and $f = 1$, this approach can ideally reduce the storage cost from $3D$ to $(k+1)D/k$ bits. If a large k is used, the redundancy overhead with erasure codes is diminished, and the gap becomes significant.

While this approach is appealing, a closer look at existing erasure-coded storage solutions reveals that they incur extra storage costs somewhere else. Specifically, the use of coding in lieu of replication creates a need for storing multiple versions of each data block, and the number of version *can grow without bound* if concurrency is high [42, 23]. To avoid it, some keep unbounded information in channels [30, 21] while others assume periods of synchrony [6] or allow returning obsolete values [77].

In Chapter 3 we study to what extent such a storage blowup is inevitable. We investigate the fundamental limitations of coded storage and the tradeoffs between replication

and coding. In particular, we first prove a lower bound on the storage cost, in terms of bits, for asynchronous algorithms under some assumptions stated in Chapter 3. Our lower bound is presented as a function of the number of storage node failures tolerated, the concurrency allowed by the algorithm, and the data size. Then, we also present in Chapter 3 an algorithm that combines erasure-codes with replication to achieve an asymptotically tight upper bound.

1.2 Dynamic reconfiguration

A second key challenge for distributed storage is *reconfiguration*. Clearly, any production storage system that provides data reliability and availability for long periods *must* be able to reconfigure in order to remove failed or old nodes and add healthy or new ones. Reconfiguration is also essential in order to realize the greatest advantage of distributed storage over traditional monolithic enterprise storage solutions, namely, that it supports *elastic incremental growth*– the main motivation for the economic model of cloud computing.

The study of reconfigurable replication has been active since at least the early 1980s, with the development of group communication and virtual synchrony (see survey in [28]). But despite its vital real-world importance, the issue of online reconfiguration of storage services is still not well understood; theory for this problem is in its infancy, and most distributed storage systems in production are reconfigured manually during complete system shutdown.

Until recently, no liveness (progress) properties for a reconfigurable storage service were formally specified. While it has long been well-known that, in a static system, asynchronous replication requires a majority of replicas to be operational for liveness, no similar criterion had been formulated for the case where the set of replicas dynamically changes. The DynaStore [8] algorithm was the first to formulate *sufficient* conditions for liveness of reconfigurable storage. Nevertheless, these conditions were never shown to be necessary. For example, DynaStore, like its predecessors [40, 41, 53, 24], ensures progress only if the number of reconfigurations is finite.

Liveness of Dynamic Storage In Chapter 4, we prove that this is indeed inherent for asynchronous API-based storage solutions, where the service's API is extended with a reconfiguration operation for changing the current configuration of participating processes [66, 8, 50, 37, 40, 41, 24], which can be only invoked by members of the current configuration. In addition, we define in Chapter 4 a dynamic failure detector that can be easily implemented in eventually synchronous systems, and use it to implement dynamic storage algorithm that ensures liveness for all operations without restricting the reconfiguration rate.

Dynamic fault model. Moreover, DynaStore's restrictions on failures are not defined in an intuitive way, and do not create a clear distinction between user-controlled actions (like removing a process from the current configuration) and ones stemming from the environment (like process failures). In Chapter 5 we first define a dynamic model with a clean failure condition that allows an administrator to reconfigure the system and switch off a server once the reconfiguration operation removing it completes. Our dynamic fault model is defined as an interplay between the object's implementation and its environment.

Reconfiguration abstraction and optimal solution. DynaStore is quite complex, the reconfiguration mechanism is not clearly abstracted away from the intricacies of data replication, its operation complexity is exponential, and it does not explicitly deal with clients and only presents a server-side algorithm. (Follow-up work [66] extends DynaStore to support client operations, but this is systems work that does not deal with the topic formally). In Chapter 5 we also define a Reconfiguration abstraction in the client-server model, show how it can be used to build dynamic storage, and give an optimal asynchronous algorithm implementing the Reconfiguration abstraction, which in turn leads to the first asynchronous storage emulation with optimal linear complexity.

Chapter 2

The Power of Primitives for Fault-Tolerant Register Emulations

Reliable storage emulations seek to construct fault-tolerant shared objects, such as read/write registers, using a collection of *base objects* hosted on failure-prone servers. Such emulations are core enablers for many modern storage services and applications, including cloud-based online data stores [29, 61, 60, 47, 58] and Storage-as-a-Service offerings [63, 67, 33, 75].

Most existing storage emulation algorithms are constructed from storage services capable of supporting custom-built read-modify-write (RMW) primitives [10, 35, 35, 41, 31, 39, 7, 64, 57], and perhaps the most famous one is ABD [10]. This algorithm emulates an f-tolerant atomic wait-free register, accessed by an unbounded number of processes (readers and writers), on top of $2f + 1$ servers, each of which stores a single RMW object. Since f-tolerant register emulation is impossible with less than $2f + 1$ servers [14, 55], the ABD algorithm's space complexity is optimal.

However, support for atomic RMW is not always available: the operations exposed by network-attached disks are typically limited to basic read/write capabilities, and the interfaces exposed by cloud storage services sometimes augment this with simple conditional update primitives similar to Compare-and-Swap (CAS). A natural question that arises is therefore how the ABD results generalize when only weaker primitives (e.g., read/write registers) are available. More specifically, we are interested whether reliable storage emulations are possible with weaker primitives, and if so, what their space complexity is, and in particular, does their complexity depend on the number of writers and the number of servers. These questions lead us to define a new space hierarchy of fault tolerant register emulations.

Base object	Lower bound (WS-Safe, obstruction-free)	Upper bound (WS-Regular, wait-free)
max-register	$2f+1$	$2f+1$
CAS	$2f+1$	$2f+1$
read/write register	$kf + \left\lceil \frac{k}{\frac{n-(f+1)}{f}} \right\rceil (f+1)$	$kf + \left\lceil \frac{k}{\left\lfloor \frac{n-(f+1)}{f} \right\rfloor} \right\rceil (f+1)$

Table 2.1: **Space hierarchy:** Number of base objects for f-tolerant register emulation with k writers and $n \geq 2f+1$ servers.

Space hierarchy. Herlihy's well-known consensus hierarchy [44] provides a way to compare the power of different primitives (e.g., read/write register, T&S, CAS, etc), in terms of computability. However, this hierarchy tells us nothing about the space complexity of the corresponding emulations. A recent paper by Ellen et al. [34] sheds some light on this topic by presenting a hierarchy classified by the space complexity required for solving obstruction-free consensus. We follow Ellen et al. [34], and make another step towards understanding the space requirements induced by different primitives.

We introduce a new space hierarchy, which classifies primitives by the space required for reliable register emulations on top of fault-prone servers that support the primitives. More specifically, we assume a collection of n fault-prone servers, each of which stores base objects supporting the given primitive. The failure granularity is servers, meaning that a server crash causes all base objects it stores to crash as well. We study three primitives: read/write register, max-register [9], and CAS. For each primitive, we are interested in the number of base objects required to emulate an f-tolerant register for k writers using n servers.

To strengthen our result, we prove the lower bound under weak liveness and safety guarantees, namely, obstruction freedom and write sequential safety (WS-Safety). The latter is a weak generalization for Lamport's notion of safety [52] to multi-writer registers, which we define in Section 2.1. Since atomicity usually requires readers to write, which may induce a dependency on the number of readers, we consider regularity for our upper bound; we define in Section 2.1 write sequential regularity (WS-Regularity), which is a weaker form of multi-writer regularity defined in [64]. The lower bound proved in [14, 55] on the number of servers required for f-tolerant register emulations can be easily generalized for WS-Safe obstruction-free emulations. Therefore, we assume that $n \geq 2f+1$ throughout the chapter.

Table 2.1 summarizes the hierarchy. Interestingly, even though both registers and max-registers have the lowest consensus number of 1 in Herlihy's hierarchy [44], they have different powers in our hierarchy, whereas CAS, which has an infinite consensus number, has the same power as max-register in our hierarchy. As an aside, we note that our hierarchy has implications for the standard shared memory model (without base object failures); for example, it implies that a max-register for k writers cannot be implemented from less than k read/write registers (proven in Theorem 3).

Results. Despite the fact that the original ABD emulation [10] assumes a general RMW base object on every server, we observe that the code executed by each server in the multi-writer ABD protocol [41, 64, 57] can be encapsulated into the *write-max* (for handling update messages) and *read-max* (for handling read messages) primitives of max-register. Therefore, the upper bound of $2f + 1$ applies to max-registers as well. In Section 2.2 we show how to emulate a max-register from a single CAS in a wait-free manner. Thus, the upper bound for max-register also applies to CAS.

Our main technical contribution is a lower bound on the number of read/write registers required to emulate an f-tolerant WS-Safe obstruction-free register for k clients using n servers. While the ABD [10] space complexity does not depend on the number of writers or the number of servers, we show in Section 2.3 (Theorem 2) that when servers support only read/write registers, the lower bound increases linearly with the number of writers and decreases (up to a certain point) with the number of available servers. In particular, our lower bound implies that at least $kf + f + 1$ registers are needed regardless of the number of available servers. We exploit asynchrony to show that an emulated write must complete even if it leaves f pending writes on base registers, forcing the next writer to use a different set of registers, even in a write-sequential run.

In Section 2.3, we also present a new upper bound construction that closely matches our lower bound (Theorem 4). Note that the two bounds are closely aligned, and in particular, coincide in the two important cases of $n = 2f + 1$ and $n \geq kf + f + 1$ where they are equal to $kf + k(f + 1)$ and $kf + f + 1$ respectively. An interesting open question is to close the remaining small gap.

Another open question is whether our lower bound is tight for stronger regularity definitions [65]. In the special case of $n = 2f + 1$ servers and k writers, a matching upper bound of $(2f + 1)k$ registers can be achieved by simply having each server implement a single k-writer max-register from k base registers [9]. The question of the general case of $n \geq 2f + 1$ remains open.

In Appendix .1, we show the following three additional results implied by an extended variant of our main lower bound construction: (1) a lower bound of k registers per server for $n = 2f + 1$ (Theorem 13); (2) a lower bound on the number of servers when the maximum number of registers stored on each server is bounded by a known constant (Theorem 14); and (3) impossibility of constructing fault-tolerant multi-writer register emulations adaptive to point contention [2, 13] (Theorem 15).

Related work. The space complexity of fault-tolerant register emulations has been explored in a number of prior works. Afek et al. [3] consider a model where faulty servers are responsive but may return arbitrary values, and present several storage efficient algorithms for reliable registers. We prove that when faulty servers are unresponsive, the additional storage cost is inherent. Aguilera et al. [5] present bounds on the number of fault-prone base registers required to support a reliable multi-writer one using *uniform* algorithms (i.e., algorithms that do not depend on a priori knowledge of the number of writers). Our results, in contrast, do not assume uniformity, and provide bounds on the

number of base registers as a function of the number of writers, servers, and the failure threshold. In Chapter 3 we consider the space complexity of reliable register emulations in terms of the amount of data bits stored on fault-prone RMW servers, and since in this chapter we are only interested in the number of stored registers and not their sizes, these results are orthogonal.

Basescu et al. [18] describe several fault-tolerant multi-writer register emulations from a collection of fault-prone read/write data stores. Their algorithms incorporate a garbage-collection mechanism that ensures that the storage cost is adaptive to the write concurrency, provided that the underlying servers can be accessed in a synchronous fashion. Our results show that asynchrony has a profound impact on storage consumption by exhibiting a *sequential* failure-free run where the number of registers that need to be stored grows linearly with the number of writers.

The proof of our main result (see Lemma 2) further extends the adversarial framework we use in Chapter 3 to exploits the notion of *register covering* (originally due to [**?**]) extended to fault-prone base registers as in [5]. Covering arguments have been successfully applied to proving numerous space lower bound results in the literature (see [12] for a survey) including the recent tight bounds for obstruction-free consensus [38, 78], which are at the heart of the space hierarchy of [34] discussed above.

2.1 Model

2.1.1 Shared Objects

A *shared object* supports concurrent execution of *operations* performed by some set, $\mathbb{C} = \{c_1, c_2, \dots\}$, of client processes. Each operation has an *invocation* and *response*. An object *schedule* is a sequence of the operation invocations and their responses. An invoked operation is *complete* in a given schedule if the operation's response is also present in the schedule, and *pending* otherwise. For a schedule σ, $ops(\sigma)$ denotes the set of all operations that were invoked in σ, and $complete(\sigma)$ (resp., $pending(\sigma)$) denotes the subset of $ops(\sigma)$ consisting of all the complete (resp., pending) operations. Also, for a set X of operations, we use $\sigma|X$ to denote the subsequence of σ consisting of all the invocation and responses of the operations in X.

An operation op *precedes* an operation op' in a schedule σ, denoted $op \prec_\sigma op'$ iff op' is invoked after op responds in σ. Operations op and op' are *concurrent* in σ, if neither one precedes the other. A schedule with no concurrent operations is *sequential*. Given a schedule σ, we use $\sigma|i$ to denote the subsequence of σ consisting of the actions client c_i. The schedule is *well-formed* if each $\sigma|i$ is a sequential schedule. In the following, we will only consider well-formed schedules.

The object's *sequential specification* is a collection of the object's sequential schedules in which all operations are complete. For an object schedule σ, a *linearization* L_σ of σ is a sequential schedule consisting of all operations in $complete(\sigma)$ along with their responses and a subset of $pending(\sigma)$, each of which being assigned a matching response, so that L_σ satisfies both the σ's operation precedence relation ($\prec_\sigma$) and the object sequential specification.

2.1.2 Registers

A *read/write register* object (or simply a *register*) supports two operations of the form $write(v)$, $v \in Vals$, and $read()$ returning *ack* and $v \in Vals$ respectively where $Vals$ is the register value domain. Its sequential specification is the collection of all sequential schedules where every *read* returns the value written by the last preceding *write* or an initial value $v_0 \in Vals$ if no such write exists.

A register is *multi-writer (MW)* (resp., *multi-reader (MR)*) if it can be written (resp., read) by an *unbounded* number of clients. A k-writer register, or simply, k-register, is a register that can be written by at most $k > 0$ distinct clients. A register is *single-writer (SW)* (resp., *single-reader (SR)*) if only one process can write (resp., read) the register. For a register schedule σ, we use $writes(\sigma)$ and $reads(\sigma)$ to denote the sets of all write and read operations invoked in σ. We say that σ is *write-sequential* if no two writes in $writes(\sigma)$ are concurrent.

2.1.3 Consistency Conditions

Consistency conditions specify the shared object behaviour when accessed concurrently by the clients. Below, we introduce a number of consistency conditions that will be used throughout this paper. They are expressed as a set of schedules C satisfying one of the following requirements:

Atomicity [45] For all schedules $\sigma \in C$, σ has a linearization.

Write-Sequential Regularity (WS-Reg) For all $\sigma \in C$, if σ is write-sequential, then for each $rd \in reads(\sigma) \cap complete(\sigma)$ there is a linearization L_{rd} of $\sigma|writes(\sigma) \cup \{rd\}$.

Write-Sequential Safety (WS-Safe) As WS-Reg, but only required to hold for complete reads that are not concurrent with any writes.

2.1.4 System Model

We consider an *asynchronous fault-prone shared memory system* [48] consisting of a set of shared *base* objects $\mathcal{B} = \{b_1, b_2, \dots\}$. The objects are accessed by a collection of clients in the set $\mathbb{C} = \{c_1, c_2, \dots\}$.

We consider a slight generalization of the model in [48] where the objects are mapped to a set $\mathcal{S} = \{s_1, s_2, \dots\}$ of servers via a function δ from $\mathcal{B}$ to $\mathcal{S}$. For $B \subseteq \mathcal{B}$, we will write $\delta(B)$ to denote the *image* of B, i.e., $\delta(B) = \{\delta(b) : b \in B\}$. Conversely, for $S \subseteq \mathcal{S}$, we will write $\delta^{-1}(S)$ to denote the *pre-image* of S, i.e., $\delta^{-1}(S) = \{b : \delta(b) \in S\}$. Note that for all $B \subseteq \mathcal{B}$, $|\delta(B)| \leq |B|$, and conversely, for all $S \subseteq \mathcal{S}$, $|\delta^{-1}(S)| \geq |S|$. Both servers and clients can fail by crashing. A crash of a server causes all objects mapped to that server to instantaneously crash[1].

We study algorithms emulating reliable k-writer registers to a set of clients. Clients interact with the emulated register via *high-level* read and write operations. To distinguish the high-level emulated reads and writes from low-level base object invocations, we refer to the former as READ and WRITE. We say that high-level operations are *invoked* and *return* whereas low-level operations are *triggered* and *respond*. A high-level operation consists of a series of trigger and respond *actions* on base objects, starting with the operation's invocation and ending with its return. Since base objects are crash-prone, we assume that the clients can trigger several operations in a row without waiting for the previously triggered operations to respond.

An emulation algorithm A defines the behaviour of clients as deterministic state machines where state transitions are associated with actions, such as trigger/response of low-level operations. A *configuration* is a mapping to states from system components, i.e., clients and base objects. An *initial configuration* is one where all components are in their initial states.

[1] Note that the original faulty shared model of [48] can be derived from our model by choosing δ to be an injective function.

A *run* of A is a (finite or infinite) sequence of alternating configurations and actions, beginning with some initial configuration, such that configuration transitions occur according to A. We use the notion of time t during a run r to refer to the configuration reached after the t^{th} action in r. A *run fragment* is a contiguous sub-sequence of a run. A run is *write-only* if it has no invocations of the high-level READ operations.

We say that a base object, client, or server is *faulty* in a run r if it fails at some time in r, and correct, otherwise. A run is *fair* if (1) for every low-level operation triggered by a correct client on a correct base object, there is eventually a matching response, and (2) every correct client gets infinitely many opportunities to both trigger a low-level operation and execute the return actions. We say that a low-lever operation on a base object is *pending* in run r if it was triggered but has no matching response in r. We assume that the base objects are atomic (as defined in Section 2.1.3)

2.1.5 Properties of the Emulation Algorithms

Safety The emulation algorithm safety will be expressed in terms of the consistency conditions specified in Section 2.1.3. An emulation algorithm A *satisfies* a consistency condition C if for all A's runs r, the subsequence of r consisting of the invocations and responses of the high-level read and write operations is a schedule in C.

Liveness We consider the following liveness conditions that must be satisfied in fair runs of an emulation algorithm. A *wait-free* object is one that guarantees that every high-level operation invoked by a correct client eventually returns, regardless of the actions of other clients. An *obstruction-free* object guarantees that every high-level operation invoked by a correct client that is not concurrent to any other operation by a correct client eventually returns.

Fault-Tolerance The emulation algorithm is f-tolerant if it remains correct (in the sense of its safety and liveness properties) as long as at most f servers crash for a fixed $f > 0$.

Complexity measures The *resource consumption* of an emulation algorithm A in a (finite) run r is the number of base objects used by A in r. The *resource complexity* [48] of A is the maximum resource consumption of A in all its runs.

2.2 A Max-Register Emulation With One CAS

We present here a wait-free emulation of an atomic max-register on top of a single CAS object. The pseudocode appears in Algorithm 1. The CAS object supports one operation with two parameters, exp and new; if exp is equal to the object's current value, then the value is set to new. In any case, the operation returns the old value.

A max-register supports two operations, *max-write*(v) for some v from some domain of ordered values $\mathbb{V}$ that returns ok, and *max-read*$()$ that returns a value from $\mathbb{V}$. The sequential specification of max-register is the following: A *max-read* returns the highest value among those written by *max-write* before it, or v_0 in case no such values.

Algorithm 1 Max-register emulation from a single CAS object b

```
Local variables:
    tmp ∈ 𝕍, initially v_0
operation b.CAS(exp, new), exp, new ∈ 𝕍
    prev ← b
    if exp = b then
        b ← new
    return prev

1: operation Max-write(v)
2:     while true do
3:         tmp ← b.CAS(v_0, v_0)
4:         if tmp ≥ v then
5:             return ok
6:         b.CAS(tmp, v)

7: operation Max-read()
8:     tmp ← b.CAS(v_0, v_0)
9:     return tmp
```

2.2.1 Correctness

We say that a $b.CAS(exp, new)$ operation is *successful* if b is set to *new*.

Observation 1. *Consider a successful $b.CAS(exp, new)$ operation for some exp and new, the the next $b.CAS(exp', new')$ operation for some exp' and new' returns new.*

The following observation follows immediately from the fact that $b.CAS(exp, new)$ is called only with $new \geq exp$.

Observation 2. *The values returned by $b.CAS(exp, new)$ are monotonically increasing.*

We next define linearization points:

Definition 1. *(linearization points)*

***max-read*:** *The linearization point is line 8.*

***max-write*:** *If the operation performs a successful $b.CAS(tmp, v)$ in line 6, then the linearization point is the last time Line 6 is performed. Otherwise, the linearization point is last time line 3 is performed.*

Lemma 1. *For every run r of Algorithm 1, the sequential run σ_r, produced by the linearization points of operations in r, is a linearization of r.*

Proof. The real time order of r is trivially preserved in σ_r. We need to show that σ_r preserves max-register's sequential specification. Let mr be a *max-read()* in r that returns a value v, and let t_{mr} be the time when $b.CAS(v_0, v_0)$ (that returns v) is called by mr (line 8). We need to show that (1) there is no *max-write*(v') that precedes mr in σ_r s.t. $v' > v$, and (2) if $v \neq v_0$, there is a *max-write*(v) that precedes mr in σ_r

1. Assume by way of contradiction that there is a *max-write*(v') w that precedes mr in σ_r s.t. $v' > v$. Denote t_w to be the linearization point of w in r, and note that $t_w < t_{mr}$. Now consider two case:

   (a) w performs line 6 at time t_w (line 6 is w's linearization point). In this case, w performs a successful $b.CAS(tmp, v')$ in line 6 for some tmp at some time $t' \leq t_w < t_{mr}$. Therefore, by Observations 1 and 2, $b.CAS(v_0, v_0)$ called by mr in line 8 at time t_{mr} returns $v'' \geq v' > v$. Hence, mr returns $v'' \geq v' > v$. A contradiction.

   (b) w performs line 3 at time t_w (line 3 is w's linearization point). Thus, t_w is the last time w calls $tmp \leftarrow b.CAS(v_0, v_0)$ in lime 3. Therefore, $b.CAS(v_0, v_0)$, called at time t_w returns a value bigger than or equal to v'. Therefore, by Observations 1 and 2, $b.CAS(v_0, v_0)$ called by mr in line 8 at time t_{mr} returns $v'' \geq v' > v$. Hence, mr returns $v'' \geq v' > v$. A contradiction.

2. Assume that $v \neq v_0$. By the code and by the CAS properties, there is a *max-write(v)* operation w that calls a successful $b.CAS(tmp, v)$ (line 6) for some tmp at time $t_w < t_{mr}$. Therefore, by Observations 1 and 2, the next call to $b.CAS(v_0, v_0)$ (in line 3) by w returns $tmp \geq v$, and thus w does not perform line 6 again. We get that t_w is the linearization point of w, and thus w precedes mr in σ_r.

□

Theorem 1. *Algorithm 1 emulates wait-free atomic max-register.*

Proof. Atomicity follows from Lemma 1. We left to show wait-freedom:

- **Max-read():** Since $b.CAS(exp, new)$ is wait free, Max-read() is wait-free.
- **Max-write(v):** Note that *Max-write*(v) returns in the iteration in which $b.CAS(v_0, v_0)$ in line 3 returns a value bigger than or equal to v. Therefore, by Observations 1 and 2, *Max-write*(v) returns in the following iteration after a successful $b.CAS(tmp, v)$ in lime 6. Now assume in a way of contradiction that *Max-write*(v) do not have a successful $b.CAS(tmp, v)$ in lime 6. By the code and Observations 1 and 2, if $b.CAS(tmp, v)$ in lime 6 do not succeed, then the following $b.CAS(v_0, v_0)$ in line 3 returns a bigger value that what it returned in the previous iteration. Now let v' be the value returned by the first $b.CAS(v_0, v_0)$ in line 3, and assume w.l.o.g. that there are k values bigger than v' and smaller than v in $\mathbb{V}$. Therefore, after at most k iteration *Max-write*(v) returns.

□

2.3 Resource Complexity of Write-Sequential k-register Emulation

In Section 2.3.1 we give an overview and intuition for our lower bound, and in Section 2.3.2 we prove it. In Section 2.3.3 we present a closely matching upper bound algorithm.

2.3.1 Lower bound overview

We prove that any f-tolerant emulation of an obstruction-free WS-Safe k-register from of a collection of MWMR atomic registers stored on a collection $\mathcal{S}$ of crash-prone servers has resource complexity of at least $kf + \left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil (f+1)$.

Our proof exploits the fact that the environment is allowed to prevent a pending low-level write from taking effect for arbitrarily long [5]. As a result, a client executing a high-level WRITE operation cannot reliably store the requested value in a base register that has a pending write as this write may take effect at a later time thus erasing the stored value. At the same time, the client cannot wait for all base registers on which it triggers low-level operations to respond, since up to f of them may reside on faulty servers. It therefore must be able to complete a high-level write without waiting for responses from up to f registers. Consequently, the next high-level write (by a different client) cannot reliably use these registers (as they might have outstanding low-level writes), and is therefore forced to use additional registers thus causing the total number of registers grow with each subsequent write.

In our main lemma (Lemma 2), we formalize this intuition as follows: Starting from a run r_0 consisting of an initial configuration, we build a sequence of consecutive extensions $r_1, \ldots, r_k$ so that r_i is obtained from r_{i-1} by having a new client invoke a high-level write W_i of some (not previously used) value. We then let the environment behave in an adversarial fashion (Definition 4) by blocking the responses from the writes triggered on at most f base registers as well as the prior pending low-level writes. In Lemma 4, we show that W_i must terminate without waiting for these responses to arrive. Furthermore, in Lemma 5, we show that W_i must invoke low-level writes on at least $2f+1$ base registers (residing on $\geq 2f+1$ different servers) that do not have any prior pending writes. This, combined with Lemma 4, implies that by the time W_i completes, there are at least f more registers on at least f servers with pending writes after W_i completes. Thus, by the time the kth high-level write completes, the total number of covered registers is at least kf (see Lemma 2(a)).

To obtain a stronger bound, our construction is parameterized by an *arbitrary* subset F of servers such that $|F| = f+1$. We show that the extra storage available on these servers cannot in fact, be utilized by an emulation (see Lemma 2(b)) forcing it to use at least kf registers on the remaining $\mathcal{S} \setminus F$ servers to accommodate the same number k of writers. We use this result in Theorem 2, to show that the number of base registers

required for the emulation is a function of k and $|\mathcal{S}|$.

2.3.2 Lower Bounds

For any time t (following the t^{th} action) in a run r of the emulation algorithm we define the following:

- *Covering write*: Let w be a low-level write triggered on a base register b at times $\leq t$. We will refer to w as *covering at* t, and to b as being *covered by* w *at* t if w is pending at t.
- $C(t) \subseteq \mathbb{C}$: the set of all clients that have completed a high-level write operation at times $\leq t$.
- $Cov(t) \subseteq \mathcal{B}$: the set of all base registers being covered by some low-level write at time t.

We first prove the following key lemma:

Lemma 2. *For all $k > 0$, $f > 0$, let A be an f-tolerant algorithm that emulates a WS-Safe obstruction-free k-register using a collection $\mathcal{S}$ of servers storing a collection $\mathcal{B}$ of wait-free MWMR atomic registers. Then, for every $F \subseteq S$ such that $|F| = f + 1$, there exist k failure-free runs r_i, $0 \leq i \leq k$, of A such that (1) r_0 is a run consisting of an initial configuration and $t_0 = 0$ steps, and (2) for all $i \in [k]$, r_i is a write-only sequential extension of r_{i-1} ending at time $t_i > 0$ that consists of i complete high-level writes of i distinct values $v_1, \ldots, v_i$ by i distinct clients $c_1, \ldots, c_i$ such that:*

a) $|Cov(t_i)| \geq if$ *b)* $\delta(Cov(t_i)) \cap F = \emptyset$

Fix arbitrary $k > 0$, $f > 0$, and a set F of servers such that $|F| = f + 1$. We proceed by induction on i, $0 \leq i \leq k$. **Base:** Trivially holds for the run r_0 of A consisting of $t_0 = 0$ steps. **Step:** Assume that r_{i-1} exists for all $i \in [k-1]$. We show how r_{i-1} can be extended up to time $t_i > t_{i-1}$ so that the lemma holds for the resulting run r_i. For the remainder of the Lemma 2's proof, we will assume without loss of generality that every low-level write operation is linearized simultaneously with its respond step. Formally:

Assumption 1 (Write Linearization)**.** *For every extension r of r_{i-1} and a base object $b \in \mathcal{B}$, let $L_{r|b}$ be a linearization of $r|b$. Then, $L_{r|b}$ does not include any low-level write operations in $pending(r|b)$, and for any two low-level writes $w_1, w_2 \in complete(r|b)$ such that $respond(w_1) \prec_{r|b} respond(w_2)$, $w_1 \prec_{L_{r|b}} w_2$.*

Note that the above implies that no low-level write w that is covering some register b at a time t in r will be observed by any low-level reads from b as having taken effect until after w's respond event occurs.

We proceed by introducing the following notation:

Definition 2. *Let r be an extension of r_{i-1}. For all times $t \geq t_{i-1}$ in r, let*

1. *$Tr_i(t)$: the set of all base registers which have a low-level write triggered on between t_{i-1} and t.*
2. *$Rr_i(t) \subseteq Tr_i(t)$ be the set of all base registers which had a low-level write triggered on and responded (took effect) between t_{i-1} and t.*
3. *$Cov_i(t) = Cov(t) \setminus Cov(t_{i-1})$ be the set of all base registers that have been newly covered between t_{i-1} and t. Note that $Cov_i(t) \subseteq Tr_i(t)$.*
4. *$Q_i(t) \subseteq \mathcal{S}$ be the set of all servers such that $Q_i(t) = \delta(Cov_i(t)) \setminus F$ if $|\delta(Cov_i(t)) \setminus F| \leq f$, and $Q_i(t) = Q_i(t-1)$, otherwise. Intuitively, $Q_i(t)$ is a set of at most f servers, each of which is not in F and has at least one newly covered registers (between t_{i-1} and t).*
5. *$F_i(t) \triangleq \{s \in F \mid \delta^{-1}(\{s\}) \cap Rr_i(t) \neq \emptyset\}$, i.e., $F_i(t)$ is the set of all servers in F having a register that responded to a low-level write invoked after t_{i-1}.*
6. *$M_i(t) \triangleq \delta(Cov_i(t)) \cap (F \setminus F_i(t))$, i.e., $M_i(t)$ is the set of all servers in F with at least one register covered by a low-level write invoked after t_{i-1} and without registers that have responded to the low-level writes invoked after t_{i-1}.*
7. *$G_i(t) \subseteq \mathcal{S}$ be the set of all servers such that $G_i(t) = M_i(t)$ if $|Q_i(t)| < |F_i(t)|$, and $G_i(t) = \emptyset$, otherwise.*

Below we will introduce the adversary Ad_i, which causes A to gradually increase the number of base registers covered after t_{i-1} by delaying the respond actions of some of the previously triggered low-level writes.

Definition 3 (Blocked Writes)**.** *Let r be an extension of r_{i-1}. For all times $t \geq t_{i-1}$ in r, let $BlockedWrites_i(t)$ be the set of all low-level covering writes w satisfying either one of the following two conditions:*

1. *w was triggered by a client in $C(t_{i-1})$, or*
2. *w was triggered on a base register in $\delta^{-1}(Q_i(t) \cup G_i(t))$.*

We say that a pending low-level write w is *blocked* in an extension r of r_{i-1} if there exists a time $t \geq t_{i-1}$ such that for all $t' > t$ in r, $op \in \text{BlockedWrites}_i(t')$. The following definition specifies the set of the environment behaviours that are allowed by Ad_i in all extensions of r_{i-1}:

Definition 4 (Ad_i)**.** *For an extension r of r_{i-1} we say that the environment* behaves like Ad_i *after r_{i-1} in r if the following holds:*

1. *There are no failures after time t_{i-1} in r.*

2. *For all $t \geq t_{i-1}$ in r, if a low-level write $w \in BlockedWrites_i(t)$, then w does not respond at t.*

3. *If r is infinite then:*

   (a) *Every pending low-level read or write that is not blocked in r eventually responds.*

   (b) *Every trigger or return action that is ready to be executed at a client c in r is eventually executed.*

For a finite extension r of r_{i-1}, we will write $\langle r, Ad_i \rangle$ to denote the set of all extensions of r in which the environment behaves like Ad_i after r_{i-1}; and we will write $\langle r, Ad_i, t \rangle$ to denote the subset of $\langle r, Ad_i \rangle$ consisting of all runs having exactly t steps. For $X \in \{Q_i, F_i, M_i\}$ and a run $r \in \langle r_{i-1}, Ad_i, t \rangle$, we say that X is *stable* after r if for all $t' \geq t$ for all extensions $r' \in \langle r, Ad_i, t' \rangle$, $X(t') = X(t)$.

The following lemma asserts several technical facts implied directly by Definitions 2 and 4.

Lemma 3. *For all $t \geq t_{i-1}$ and $r \in \langle r_{i-1}, Ad_i, t \rangle$, all of the following holds at time t in r:*

1. $Q_i(t) \subseteq \delta(Cov_i(t)) \setminus F$
2. $Q_i(t) \subseteq Q_i(t+1)$
3. $F_i(t) \subseteq F_i(t+1)$
4. $|F_i(t)| - |Q_i(t)| \leq 1$
5. $|Q_i(t)| \leq f$
6. $|F_i(t)| \leq f + 1$
7. $F_i(t) = F_i(t+1) \implies M_i(t) \subseteq M_i(t+1)$
8. $|M_i(t)| \leq f + 1$
9. $|\delta(Cov_i(t)) \setminus F| \geq f \implies |Q_i(t)| \geq f$
10. $|\delta(Cov_i(t)) \setminus F| < f \implies \delta(Rr_i(t)) \setminus F = \emptyset$
11. $(Q_i(t) \cup M_i(t)) \cap \delta^{-1}(Rr_i(t)) = \emptyset$

Proof. By induction on $t \geq t_{i-1}$.
Base: If $t = t_{i-1}$, then $Tr_i(t) = Rr_i(t) = Cov_i(t) = F_i(t) = \emptyset$. Furthermore, since $|\delta(Cov_i(t)) \setminus F| = 0 \leq 1 \leq f$, $Q_i(t) = \delta(Cov_i(t)) \setminus F$. Thus, all the claims hold.
Induction step: Suppose all the claims hold for all $t \geq t_{i-1}$, and consider the time $t+1$:
3.1: If $|\delta(Cov_i(t+1)) \setminus F| \leq f$, then by Definition 2.4, $Q_i(t+1) = \delta(Cov_i(t+1)) \setminus F$ as needed. Otherwise, $Q_i(t+1) = Q_i(t)$. Consider an arbitrary server $s \in Q_i(t+1)$, and towards a contradiction, suppose that $s \notin \delta(Cov_i(t+1)) \setminus F$. Since $s \in Q_i(t+1) = Q_i(t)$, by the induction hypothesis, $s \in \delta(Cov_i(t)) \wedge s \notin F$. Since $s \notin \delta(Cov_i(t+1)) \setminus F$, we get that either (1) $s \notin \delta(Cov_i(t+1))$ or (2) $s \in \delta(Cov_i(t+1)) \cap F$. Since $s \notin F$, (2) is false, and therefore, $s \notin \delta(Cov_i(t+1))$. Hence, there exists a base register in $\delta^{-1}(\{s\})$ that responded at t to a low-level write w triggered after t_{i-1}. Since $s \in Q_i(t)$, by Definition 3,

$w \in \text{BlockedWrites}(t)$. However, since the environment behaves like Ad_i after r_{i-1}, by Definition 4.2, w does not respond at t. A contradiction.

3.2: Towards a contradiction, suppose that there exists $s \in \mathcal{S}$ such that $s \in Q_i(t) \wedge s \notin Q_i(t+1)$. By Definition 2.4, $|\delta(Cov_i(t+1)) \setminus F| \leq f$ as otherwise, $Q_i(t+1) = Q_i(t)$ contradicting the assumption. Thus, $Q_i(t+1) = \delta(Cov_i(t+1)) \setminus F$, and therefore, either (1) $s \notin \delta(Cov_i(t+1))$ or (2) $s \in \delta(Cov_i(t+1)) \cap F$. By the induction hypothesis for 3.1, $s \in \delta(Cov_i(t)) \wedge s \notin F$. Hence, (2) is false, and it is only left to consider the case $s \notin \delta(Cov_i(t+1))$. Thus, $s \in \delta(Cov_i(t))$ and $s \notin \delta(Cov_i(t+1))$, which implies that there exists a base register in $\delta^{-1}(\{s\})$ that responded at time t to a low-level write w invoked after t_{i-1}. Since $s \in Q_i(t)$, by Definition 3, $w \in \text{BlockedWrites}(t)$. However, since the environment behaves like Ad_i after r_{i-1}, by Definition 4.2, w does not respond at t. A contradiction.

3.3: Let $s \in F_i(t)$. By Definition 2.5, there exists a base register $b \in \delta^{-1}(\{s\})$ that responded to a low-level write triggered on b after t_{i-1} at time t' such that $t_{i-1} < t' \leq t < t+1$. Since $t < t+1$, the b's response has also occurred before $t+1$, and therefore, $b \in Rr_i(t+1)$. Hence, $b \in (\delta^{-1}(S) \cap Rr_i(t+1)) = F_i(t+1)$ as needed.

3.4: Toward a contradiction, suppose that $|F_i(t+1)| - |Q_i(t+1)| > 1$. Since we already proved that $Q_i(t) \subseteq Q_i(t+1)$, $|Q_i(t)| \leq |Q_i(t+1)|$. In addition, we know that $||Q_i(t+1)| - |Q_i(t)|| \leq 1$, $||F_i(t+1)| - |F_i(t)|| \leq 1$, and by the induction hypothesis $|F_i(t)| - |Q_i(t)| \leq 1$. Thus, $|F_i(t+1)| - |Q_i(t+1)| > 1$ implies that (1) $|F_i(t)| - |Q_i(t)| = 1$ (i.e., $|F_i(t)| > |Q_i(t)|$), (2) $|Q_i(t+1)| = |Q_i(t)|$, and (3) $|F_i(t+1)| = |F_i(t)| + 1$. Since we already proved that $F_i(t+1) \supseteq F_i(t)$, (3) implies that there exists $s \in \mathcal{S}$ such that $s \in F_i(t+1) \setminus F_i(t)$. Since by Definition 2.5, $F_i(t+1) \subseteq F$, $s \in F$. Thus, $s \in F \setminus F_i(t)$. This means that either no low-level writes have been triggered on registers in $\delta^{-1}(\{s\})$ after t_{i-1}, or there is a register $b \in \delta^{-1}(\{s\})$ that responds to a low-level write triggered on b after t_{i-1}. In the first case, no register on s can respond at time t, and therefore, $s \notin F_i(t+1)$, which is a contradiction. In the second case, we obtain that b satisfies $b \in Cov_i(t)$, $b \in \delta^{-1}(\{s\})$, $s \in F \setminus F_i(t)$, and b responds at t to a covering write w triggered after t_{i-1}. Thus, by Definition 2.6, $s \in M_i(t)$ and since $|F_i(t)| > |Q_i(t)|$, by Definition 2.7, $s \in G_i(t)$. Thus, by Definition 3, $w \in \text{BlockedWrites}(t)$, and since the environment behaves like Ad_i after r_{i-1}, by Definition 4.2, w does not respond at t. A contradiction.

3.5: Assume by contradiction that $|Q_i(t+1)| > f$. Since by 3.1, $|Q_i(t+1)| \subseteq \delta(Cov_i(t+1)) \setminus F$, $|\delta(Cov_i(t+1)) \setminus F| > f$. By Definition 2.4, $Q_i(t+1) = Q_i(t)$, and therefore, $|Q_i(t)| > f$. A contradiction to the inductive assumption.

3.6: By Definition 2.5, $F_i(t+1) \subseteq F$. Since $|F| = f+1$, $|F_i(t+1)| \leq f+1$.

3.7: Suppose $F_i(t) = F_i(t+1)$. Consider $s \in M_i(t)$, and toward a contradiction, suppose that $s \notin M_i(t+1)$. Since by Definition 2.5, $F_i(t) \subseteq F$ and $F_i(t+1) \subseteq F$, $F \setminus F_i(t) = F \setminus F_i(t+1)$. This together with the fact that $s \in F \setminus F_i(t)$ implies that $s \in F \setminus F_i(t+1)$ as well. Thus, it must be the case that $s \in \delta(Cov_i(t)) \wedge s \notin \delta(Cov_i(t+1))$. Thus, by Definition 2.3, there exists a base register $b \in \delta^{-1}(\{s\})$ that responds to a low-

level write invoked after t_{i-1} at time t which implies that $b \in Rr_i(t+1)$. Hence, by Definition 5, $s \in F_i(t+1)$. However, since $s \in F \setminus F_i(t+1)$, $s \notin F_i(t+1)$. A contradiction.

3.8: Since F is fixed in advance and $|F| = f+1$, we receive $|M_i(t+1)| = |\delta(Cov_i(t+1)) \cap (F \setminus F_i(t+1))| \leq |F \setminus F_i(t+1)| \leq |F| = f+1$.

3.9: If $|\delta(Cov_i(t)) \setminus F| < f$ and $|\delta(Cov_i(t+1)) \setminus F| \geq f$, then there exists a base register on a server in $\mathcal{S} \setminus F$ that is newly covered after t. Thus, we get $|\delta(Cov_i(t)) \setminus F| = f-1$ and $|\delta(Cov_i(t+1)) \setminus F| = f$. By Definition 2.4, $Q_i(t+1) = \delta(Cov_i(t+1)) \setminus F$, and therefore, $|Q_i(t+1)| = f$. Otherwise, by the induction hypothesis, $|Q_i(t)| \geq f$. Since $|\delta(Cov_i(t+1)) \setminus F| \geq f$, we have that either (1) $|\delta(Cov_i(t+1)) \setminus F| = f$, or (2) $|\delta(Cov_i(t+1)) \setminus F| > f$. Applying Definition 2.4 to (1) and (2), we get the following: for (1), $Q_i(t+1) = \delta(Cov_i(t+1)) \setminus F$, which implies $|Q_i(t+1)| = |\delta(Cov_i(t+1)) \setminus F| = f$, and for (2), $Q_i(t+1) = Q_i(t)$, and therefore, $|Q_i(t+1)| \geq f$.

3.10: Toward a contradiction, suppose that $|\delta(Cov_i(t+1)) \setminus F| < f$ and $\delta(Rr_i(t+1)) \setminus F \neq \emptyset$. By the induction hypothesis, $|\delta(Cov_i(t)) \setminus F| < f \implies \delta(Rr_i(t)) \setminus F = \emptyset$. We first consider the case $|\delta(Cov_i(t)) \setminus F| \geq f$. Thus given $|\delta(Cov_i(t+1)) \setminus F| < f$, there exists a server in $\delta(Cov_i(t)) \setminus F$ such that some register on that server responds to a low-level write w that was triggered after t_{i-1}. Moreover, $|\delta(Cov_i(t)) \setminus F| = f$, and thus, by Definition 2.4, $Q_i(t) = \delta(Cov_i(t)) \setminus F$. Since the environment behaves like Ad_i after r_{i-1}, by Definition 3, $w \in \text{BlockedWrites}(t)$, and therefore, by Definition 4.2, w does not respond at t. A contradiction.

Thus, we know that $|\delta(Cov_i(t)) \setminus F| < f$ and $\delta(Rr_i(t)) \setminus F = \emptyset$. And since $\delta(Rr_i(t+1)) \setminus F \neq \emptyset$, there exists a server $s \in \delta(Cov_i(t)) \setminus F$ such that some object on s responded at t to a low-level write w triggered after t_{i-1}. Since $|\delta(Cov_i(t)) \setminus F| < f$, by Definition 2.4, $Q_i(t) = \delta(Cov_i(t)) \setminus F$. Thus, $s \in Q_i(t)$. However, by Definition 3, $w \in \text{BlockedWrites}(t)$, and since the environment behaves like Ad_i after t_{i-1}, by Definition 4.2, w does not respond at t. A contradiction.

3.11: Toward a contradiction, suppose that $(Q_i(t+1) \cup M_i(t+1)) \cap \delta(Rr_i(t+1)) \neq \emptyset$. We will consider the following two cases separately: (1) $Q_i(t+1) \cap \delta(Rr_i(t+1)) \neq \emptyset$, and (2) $M_i(t+1) \cap \delta(Rr_i(t+1)) \neq \emptyset$.

(1) Suppose $Q_i(t+1) \cap \delta(Rr_i(t+1)) \neq \emptyset$, and let $s \in Q_i(t+1) \cap \delta(Rr_i(t+1))$. If $s \in Q_i(t)$, then by the induction hypothesis $s \notin \delta(Rr_i(t))$. This means that either (a) $\delta^{-1}(\{s\}) \cap Tr(t) = \emptyset$, or (b) there exists a base register in $\delta^{-1}(\{s\})$ that responds to a low-level write w triggered after t_{i-1}. If (a) holds, then no base register can respond to a low-level write before $t+1$, and therefore, $s \notin \delta(Rr_i(t+1))$, which is a contradiction. If (b) is the case, then since $s \in Q_i(t)$, by Definition 3, $w \in \text{BlockedWrites}(t)$. Since the environment behaves like Ad_i after t_{i-1}, by Definition 4.2, w does not respond at t, which is also a contradiction.

If $s \notin Q_i(t)$, then since $s \in Q_i(t+1)$, by Definition 2.4, the only action that can follow t is a trigger of a low-level write on some register in $\delta^{-1}(\{s\})$. Since $s \in \delta(Rr_i(t+1))$, and the action executed at t is not a respond, $s \in \delta(Rr_i(t))$. Thus, $Q_i(t+1) \neq Q_i(t)$ which by Definition 2.4, implies that $Q_i(t+1) = \delta(Cov_i(t+1)) \setminus F$, and $|Q_i(t+1)| =$

$|\delta(Cov_i(t+1)) \setminus F| \leq f$. Since $Q_i(t) \subset Q_i(t+1)$, $|Q_i(t)| < f$, and by the induction hypothesis for 3.9, we have $|\delta(Cov_i(t+1)) \setminus F| < f$. Thus, by the induction hypothesis for 3.10, we conclude that no registers on servers in $\mathcal{S} \setminus F$ have responded to any low-level writes triggered between t_{i-1} and t. However, since $s \in \delta(Rr_i(t))$ and, by the induction hypothesis for 3.1, $s \notin \delta(Rr_i(t))$, $s \in \mathcal{S} \setminus F$ has a register that responded to a low-level write triggered after t_{i-1}. A contradiction.

(2) Suppose that $M_i(t+1) \cap \delta(Rr_i(t+1)) \neq \emptyset$, and let $s \in M_i(t+1) \cap \delta(Rr_i(t+1))$. By Definition 2.5, we know that $s \in F \setminus F_i(t+1)$, and therefore, $s \in F$ and $s \notin F_i(t+1)$. Thus, $s \in F \cap \delta(Rr_i(t+1))$, and therefore, by Definition 2.5, $s \in F_i(t+1)$. A contradiction. □

The following corollary follows immediately from the claims 2–3 and 5–8 of Lemma 3.

Corollary 1. *There exists a run $r \in \langle r_{i-1}, Ad_i \rangle$ such that F_i, Q_i, and M_i are all stable after r.*

We first show that r_{i-1} can be extended with a complete high-level write W_i by a new client c_i such that the environment behaves like Ad_i until W_i returns. Roughly, the reason for this is that Ad_i guarantees that after r_{i-1}, c_i would only miss responses from for the writes invoked on at most f servers (see Claim 1) as well as those that might have been invoked in r_{i-1} by other clients $\{c_1, \ldots, c_{i-1}\}$, which c_i is unaware of. As a result, the involved servers and clients would appear to c_i as faulty after r_{i-1}, and therefore, to ensure obstruction freedom, it must complete W_i without waiting for the outstanding writes to respond.

Lemma 4. *Let $r \in \langle r_{i-1}, Ad_i \rangle$ be a run consisting of r_{i-1} followed by a high-level write invocation W_i by client $c_i \notin C(t_{i-1})$. Then, there exists a run $r_r \in \langle r, Ad_i \rangle$ in which W_i returns.*

By Corollary 1, there exists an extension $r' \in \langle r, Ad_i, t' \rangle$ where $t' > t_{i-1}$ such that Q_i, F_i, and M_i are all stable after r'. If W_i returns in r', we are done. Otherwise, we will first bound the number of servers $|Q_i(t') \cup M_i(t')|$ controlled by Ad_i as follows:

Claim 1. *Consider a time $t > t_{i-1}$, and a run $r \in \langle r_{i-1}, Ad_i, t \rangle$. If M_i is stable after r, then $|Q_i(t) \cup M_i(t)| \leq f$.*

Proof. By Lemma 3.5, $|Q_i(t)| \leq f$. Thus, if $M_i(t) = \emptyset$, then $|Q_i(t) \cup M_i(t)| \leq f$ as needed. Otherwise ($M_i(t) \neq \emptyset$), we show that $|F_i(t)| = |Q_i(t)| + 1$. Suppose to the contrary that $|F_i(t)| \neq |Q_i(t)| + 1$. Since by Lemma 3.4, $|F_i(t)| \leq |Q_i(t)| + 1$, the only possibility for $|F_i(t)| \neq |Q_i(t)| + 1$ is if $|F_i(t)| \leq |Q_i(t)|$. Thus, by Definition 2.7, $G_i(t) = \emptyset$, and hence, by Definition 3, no writes on the registers in $\delta^{-1}(M_i(t))$ are blocked. However, since $M_i(t) \neq \emptyset$, at least one register in $\delta^{-1}(M_i(t))$ must have an outstanding write. Therefore, by Definition 4.3(a), there exists time t', and an extension $r' \in \langle r, Ad_i, t' \rangle$ such that one of the registers on some server $s \in M_i(t)$ responds at time t'. Thus, $s \in F_i(t')$, and therefore, $s \notin M_i(t')$. Hence, M_i is not stable after r. A contradiction to the assumption.

Since $F_i(t) \subseteq F$, $|F \setminus F_i(t)| + |F_i(t)| = |F| = f + 1$. Hence, $|F \setminus F_i(t)| = f + 1 - |F_i(t)| = f - |Q_i(t)|$. Since by Definition 2.6, $M_i(t) \subseteq (F \setminus F_i(t))$, $|M_i(t)| \leq |F \setminus F_i(t)| = f - |Q_i(t)|$. Thus, we receive $|Q_i(t) \cup M_i(t)| \leq |Q_i(t)| + |M_i(t)| \leq f$ as needed. □

We are now ready to complete the proof of Lemma 4:

Proof of Lemma 4. By Claim 1, $|Q_i(t') \cup M_i(t')| \leq f$, and by Lemma 3.11, no base registers on servers in $Q_i(t') \cup M_i(t')$ have ever responded to any low-level writes issued after t_{i-1}. Thus, there exists a finite run r'', which is identical to r' except all servers in $Q_i(t') \cup M_i(t')$ crash immediately after r and each client $c_1, \ldots, c_{i-1}$ fails before any of its covering writes on registers in $Cov(t_{i-1})$ responds. By f-tolerance and obstruction freedom, there exists a fair extension $r''\sigma$ of r'' (i.e., $r''\sigma \notin \langle r'', Ad_i \rangle$) such that W_i returns in $r''\sigma$. Since $Q_i \cup M_i$ is stable after r', the set of registers precluded by Ad_i from responding in r' is identical to that in r'', and by Assumption 1, no write with a missing response is linearized, r' is indistinguishable from r'' to c_i. Thus, $r'\sigma \in \langle r, Ad_i \rangle$, and since σ includes the return event of W_i, $r'\sigma$ satisfies the lemma.

□

We next show that in order to guarantee safety in the face of the environment behaving like Ad_i, W_i must trigger a low-level write on at least one non-covered register on each server in a set of $2f + 1$ servers. An illustration of the runs constructed in the proof appears in Figure 2.1.

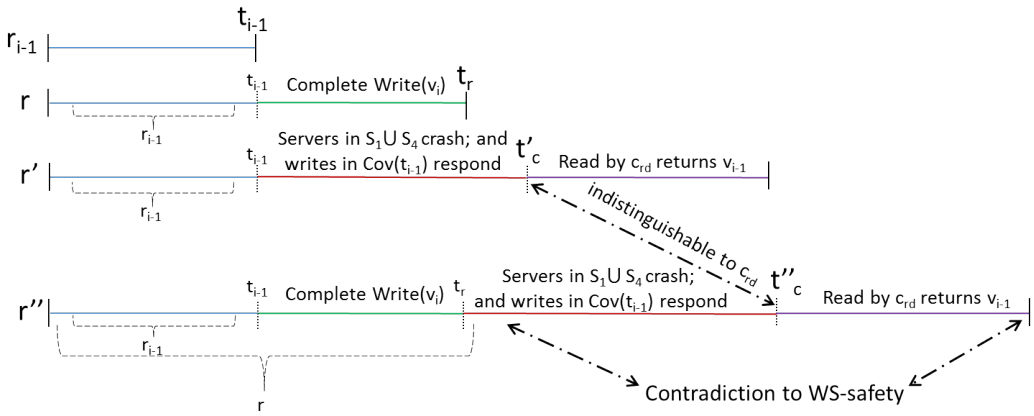


Figure 2.1: An illustration of the runs constructed in Lemma 5.

Lemma 5. *Consider a run $r \in \langle r_{i-1}, Ad_i, t_r \rangle$ where $t_r > t_{i-1}$, consisting of r_{i-1} followed by a complete high-level write invocation by client $c_i \notin C(t_{i-1})$ that returns at time t_r. Then, $|\delta(Tr_i(t_r) \setminus Cov(t_{i-1}))| > 2f$.*

Proof. Denote $X \triangleq \delta(Tr_i(t_r) \setminus Cov(t_{i-1}))$, and assume by contradiction that $|X| \leq 2f$. Let $S_1 = F_i(t_r)$, $S_2 = Q_i(t_r)$, $S_3 = X \cap (F \setminus F_i(t_r))$ and $S_4 = X \setminus (S_1 \cup S_2 \cup S_3)$. Note that S_1, S_2, S_3, S_4 are pairwise disjoint, and $X = S_1 \cup S_2 \cup S_3 \cup S_4$.

We first show that $|S_1 + S_4| \leq f$. By Lemma 3.6, $|S_1| \leq f + 1$. However, if $|S_1| = f + 1$, then by Lemma 3.4, $|S_1| - |S_2| = f + 1 - |S_2| \leq 1$, and therefore, $|S_1| + |S_2| \geq 2f + 1$ violating the assumption. Hence, $|S_1| \leq f$. By Lemma 3.5, $|S_2| \leq f$. If $|S_2| = f$, then by assumption, $|S_1 \cup S_3 \cup S_4| = |S_1| + |S_3| + |S_4| \leq f$, and therefore, $|S_1 + S_4| = |S_1| + |S_4| \leq f$. And if $|S_2| < f$, then by Definitions 2.4 and 4, $|S_4| = 0$. Hence, $|S_1 + S_4| = |S_1| + |S_4| \leq f$.

Now let r' be a fair extension of r_{i-1} consisting of t'_c steps in which r_{i-1} is followed by (1) the crash events of all servers in $S_1 \cup S_4$, and (2) the respond steps of all the covering writes in r_{i-1} and (and no other steps). Extend r' with an invocation of a high-level read operation R by client $c_{rd} \neq c_i$ at time t'_c. Since $|S_1 + S_4| \leq f$, by obstruction freedom and f-tolerance, there exists time $t_{rd} > t_{i-1}$ at which R returns. Since r' is write-sequential, by WS-Safety, R must return v_{i-1}.

Next, let r'' be an extension of r consisting of all steps in r up to the time t_r followed by (1) the crash events of all servers in the set $S_1 \cup S_4$, and (2) the respond steps of all covering writes in r_{i-1} (and no other steps). Let $t''_c > t_r$ be the number of steps in r''. By Assumption 1, the values that can be read from the base registers in $Cov(t_{i-1})$ at time t''_c in r'' are identical to those that can be read at time t'_c in r'. Furthermore, by definitions 2.5 and 4, low-level writes triggered on registers in $\delta^{-1}(S_2 \cup S_3)$ do not respond before t_r in r. Thus, by Assumption 1, the values that can be read from the base registers in $\delta^{-1}(S_2 \cup S_3)$ at time t'_c in r' are also the same as those that can be read at time t''_c in r''. Thus, all registers in non-faulty servers at time t'_c in r' will appear to the subsequent reads as having the same content as at the time t''_c in r''_c.

We now extend r'' by letting client c_{rd} invoke high-level read R at time t''_c. Since r' is indistinguishable from r'' to c_{rd}, and R has no concurrent high-level operations, we get that R returns v_{i-1} in r''. However, since W_i is the last complete write preceding R in r'', by WS-Safety, the R's return value must be $v_i \neq v_{i-1}$. A contradiction.

□

The following corollary follows immediately from Lemma 5, Definitions 2.4 and 4, and the choice of $|F| = f + 1$:

Corollary 2. *Consider a run $r \in \langle r_{i-1}, Ad_i, t_r \rangle$ where $t_r > t_{i-1}$, consisting of r_{i-1} followed by a complete high-level write invocation by client $c_i \notin C(t_{i-1})$ that returns at time t_r. Then, $|Q_i(t_r)| = f$.*

We are now ready to complete the proof of the induction step of Lemma 2:

Proof of the induction step (Lemma 2). By Lemma 4, there exists a run $r \in \langle r_{i-1}, Ad_i, t_r \rangle$, $t_r > t_{i-1}$, in which r_{i-1} is followed by a complete high-level write invocation W_i by client $c_i \neq c_{i-1}$ writing a value $v_i \neq v_{i-1}$ and returning at time t_r. By Corollary 2, $|Q_i(t_r)| = f$, and therefore, by Lemma 3.4, $|F_i(t_r)| = f + 1$. Since $F_i(t_r) \subseteq F$ and $|F| = f + 1$, we conclude that $F_i(t_r) = F$. Hence, by Definition 2.6, $M_i(t_r) = \emptyset$, which by Definition 2.7, implies that $G_i = \emptyset$. Thus, by Definition 3, no writes on the registers in $\delta^{-1}(F)$ triggered after t_{i-1} are blocked.

Hence, by Definition 4.3(a), there exists an extension $r' \in \langle r, Ad_i, t' \rangle$, for some $t' \geq t_r$, such that $\delta(Cov_i(t')) \cap F = \emptyset$. We now show that $r_i = r'$ and $t_i = t'$ satisfy the lemma. By the induction hypothesis and the construction of extension r', r' is a write-only failure-free sequential extension of r_{i-1} ending at time t' that consists of i complete high-level writes of values $v_1, \ldots, v_i$ by i distinct clients $c_1, \ldots, c_i$. It remains to show that the implications (a)–e) hold for $t_i = t'$:

a) $|Cov(t')| \geq if$: By the induction hypothesis $|Cov(t_{i-1})| \geq (i-1)f$, and by Definition 4, $Cov(t_{i-1}) \subseteq Cov(t')$. Therefore, we left to show that $|Cov(t') \setminus Cov(t_{i-1})| \geq f$. Since by Corollary 2, $|Q_i(t_r)| = f$, and by Lemma 3.2, $Q_i(t_r) \subseteq Q_i(t')$, we get $|Q_i(t')| = f$. By Definition 2.4, $|Cov_i(t')| \geq |Q_i(t')|$, and by Definition 2.3, $|Cov(t') \setminus Cov(t_{i-1})| = |Cov_i(t')|$. Therefore, we get $|Cov(t') \setminus Cov(t_{i-1})| \geq f$.

b) $\delta(Cov(t')) \cap F = \emptyset$: By the induction hypothesis we get that $\delta(Cov(t_{i-1})) \cap F = \emptyset$, and by construction of r' we get that $\delta(Cov_i(t')) \cap F = \emptyset$. By Definition 2.3, $Cov(t') = Cov_i(t') \cup Cov(t_{i-1})$. Therefore, $\delta(Cov(t')) \cap F = \emptyset$.

□

Resource Complexity. We will now use Lemma 2 to characterize the minimum resource complexity of the algorithms implementing an f-tolerant obstruction-free WS-Safe k-register as a function of the number $|\mathcal{S}|$ of available servers. First, it is easy to see that if $|\mathcal{S}| \leq 2f$, then no such algorithm can exist. This result is implied by an extended statement of Lemma 2 (see Theorem 12 in Appendix **??**), and can also be shown directly by a straightforward application of a partitioning argument as discussed in [55, 14]. If $|\mathcal{S}| > 2f$, then we have the following:

Theorem 2. *For all $k > 0$, $f > 0$, let A be an f-tolerant algorithm emulating an obstruction-free WS-Safe k-register using a collection $\mathcal{S}$ of servers such that $|\mathcal{S}| \geq 2f + 1$. Then, A uses at least $kf + \left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil (f+1)$ base registers (i.e., $|\delta^{-1}(\mathcal{S})| \geq kf + \left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil (f+1)$).*

Proof. Let $G \subseteq \mathcal{S}$ be the set consisting of all servers that store at least $\left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil$ base registers (i.e., $\forall s \in G$, $|\delta^{-1}(\{s\})| \geq \left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil$ and $\forall s \in \mathcal{S} \setminus G$, $|\delta^{-1}(\{s\})| < \left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil$). We first show that $|G| \geq f + 1$. Suppose toward a contradiction that $|G| < f + 1$, and pick a set F, such that $|F| = f + 1$ and $\mathcal{S} \supset F \supset G$. By Lemma 2.a)-b), there exists a run r of A consisting of t steps such that $|Cov(t)| \geq kf$ and $\delta(Cov(t)) \cap F = \emptyset$. Thus, by the pigeonhole argument, and since $|S \setminus F| = |S| - (f + 1)$, there is at least one server in $\mathcal{S} \setminus F$ that stores at least $\left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil$. Therefore, since $F \supset G$ and the number of objects stored on a server is an integer, we get that there is at least one server in $\mathcal{S} \setminus G$ that stores at least $\left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil$ base registers. A contradiction.

We get that $|\delta^{-1}(G)| \geq \left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil (f+1)$. Now, again by Lemma 2.a)-b), there exists a run r' of A consisting of t' steps such that $|Cov(t')| \geq kf$ and $\delta(Cov(t')) \cap G = \emptyset$,

meaning that $|\delta^{-1}(\mathcal{S} \setminus G)| \geq kf$. Therefore, we get that $|\delta^{-1}(\mathcal{S})| \geq kf + \left\lceil \frac{kf}{|\mathcal{S}|-(f+1)} \right\rceil (f + 1)$. □

The following bound on the number of registers required to emulate a single (i.e., non-fault-tolerant) max-register is a direct consequence of Theorem 2:

Theorem 3 (Resource Complexity of k-max-register)**.** *For all $k > 0$, any algorithm implementing a wait-free k-writer max-register from a collection of wait-free MWMR atomic base registers uses at least k base registers.*

Proof. Suppose to the contrary that there exists an algorithm A implementing a k-writer max-register using $\ell < k$ base MWMR wait-free atomic registers. Consider a fault-prone shared memory system consisting of $n = 2f + 1$ servers each of which stores ℓ MWMR wait-free atomic registers. Run n copies $A_1, \ldots, A_n$ of A, one on each server, to obtain $n = 2f + 1$ copies of k-writer max-register. Run a generic protocol of [64] to obtain an f-tolerant emulation $\mathcal{A}$ of a wait-free k-writer regular register. By assumption, the resource complexity of $\mathcal{A}$ is $(2f + 1)\ell < (2f + 1)k$ base registers. However, by Theorem 2, for $n = 2f + 1$, it must be at least $kf + k(f + 1) = (2f + 1)k$. A contradiction. □

In Appendix .1, we prove an extended statement of Lemma 2, and use it to show three additional lower bounds as discussed above.

2.3.3 Upper Bound

In this section we present an f-tolerant construction emulating a *wait-free WS-Regular* k-register for all combinations of values of the parameters $k > 0$, $f > 0$, and n where $n > 2f$. Our construction is carefully crafted to deal with the adversarial behaviour (Definition 4) that was exploited in the proof of Lemma 2 while minimizing the resource complexity. Similarly to multi-writer ABD [41, 64, 57], our algorithm uses *read* and *write quorums* to read from and write to registers. However, since RMW objects are replaced with read/write registers, and covering low-level writes belonging to old WRITEs can overwrite registers at any time, the quorums in our case must have a larger intersection.

Let $z \triangleq \left\lfloor \frac{n-(f+1)}{f} \right\rfloor$ and $y \triangleq zf + f + 1$, we construct a collection $\mathcal{R}$ of $m = \left\lfloor \frac{k}{z} \right\rfloor$ disjoint sets $R_0, \ldots, R_{m-1}$, each of which consist of y registers, and if k/z is not an integer, then we add to $\mathcal{R}$ another disjoint set R_m of $(k - \left\lfloor \frac{k}{z} \right\rfloor z)f + f + 1$ registers. Intuitively, z is the maximum number of writers that can be supported by a single set of y registers as can be deduced from Lemma 2's argument. If z divides k, then exactly k/z such sets are needed to accommodate the total of k writers. Otherwise, the remaining k mod z writers are moved to an overflow set R_m. Note that for all $R_i \in \mathcal{R}$, $2f + 1 \leq |R_i| \leq n$. Then, we distribute the registers in each set R_i on servers in $\mathcal{S}$ so that every register in R_i is mapped to a different server (i.e., $|\delta(R_i)| = |R_i|$). Figure 2.2 demonstrates a possible mapping from registers to servers. All in all, we use $\Sigma_{R_i \in \mathcal{R}} |R_i| = \left\lfloor \frac{k}{z} \right\rfloor y + (k - \left\lfloor \frac{k}{z} \right\rfloor z)f + (f + 1)(\left\lceil \frac{k}{z} \right\rceil - \left\lfloor \frac{k}{z} \right\rfloor) = \cdots = kf + \left\lfloor \frac{k}{z} \right\rfloor (f + 1) = kf + \left\lceil \frac{k}{\left\lfloor \frac{n-(f+1)}{f} \right\rfloor} \right\rceil (f + 1)$ registers.

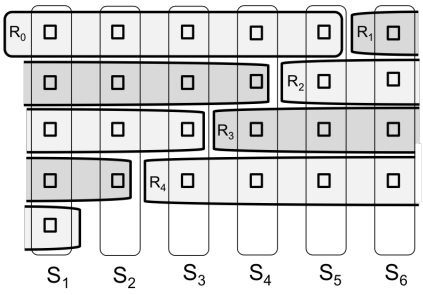


Figure 2.2: A possible mapping from $\mathcal{R}$ to $\mathcal{S}$ in case $n = 6, k = 5$, and $f = 2$.

The resulting layout is then used to derive the *read* and *write quorums* as follows: for every set $R_i \in \mathcal{R}$, any subset of R_i of size $|R_i| - f$ is a write quorum for all writers c_j such that $j = \lfloor \frac{i}{z} \rfloor$; and any subset of registers consisting of *all* registers mapped to $n - f$ servers is a read quorum (i.e., the set of the read quorums is $\{B \subseteq \mathcal{B} : \exists S \in \mathcal{S} \text{ s.t. } |S| = 2f + 1 \wedge \delta^{-1}(S) = B\}$). Observe that by construction of $\mathcal{R}$, for every set $R_i \in \mathcal{R}$, (1) the number of clients mapped to write quorums in R_i is $\lfloor \frac{|R_i|-(f+1)}{f} \rfloor = \frac{|R_i|-(f+1)}{f}$, and (2) any write quorum in R_i intersects with any read quorum on at least $|R_i| - f$ registers. Therefore, in a write-sequential run, the latest written value is always guaranteed to be available to subsequent READs provided every writer c executing a high-level WRITE W leaves no more than f pending low-level writes upon W's completion. To enforce the latter, c is precluded from triggering new low-level writes on registers on which it still has writes pending from preceding high-level WRITEs invocations. In addition, since the registers in every (read or write) quorum are mapped to exactly $n - f$ servers, each quorum access is guaranteed to terminate, and thus, the algorithm is wait-free. The algorithm's pseudo-code appear in Algorithm 2.

Algorithm 2 $\forall f > 0\ \forall k > 0,\ \forall n = |\mathcal{S}| \geq 2f + 1$.

Types:

$TSVal = \mathbb{N} \times \mathbb{V}$, with selectors *ts* and *val*.

$States = TSVal \times 2^{TSVal} \times 2^{\mathcal{B}} \times 2^{\mathcal{B}}$ with selectors *tsVal*, *rdSet*, *wrSet* and *coverSet*.

Base Objects and Servers:

$\forall b \in \delta^{-1}(\mathcal{S}), b \in TSVal$, initially, $\langle 0, v_0 \rangle$.

Let $z \triangleq \lfloor \frac{n-(f+1)}{f} \rfloor$, $y \triangleq zf + f + 1$, and $m \triangleq \lceil \frac{k}{z} \rceil$.

$R = \{R_0, \ldots, R_{m-1}\} \subset 2^{\delta^{-1}(\mathcal{S})}$ s.t.

1. $\forall i \in \{0, \ldots, m-2\}$, $|R_i| = y$. If $\lceil \frac{k}{z} \rceil = \lfloor \frac{k}{z} \rfloor$, then $|R_{m-1}| = y$. Else, $|R_{m-1}| = (k - \lfloor \frac{k}{z} \rfloor z)f + f + 1$.
2. $\forall R_i, R_j \in R$, $R_1 \cap R_j = \emptyset$.
3. $\forall R_i \in R$, $|\delta(R_i)| = |R_i|$.

Clients states:

$\forall i \in [k]$, $State_i \in States$, initially,
$\langle \langle \langle 0, 1 \rangle, v_0 \rangle, \emptyset, R_j, \emptyset \rangle$, where $j = \lfloor \frac{i}{z} \rfloor$.

Code for client c_i, $1 \leq i \leq k$:

```
1:  operation WRITE(v)
2:      value ← collect()
3:      State_i.tsVal.val ← v
4:      State_i.tsVal.ts ← value.ts + 1
5:      j ← ⌊i/z⌋
        ▷ do not handle responds between lines 6 to 10
6:      State_i.coverSet ← R_j \ State_i.wrSet
7:      State_i.wrSet ← ∅
8:      || for each b ∈ R_j
9:          if b ∉ State_i.coverSet
10:             trigger b.write(State_i.tsVal)
11:     wait until |State_i.wrSet| ≥ |R_j| − f
12:     return ack

13: scan(s)
14:     for each b ∈ δ^{-1}(s) do
15:         trigger b.read()
16:         wait for the matching response

17: operation READ()
18:     value ← collect()
19:     return value.val

20: collect()
21:     State_i.rdSet ← ∅
22:     || for each s ∈ S do
23:         scan(s)
24:     wait for n − f scans to complete
25:     ts ← max({ts' | ⟨ts', ∗⟩ ∈ State_i.rdSet})
26:     return ⟨v, ts'⟩ ∈ State_i.rdSet : ts' = ts

27: upon receiving b.read() respond res do
28:     State_i.rdSet ← State_i.rdSet ∪ {res}

29: upon receiving b.write(∗) respond do
30:     if b ∈ State_i.coverSet then
31:         State_i.coverSet ← State_i.coverSet \ {b}
32:         trigger b.write(State_i.tsVal)
33:     else
34:         State_i.wrSet ← State_i.wrSet ∪ {b}
```

The registers store values in $\mathbb{V}$ each of which is associated with a unique timestamp. (Note that since safety is required only in write-sequential runs, we do not need to break ties with clients' ids.) To write a value v to the emulated register, a client c_i first accesses a read quorum (via collect() in lines 22–26) and selects a new timestamp *ts* which is higher than any other timestamp that has been returned. It then proceeds to trigger low-level writes of $\langle ts, v \rangle$ on registers in $R_j = \lfloor \frac{i}{z} \rfloor$, so as to ensure that (1) $\langle ts, v \rangle$ is stored in a write quorum *wq* (lines 8–11), and (2) no more than f registers in R_j are left covered by c_i's

writes (the current and the previous operations). The latter is achieved by preventing c_i from triggering a new low-level write on every register that has not yet responded to the previously triggered one (lines 9–10). To read a value, a client simply reads all registers in a read quorum, via collect(), and returns the value having the highest timestamp.

Correctness proof

The space complexity of the algorithm is $\Sigma_{R_i \in R}|R_i| = \lfloor \frac{k}{z} \rfloor y + (k - \lfloor \frac{k}{z} \rfloor z)f + (f+1)(\lceil \frac{k}{z} \rceil - \lfloor \frac{k}{z} \rfloor) = \cdots = kf + \lfloor \frac{k}{z} \rfloor (f+1) = kf + \lceil \frac{k}{\lfloor \frac{x-(f+1)}{f} \rfloor} \rceil$ registers. Below, we prove that the algorithm satisfies wait-freedom and write-sequential regularity. The following observation follows from code and the construction of the sets in R; (1) writers never trigger low-level writes on base object with pending low-lever writes from previous WRITEs, (2) writers wait for $n - f$ base objects to reply (line 24), and for every set $R_i \in R$, the number of client that write to registers in R_i is $\lfloor \frac{|R_i|-(f+1)}{f} \rfloor = \frac{|R_i|-(f+1)}{f}$.

Observation 3. *For every $0 < i \leq k$ for every time t in a run r, if writer c_i have no pending* WRITE *at t then it covers at most f base objects at time t.*

Lemma 6. *Consider a write-sequential run r of the algorithm, and consider two sequential* WRITEs *W_i, W_j in r s.t. W_i precedes W_j. Then W_j's value is associated with a bigger timestamp than W_i's value.*

Proof. Since W_i precedes W_j, W_j starts the *collect* in line 2 after W_i returns. W_i triggers low level writes with its value and timestamp on base objects in R_l ($l = \lfloor \frac{i}{z} \rfloor$) that are not covered by its previous WRITEs, and waits for $|R_l| - f$ low level writes to respond (line 11) before it returns. Thus, since $|\delta(R_l)| = |R_l|$, W_j starts its *collect* after W_i writes its timestamp to at least $|R_l| - f$ base objects in different servers, none of which is covered by low-level write of W_i previous WRITEs.

Moreover, since the number of writers excluding W_i that write to base objects in R_j is $\lfloor \frac{|R_l|-(f+1)}{f} \rfloor - 1 = \frac{|R_i|-(f+1)}{f} - 1$, readers do not write, and each writer covers at most f base objects, we get that at least $f+1$ servers has a base object that stores W_i's timestamp when W_j begins its *collect*. Now since *collect* reads all base object in at least $n - f$ servers (line 24), W_j sees W_i's timestamp and picks a bigger one (line 4).

□

Corollary 3. *Consider a write-sequential run r of the algorithm. If* WRITE *W_i precedes* WRITE *W_j, then W_j is associated with a bigger timestamp than W_i.*

Lemma 7. *Consider a write-sequential run r of the algorithm, and a read operation rd and a* WRITE *W in r. Let ts be the timestamp associated with W. If W precedes rd, than rd returns a value associated with timestamp $ts' \geq ts$.*

Proof. Let t be the time when W returns, and assume w.l.o.g that W is performed by client c_i s.t. $\lfloor \frac{i}{z} \rfloor = j$. Before W returns it c_i triggers low level writes with its value and

timestamp on base objects in R_j that are not covered by its previous WRITEs, and waits for $|R_j| - f$ low level writes to respond. The number of clients excluding c_i that trigger low-level writes on base objects is R_j is $\lfloor \frac{|R_l|-(f+1)}{f} \rfloor - 1 = \frac{|R_i|-(f+1)}{f} - 1$, and by Observation 3, each client covers at most f base objects at time t. By Corollary 3 and since readers do not write, every low level write in r that is triggered after time t is associated with a bigger timestamp than ts. Therefore, since $|\delta(R_l)| = |R_l|$, there is a set of $f+1$ base objects, each of which mapped to a different server, s.t. at any time $t' \geq t$ the timestamp each of them stores is bigger than or equal to ts.

Since W precedes rd, rd starts the *collect* after time t. And since *collect* reads all base object in at least $n - f$ servers, rd sees at least one value associated with timestamp bigger than or equal to ts, and thus, returns a value associated with timestamp $ts' \geq ts$.

□

Definition 5. *For every write-sequential run r, for every* read *rd in r that returns a value associated with timestamp ts we define the sequential run $\sigma_{r_{rd}}$ as follows: All the completed write operations in r are ordered in $\sigma_{r_{rd}}$ by their timestamp, and rd is added after the* WRITE *operation that is associated with ts.*

In order to show that the algorithm simulates a write-sequential regular register we need to proof that for every write-sequential run r, for every *read* rd, $\sigma_{r_{rd}}$ preserves the real time order of r and the sequential specification. Note that the sequential specification is satisfied by construction, and we prove the real time order in the next lemma.

Lemma 8. *For every write-sequential run r, for every complete read rd that returns a value associated with timestamp ts in r, $\sigma_{r_{rd}}$ preserves r's operation precedence relation (real time order).*

Proof. By Corollary 3, the real time order of r between every two WRITE operations is preserved in $\sigma_{r_{rd}}$. We left to show that the real time order of r between rd and any WRITE W in $\sigma_{r_{rd}}$ is preserved. Consider two cases:

- W precedes rd in r. By Lemma 7, W is associated with a timestamp smaller than or equal to ts, and thus, by construction of $\sigma_{r_{rd}}$ the real time order between rd and W is preserved.
- rd precedes W in r. Let W_{ts} be the WRITE operation associated with timestamp ts. Since rd returns a value associated with timestamp ts, W_{ts} starts before rd completes, and since r is write-sequential, w_{ts} precedes W in r. Thus, by lemma 6, W is associated with bigger timestamp than ts. Therefore, by construction of $\sigma_{r_{rd}}$ the real time order between rd and W is preserved.

□

Corollary 4. *For every write-sequential run r, for every complete read rd in r, there is a linearization of rd and all the* WRITE *operations in r.*

Theorem 4. *For all $k > 0$, $f > 0$, and $n > 2f$, there exists an f-tolerant algorithm emulating a wait-free WS-Regular k-register using a collection of n servers storing $kf + \lceil \frac{k}{z} \rceil (f+1)$ wait-free z-writer/multi-reader atomic base registers where $z = \lfloor \frac{n-(f+1)}{f} \rfloor$.*

Proof. By Corollary 4, the code in Algorithm 2 satisfies WS-regularity. Now notice that in both WRITE and READ operations clients never wait for more than $n - f$ servers to respond, and thus, wait-freedom follows. We conclude that Algorithm 2 satisfies the theorem.

□

2.4 Discussion and Future Directions

We introduced a new hierarchy, which classifies object types by the number of base objects of a given type required to emulate an f-tolerant register, as a function of the number of writers k and the number of available servers n. Interestingly, our hierarchy can be used to derive resource complexity bounds in the standard shared memory model (i.e., without object failures) as evidenced by our proof of a lower bound on the number of registers required for implementing a max-register for k writers. Our main technical contribution comprises the lower bound of $\left\lceil \frac{k}{\frac{n-(f+1)}{f}} \right\rceil (f+1) + kf$ and the upper bound of $\left\lceil \frac{k}{\lfloor \frac{n-(f+1)}{f} \rfloor} \right\rceil (f+1) + kf$ on the resource complexity of emulating an f-tolerant k-writer register from n fault-prone servers storing read/write registers. To strengthen our lower bound, it was proved for emulations satisfying weak liveness and safety properties.

Future directions. First, for some choices of k and n, our bounds leave a small gap that can be closed. Second, an interesting question that arises is whether our lower bound is tight for stronger properties. In the special case of $n = 2f + 1$ servers, emulation with stronger regularity [65] is possible with $(2f + 1)k$ registers (tight to our lower bound). However, the question of the general case ($n \geq 2f + 1$) remains open. In addition, since atomicity usually requires readers to write, it is interesting to investigate whether the space complexity (assuming read/write registers) in this case also linearly depends on the number of readers.

Another possible direction is to extend the hierarchy with more types (e.g., multiple assignment), and to also consider the time complexity of the emulations. For example, we showed that although a max-register can be implemented from a single CAS, the time complexity of the implementation is high. An interesting open question is to determine whether this tradeoff is inherent.

Chapter 3

Space Bounds for Reliable Storage: Fundamental Limits of Coding

In recent years we have seen an exponential increase in storage capacity demands, creating a need for big data storage solutions. In this era, distributed storage plays a key role. Data is typically stored on a collection of nodes accessed asynchronously by clients over a network. By storing redundant information, data remains available following failures. The most common approach to achieve this is via replication [10]; in asynchronous settings, $2f + 1$ replicas are needed in order to tolerate f failures [10]. Given the immense size of data, the storage cost of replication is significant. Previous works have attempted to mitigate this cost via the use of erasure codes [6, 21, 42, 23, 77, 30].

Indeed, codes can reduce the storage cost as long as data is not accessed concurrently by multiple clients. For example, if the data size is D bits and a single failure needs to be tolerated, erasure-coded storage ideally requires $(k+2)D/k$ bits for some parameter $k > 1$ instead of the $3D$ bits needed for replication. But as concurrency grows, the cost of erasure-coded storage grows with it: when c clients access the storage concurrently, existing asynchronous code-based algorithms [21, 42, 23, 30] store $O(cD)$ bits in storage nodes or communication channels. Intuitively, this occurs because coded data cannot be reconstructed from a single storage node. Therefore, writing coded data requires coordination – old values cannot be deleted before ensuring that sufficiently many blocks of the new value are in place. This is in contrast to replication, where written values can always be read coherently from a single copy, and so old values may be safely overwritten without coordination.

In this Chapter we prove that this extra cost is inherent: Given three problem parameters: f, c, and D, where f is the number of storage node failures tolerated (client failures are unrestricted), c is the concurrency allowed by the algorithm, and D is the data size,

we prove that the storage complexity is $\Theta(min(f, c) \cdot D)$. Asymptotically, this means either a storage cost as high as that of replication, or as high as keeping as many versions of the data as the concurrency level.

Lower bound Our results are proven for emulations of a lock-free multi-reader multi-writer regular register [52, 65]; see Section 3.1 for definitions. Interestingly, the lower bound does not hold for the weaker safe register semantics; in Section 3.5 we present a simple storage-efficient wait-free algorithm that ensures safe semantics, but not regularity. we consider algorithms that use (arbitrary) *black-box* encoding schemes, i.e., produce and manipulate code blocks of a given value independently of other values and meta-data; as formalized in Section 3.1.2. The storage consists of such code blocks, in addition to possibly unbounded data-independent meta-data, (e.g., timestamps), which we do not count as part of the storage cost. Our black-box assumption excludes storage-reduction techniques like de-duplication, which do require data-dependent meta-data. In Section 3.2 we survey how this assumption holds in related work on popular storage algorithms [6, 21, 42, 23, 30, 43], and compare it with assumptions made in proving other lower bounds [22]. Yet, the question whether there is a more storage-efficient algorithm that circumvents our result by taking stored values into consideration remains open; see further discussion in Section 3.6.

We prove the bound in Section 3.3: we first use a fundamental pigeonhole argument to show that as long as no ongoing write operation contributes code blocks consisting of D or more bits to the storage, no write operation can complete. We then define a parameter $0 < \ell \leq D$. For a given ℓ, we devise a particular adversary behavior, which we prove drives the storage to a state where either (1) $f + 1$ storage nodes hold at least ℓ bits each, or (2) the storage holds more than $D - \ell + 1$ bits in distinct code blocks for each of c different operations. Now, picking $\ell = D/2$ implies our lower bound.

Algorithm To prove our bound tight, we present in Section 3.4 a reliable storage algorithm whose storage cost is $O(min(f, c) \cdot D)$. We achieve this by combining the advantages of replication and erasure coding. Our algorithm does not assume any a priori bound on concurrency; rather, it uses erasure codes when concurrency is low and adaptively switches to replication when it is high.

3.1 Model

3.1.1 Preliminaries

We consider an asynchronous fault-prone shared memory system [4, 1, 48] consisting of set $B = \{bo_i, \ldots, bo_n\}$ of n base objects (typically residing at distinct storage nodes) supporting arbitrary atomic *read-modify-write* (RMW) access by clients from some infinite set Π (see Figure 3.1). Any f out of n base objects and any number of clients may fail by crashing, for some predefined $f < n/2$.

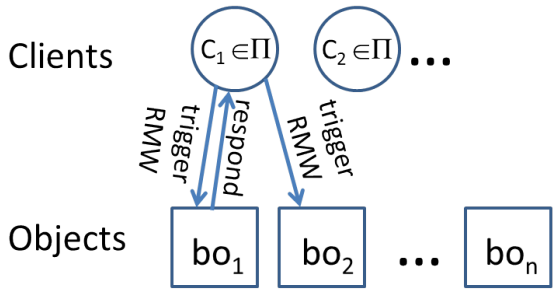


Figure 3.1: Clients and base objects.

An *algorithm* defines the behavior of clients as deterministic state machines, where state transitions are associated with actions such as RMW trigger/response. A *configuration* is a mapping to states from system components, i.e., clients and base objects. An *initial configuration* is one where all components are in their initial states.

A *run* of algorithm A is a (finite or infinite) alternating sequence of configurations and actions, beginning with some initial configuration, such that configuration transitions occur according to A. For a run r, *trace(r)* is the subsequence of r consisting of all the operation invocation and returns in r. We use the notion of time t during a run r to refer to the configuration reached after the t^{th} action in r. A *run fragment* is a contiguous subsequence of a run starting and ending with a configuration. We assume that runs are *well-formed*, in that each client's first action is an invocation, and a client has at most one outstanding operation at any time.

We say that a base object or client is *faulty* in a run r if it fails any time in r, and otherwise, it is *correct*. A run is *fair* if (1) for every RMW triggered by a correct client on a correct base object, there is eventually a matching response, (2) every correct client gets infinitely many opportunities to trigger RMWs.

We study algorithms that emulate a shared *register* [52], which stores a value v from some domain $\mathbb{V}$, where $D = \log_2 |\mathbb{V}|$. Initially, the register holds some initial value $v_0 \in \mathbb{V}$. Clients interact with the emulated register via high-level *read* and *write operations*. A client that performs a write operation is called a *writer*, and a client performing a read is a *reader*.

To distinguish the high-level emulated operations from low-level base object access, we refer to the latter as *RMWs*. We say that RMWs are *triggered* and *respond*, whereas operations are *invoked* and *return*. A (high-level) operation is emulated via a series of trigger and respond *actions* on base objects, starting with the operation's invocation and ending with its return. In the course of an operation, a client triggers RMWs separately on each $bo_i \in B$. The state of each $bo_i \in B$ changes atomically, according to the RMW triggered on it, at some point after the time when the RMW is triggered but no later than the time when the matching response occurs. To distinguish incomplete invocations to the emulated register from incomplete RMWs triggered on base objects, we refer to the former as *outstanding* operations and to the latter as *pending* RMWs.

A parameter c defines the write concurrency level, that is, at most c write operations are outstanding at a given time. We next define the safety and liveness properties we use in this chapter.

Liveness There is a range of possible liveness conditions, which need to be satisfied in fair runs. A *wait-free* object is one that guarantees that every correct client's operation completes, regardless of the actions of other clients. A *lock-free* object guarantees progress: if at some point in a run there is an outstanding operation of a correct client, then *some* operation eventually completes. An *FW-terminating* [1] register is one that has wait-free *write* operations, and in addition, if there are finitely many *write* invocations in a run, then every *read* operation completes.

Safety In order to define regularity, we first introduce some terminology: Operation op_i *precedes* operation op_j in a run r, denoted $op_i \prec_r op_j$, if op_i's return occurs before op_j's invoke in r. Operations op_i and op_j are *concurrent* in a run r if neither one precedes the other. A run with no concurrent operations is *sequential*. Two runs are *equivalent* if every client performs the same sequence of operations in both, where operations that are outstanding in one can either be included in or excluded from the other. A *linearization* of a run r is an equivalent sequential run that preservers r's operation precedence relation and the object's sequential specification. The sequential specification for a register is as follows: A read returns the latest written value, or v_0 if none was written. A *write* w in a run r is *relevant* to a *read* rd in r [65] if $rd \not\prec_r w$; *rel-writes*(r, rd) is the set of all *writes* in r that are relevant to rd.

Following Lamport [52], we consider a hierarchy of safety notions. Lamport [52] defines *regular* and *safe* single-writer registers. Shao et al. [65] extend Lamport's notion of regularity to MWMR registers, and give four possible definitions. Here we use two of them. The first is the weakest definition, and we use it in our lower bound proof. The second, which we use for our algorithm, is the strongest definition that is satisfied by ABD [10] in case readers do not change the storage (no *write-back*):

A MWMR register is *weakly regular*, (called *MWRegWeak* in [65]), if for every run r and *read* rd that returns in r, there exists a linearization of the subsequence of r consisting of

rd and the writes in *r*. A MWMR register is *strongly regular*, (called *MWRegWO* in [65]), if it satisfies weak regularity and the following condition: For all *reads* rd_1 and rd_2 that return in *r*, for all writes w_1 and w_2 in *rel-writes*$(r, rd_1) \cap$ *rel-writes*(r, rd_2), it holds that $w_1 \prec_{L_{rd_1}} w_2$ if and only if $w_1 \prec_{L_{rd_2}} w_2$.

We extend the safe register definition and say that a MWMR register is *strongly safe* if there exists a linearization σ_w of the subsequence of *r* consisting of the *write* operations in *r*, and for every *read* operation *rd* that has no concurrent *writes* in *r*, it is possible to add *rd* at some point in σ_w so as to obtain a linearization of the subsequence of *r* consisting of the write operations in *r* and *rd*.

3.1.2 Storage algorithm model and assumptions

We first give a formal model for coded storage algorithms, then define the notion of storage cost in this model, and finally state our assumptions that the encoding is symmetric and algorithms use it as a black-box.

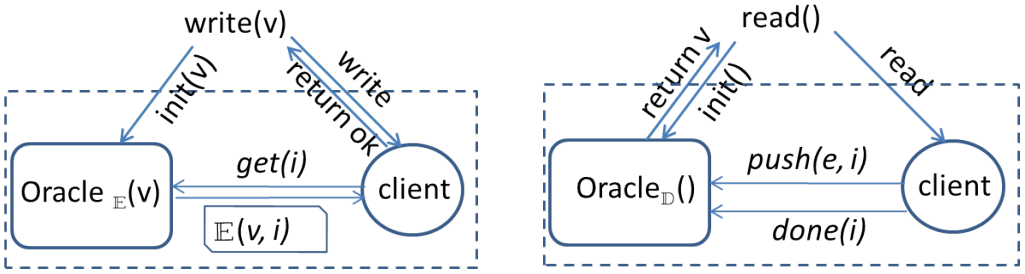
We consider algorithms that use (arbitrary) encoding schemes, which produce code blocks in some domain $\mathcal{E}$, so that each value is coded independently of other values. The coding scheme is based on two functions: The encoding function $\mathbb{E} : \mathbb{V} \times \mathbb{N} \rightarrow \mathcal{E}$ maps value/natural number pairs to code blocks. We denote the number of bits in block $e \in \mathcal{E}$ as $|e|$. The decoding function $\mathbb{D} : 2^{\mathcal{E}} \rightarrow \mathbb{V} \cup \{\perp\}$ takes as a parameter a set of code blocks and returns a value in $\mathbb{V}$, or $\perp$ in case no value can be decoded. For example, in a replication approach, each block *e* can be a full value *v*, so $\mathbb{D}(\{e\})$ simply returns *v*. Another example is *k-of-n* erasure codes, where for any value *v* and any subset *S* of size *k* of the set $\{e_i \mid e_i = E(v, i),\ 1 \leq i \leq n\}$, $\mathbb{D}(S) = v$. We capture rateless codes [62], in which an encoder can generate a limit-less sequence of blocks, by using $\mathbb{N}$ as the domain for block numbers.

We encapsulate the encoder and decoder into two oracles, $oracle_{\mathbb{E}}$ and $oracle_{\mathbb{D}}$ as illustrated in Figure 3.2. The interaction with these oracles is as follows:

Definition 6 (Encoding/Decoding Oracles)**.** *A* w=write(v) (read()*) invocation at a client* c_k *initializes an* $oracle_{\mathbb{E}}(c_k, w)$ *(*$oracle_{\mathbb{D}}(c_k, w)$*, resp.), which expires when w completes.* $oracle_{\mathbb{E}}(c_k, w)$ *exposes a* $get(i)$ *operation, which returns* $\mathbb{E}(v, i)$ *for* $i \in \mathbb{N}$*; and* $oracle_{\mathbb{D}}(c_k, w)$ *exposes two operations,* $push(e, i)$ *and* $done(i)$*, such that for all* $i \in \mathbb{N}$*, if* c_i *calls* $done(i)$*, then its read operation completes and returns* $\mathbb{D}(\{e \mid push(e, i)$ *previously occurred*$\})$*. We omit the parameters* c_k, w *when they are clear from the context.*

Writers produce code blocks via $oracle_{\mathbb{E}}$ and store them in the storage, whereas readers try to obtain enough blocks to decode legal values via $oracle_{\mathbb{D}}$. In addition to code blocks, clients and base objects can store unbounded meta-data, e.g., program counters and timestamps. But to avoid trivializing the problem, the meta-data must be data-independent, as formally defined below.

Information is represented as list of code blocks and meta-data, $\langle e_1, e_2, \ldots, e_k; m \rangle$, where $\forall i,\ e_i \in \mathcal{E}$ and the meta-data *m* is from some arbitrary domain. The *state of a*



(a) A writer and its oracle. (b) Run r_v

Figure 3.2: A model for code-based storage. Encoding and decoding are captured by oracles.

client that has an outstanding operation consists of the information stored at the client as well the parameters of its pending RMWs that have not yet taken effect. The state of a client with no outstanding operation is empty. A *base object's state* consists of the information stored at the base object and all the responses of pending RMWs that took effect on it. For a base object bo_i (client c_i), we denote the list of code blocks in bo_i's (c_i's) state at time t in run r as $bo_i^r(t)$ (resp. $c_i^r(t)$).

Let $\mathcal{S}$ be an ordered set including all base objects and clients, i.e., $B \cup \Pi$ ordered in some arbitrary way. For $S = \{bo_1, \ldots, bo_k, c_1, \ldots\} \subseteq \mathcal{S}$, $S^r(t)$ is the list of lists $bo_1^r(t), \ldots, bo_k^r(t), c_1^r(t), \ldots$ sorted according to their order in $\mathcal{S}$. A *block instance* $b \in S^r(t)$ is a triple $\langle i, j, e \rangle$ so that e is stored in the j^{th} position in the i^{th} list in S. We refer to the block contents as $b.e$.

Storage cost We count the number of bits stored in blocks in base objects as well as in clients, and neglect meta-data size. Note that oracle states are not counted as part of the storage cost, since we wish to measure the additional space required for making the data available for shared access, beyond its (trivial) existence at its sources and readers.

Definition 7 (Storage Cost)**.** *The* storage cost at time t in a run r *is* $\Sigma_{b \in \mathcal{S}^r(t)} |b.e|$. *The* storage cost of an algorithm *A is the maximum storage cost at any point t in any run r of A.*

Assumptions To make sure that the encoding does not leak information using block sizes, we assume *symmetry*, in the sense that output block sizes do not depend on input values. (Otherwise, we could for example, represent three values 0, 1, and 10 using a single coded block e_1 of size at most 1 bit by having $|e_1| = 0$ encode 10). Formally:

Definition 8 (Symmetric Encoding)**.** *An encoding function* $\mathbb{E}$ *is* symmetric *if for every* $v, v' \in \mathbb{V}$ *and for all* $i \in \mathbb{N}$, $|\mathbb{E}(v, i)| = |\mathbb{E}(v', i)|$. *We denote size*$(i) \triangleq |\mathbb{E}(v, i)|$.

Note that different block numbers (of all values) may have different sizes.

(a) Run r

(b) Run r_v

Figure 3.3: Black-box coding. Runs r and r_v have the same trace except that write w is invoked with u in r and with v in r_v; and each base object bo_i's state (blocks and metadata) is identical at all times in both runs, except that blocks produced by w's oracle in r are replaced in r_v by the corresponding blocks of v.

We next state our assumption that the storage treats the coding as a black-box. First, we define the notion of a source function, which we shall use to prohibit generation of code blocks by any source other than $oracle_{\mathbb{E}}$:

Definition 9 (Source Function)**.** *A function is a* source function *for a run r if it maps every (b, t) s.t. $b \in \mathcal{S}^r(t)$ to a pair $\langle w, i \rangle$ s.t. $b.e$ was returned by $get(i)$ in $oracle_{\mathbb{E}}(w)$.*

We use a source function to trace blocks in the storage to operations that produced them. To capture the restriction that the algorithm's decision what to store does not rely on block contents, we stipulate that we can replace the value written by a write operation w in a run r by an arbitrary value v, yielding the same sequence of states and actions, except that all stored block instances whose source is $\langle w, i \rangle$ are replaced with $\mathbb{E}(v, i)$. For clarity, we refer to the operation as w in both runs (see Figure 3.3).

Definition 10 (Black-Box Coding)**.** *An algorithm A is* black-box coding *if for every run r there is a source function $source^r$ s.t. for every $w = write(u)$ operation in r, $\forall u \in \mathbb{V}$, there is run r_v satisfying the following:*

1. *r_v has the same sequences of invocations and returns as r except that $w = write(v)$ (possibly with no change) and return values of read operations may be different; and*
2. *client and base object states at every time t in r_v are the same as at time t in r except that the contents of every $b \in \mathcal{S}^r(t)$ s.t. $source^r(b, t) = \langle w, i \rangle$ for some i is replaced by $e' = \mathbb{E}(v, i)$.*

In the following, we will only consider source functions satisfying Definition 10. In case multiple such source functions for r exist, we fix an arbitrary one and refer to it as $source^r$.

3.2 Related work

Our model captures numerous existing distributed storage algorithms, including ones that use replication [10], and erasure codes [6, 21, 42, 23, 30, 43]. We note that some

of them report a storage cost below $O(cD)$ [6, 21, 77, 30]. This is sometimes achieved by assuming periods of synchrony [6]. Other works shift the cost from storage nodes to the network and keep unbounded information in channels [30, 21]. However, since we define parameters and responses of pending RMWs to be part of clients' and base objects' states, information in channels is counted in our storage cost model and hence these algorithms are subject to our bound. The only non-black-box storage algorithm we are aware of is [77], where multiple values are encoded jointly, saving space, but also forfeiting regular register semantics. It is as of now unclear whether lifting the black-box assumption suffices in order to circumvent our result.

In [68] we showed a special case of the result in this chapter for infinite concurrency. Cadambe et al. [22] prove closely related lower bounds for coded storage algorithms. First, they show that asynchronous fault-tolerant storage algorithms require strictly more storage than synchronous erasure-coded algorithms. Second, similarly to this work, they extend the result given in In [68] to show that the storage cost must grow linearly with $\min(f, c)$, but their result is proven under a different set of assumptions than ours. In particular, while both papers make certain "black box" assumptions about the storage, [22] does not rule out joint coding as we do, but instead restricts protocol actions in a way that forbids them from depending on a written value in more than one communication round; this affords the protocol more freedom than our model in one communication round, and less freedom in all other rounds. On the face of it, the two sets of assumptions appear to be incomparable, though they both achieve the same end result, as we discuss in Section 3.6 below. Additionally, our bound allows algorithms to use unbounded (data-independent) meta-data and is proven for lock-free register emulations, whereas the bound in [22] includes meta-data and is shown for wait-free registers.

In Chapter 2 we showed that the number of fault-prone read/write registers needed to emulate a reliable multi-writer register grows linearly with the number of clients that can write to the register (even in sequential runs). Here, on the other hand, we consider storage nodes supporting fully general read-modify-write, for which that lower bound does not apply.

The challenge of providing a lower bound on stored data when meta-data is potentially unbounded was also previously addressed in the context of byzantine storage [25]. That paper has shown that certain storage algorithms cannot be "amnesic", i.e., cannot "forget" values written to them. Like our black-box assumption, the notion of amnesia was defined in terms of runs. However, it did not yield explicit bound on storage cost.

3.3 Storage Lower Bound

We now show a lower bound of $O(min(f, c) \cdot D)$ bits on the storage cost of any lock-free algorithm that uses symmetric black-box coding to simulate a weakly regular register:

Theorem 5. *Consider a lock-free algorithm A that uses symmetric black-box coding to simulate a weakly regular register. The storage cost of A is $\Omega(min(f, c) \cdot D)$.*

For the sake of our proof, we quantify the number of bits in blocks contributed by client c_i's operation w to base objects and clients other than c_i.

Definition 11. *Let $S \subset \mathcal{S}$, and consider a time t and an operation w by client c_j in a run r. We define $S^r(t,w) \triangleq \{i \in \mathbb{N} \mid \exists b \in (S \setminus \{c_j\})^r(t)$: $source^r(b,t) = \langle w, i\rangle\}$, and $||S^r(t,w)|| \triangleq \Sigma_{i \in S^r(t,w)} size(i)$.*

For $I \subseteq \mathbb{N}$, we say that two values $v' \neq v''$ in $\mathbb{V}$ are *I-colliding* if $\forall i \in I$, $\mathbb{E}(v', i) = \mathbb{E}(v'', i)$. We next use the pigeonhole argument and the assumption of symmetric black-box coding in order to show that write operations cannot return until some write stores enough bits in different blocks in every set of $n - f$ base objects.

Claim 2. *Let w be a write operation invoked in a run r of A, and t be a point in r. Consider a set of values $U \subset \mathbb{V}$, $|U| < 2^{D-1}$, and a set of base objects $S \subset \mathcal{S}$. If $||S^r(t,w)|| < D$, then there are two $S^r(t,w)$-colliding values $u \neq u'$ in $\mathbb{V} \setminus U$.*

Proof. Since $|\mathbb{V} \setminus U| > 2^{D-1}$ and $||S^r(t,w)|| < D$, the claim follows from the pigeonhole argument.

□

Lemma 9. *Consider a run r of algorithm A that begins with the invocation of c concurrent write operations. Let S be a set of at least $n - f$ base objects and assume that at every time t in r for every operation w in r, $||S^r(t,w)|| < D$. Then no write operation returns in r.*

Proof. Let $W_{ops} = \{w_1, \ldots, w_c\}$ be the set of c concurrent writes invoked in r. Assume by contradiction that there exists a complete write in W_{ops}. Let w be the first such write, and t be the time when it returns. Next we inductively build a sequence of sets of values $U_0, U_1, \ldots, U_c$, where $|U_i| = i$:

- $U_0 = \{\}$
- $\forall i \in \{0, \ldots, c-1\}$, we use U_i to build U_{i+1}. By the lemma premise, $||S^r(t, w_{i+1})|| < D$. Now since $|U_i| < c < 2^{D-1}$, by Claim 2, there are two $S^r(t, w_{i+1})$-*colliding* values $u_{w_{i+1}} \neq u'_{w_{i+1}}$ in $\mathbb{V} \setminus U_i$. We let $U_{i+1} = U_i \cup \{u_{w_{i+1}}\}$.

The set U_c contains exactly c (different) values s.t. for every operation $w_i \in W_{ops}$ there is a value $u_{w_i} \in U_c$ that has a $S^r(t, w_i)$-*colliding* value $u'_{w_i} \in \mathbb{V}$. By applying Definition 10 (c times), there is a run r' that begins with the invocation of c concurrent write operations, in which every operation $w_i \in W_{ops}$ writes u_{w_i} s.t. w returns at time t, and for every operation $w_i \in W_{ops}$, $S^r(t, w_i) = S^{r'}(t, w_i)$. Next, let clients with outstanding operations and all base objects in $B \setminus S$ fail at time t in r' (note that by assumption $|S| \geq n - f$, so $|B \setminus S| \leq f$), and let some client c_j invoke a solo read operation at time $t + 1$. By lock-freedom, c_j's read operation completes, and by regularity, it returns a value $u \in U_c$ at some time $t' > t$.

Let w' be the operation that writes u in r'. Since u has a $S^r(t, w')$-*colliding* value u' and since $S^r(t, w') = S^{r'}(t, w')$, u and u' are $S^{r'}(t, w')$-*colliding*. By Definition 10, there is

a run r'' with the same operations as in r' except that w' writes u' (instead of u) s.t. every client's and base object's state at time t in r' is identical to its state at time t in r'' (note that clients with outstanding operations and all base objects in $B \setminus S$ fail at time t) except that for every block instance $b \in \mathcal{S}^{r'}(t)$ s.t. $source^{r'}(b,t) = \langle w', i \rangle$, $b.e$ is replaced with a block $\mathbb{E}(u', i)$. In particular, states of base objects in S at time t are identical to their states at time t in r' except that for every block instance $b \in S^{r'}(t)$ s.t. $source^{r'}(b,t) = \langle w', i \rangle$, $b.e$ is replaced with a block $\mathbb{E}(u', i)$.

Now since u and u' are $S_{r'}(t, w')$-*colliding*, states of base objects in S at time t in r'' are identical to their states at time t in r'. In addition, since clients with outstanding operations and all base objects in $B \setminus S$ fail at time t, the solo reader c_j cannot distinguish between r' and r'', and thus, it pushes the same blocks to its oracle and calls *done* with the same number in r'' as in r', and therefore, its read operation returns u at time t'' in run r''. However, since the clients invoke write operations with different values in r', u is not written in r''. A contradiction to weak regularity.

□

Having shown a condition under which write operations cannot complete, we define an (unfair) adversary behavior that takes advantage of this in order to prevent progress. We introduce some notation, and then use it in order to define the adversary. We define a parameter $0 < \ell \leq D$, and for any time t in a run r of algorithm A we define the following sets, as illustrated in Figure 5.2. For convenience, from now on we omit the superscript r.

- $C(t)$: the set of outstanding write operations at time t.
- $C^-_\ell(t) = \{w \in C(t) \mid ||\mathcal{S}(t,w)|| \leq D - \ell\}$: The set of write operations each of which has at most $D - \ell$ bits in blocks, produced by its oracle with different numbers, in the storage (excluding the client performing it) at time t.
- $C^+_\ell(t) = C(t) \setminus C^-_\ell(t)$.
- $F_\ell(t) = \{bo_i \in B \mid \Sigma_{b \in \{bo_i\}(t)} |b.e| \geq \ell\}$. Base objects that store blocks that consist (together) of more than ℓ bits at time t. These are base objects that we will "freeze" in our counter-example because they are already "full", i.e., consume enough space for our lower bound.

We fix the parameter ℓ throughout the proof and omit subscript ℓ from the notation. The next observation on storage cost immediately follows from the definitions.

Observation 4. *At any point t in every run r of A, the storage cost is at least $|C^+(t)|(D - \ell + 1)$.*

We next define a particular adversary behavior that schedules actions in a way that prevents progress. Note that the adversary controls the scheduling of client actions and RMW responses.

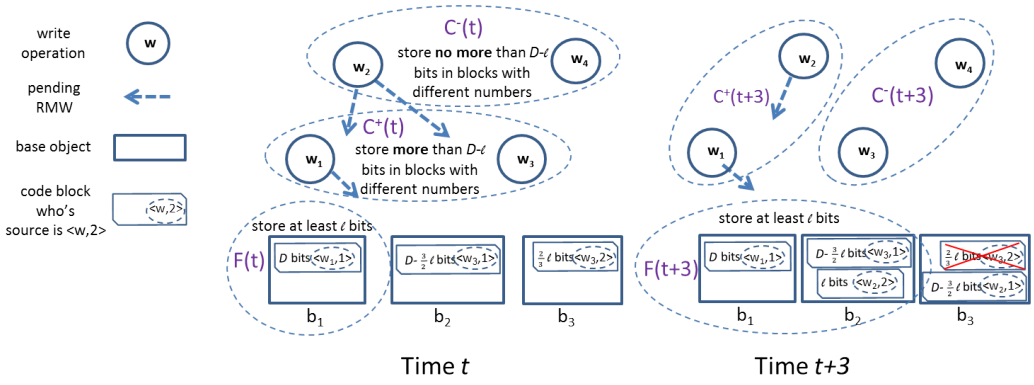


Figure 3.4: Example scenario in run of a storage algorithm with adversary Ad. In this example, $2D/5 < \ell < D$. At time t, only w_2 and w_4 are in $C^-(t)$, where w_4 has no pending RMWs and w_2 has one triggered RMW on $b_1 \in F(t)$ and one triggered RMW on $b_3 \notin F(t)$. Therefore, by the first rule, Ad schedules the response on the RMW triggered by w_2 on b_3. In this example w_2 overwrites w_3's block in b_3, thus w_3 moves from C^+ to C^-. Then, at time $t+1$, no response can be scheduled by rule 1 (no operation in $C^-(t+1)$ has a pending RMW on a base object in $N \setminus F(t+1)$), so by rule 2, Ad chooses w_2 and lets it trigger an RMW on base object b_2. Now since w_2 is the only operation that has a pending RMW on a base object not in $F(t+2)$, Ad schedules the response on the RMW triggered by w_2 on b_2 at time $t+2$. In this example w_2 adds a block with ℓ bits to b_2. Thus, c_2 is included in $C^+(t+3)$. In addition, b_2 stores more than ℓ bits at time $t+3$, so it belongs to $F(t+3)$.

Definition 12. *(Ad) At any time t, Ad schedules an action as follows:*

1. *If there is a pending RMW on a base object in $B \setminus F(t)$ by a client performing an operation in $C^-(t)$, then choose the longest pending of these RMWs, allow it to take effect on the corresponding base object, and schedule its response.*

2. *Else, choose in a fair order an operation by a client $c_i \in \Pi$ and schedule its action (trigger RMW, call its oracle, get response from its oracle, or return), without allowing it to affect the base object yet. By fair order we mean any order in which every client is chosen infinitely often (e.g., $c_1, c_1, c_2, c_1, c_2, c_3 \ldots$).*

In other words, Ad delays RMWs triggered by operations in $C^+(t)$ (for which the storage already holds $D - \ell$ bits) as well as RMWs on "frozen" base objects in $F(t)$ (which store at least ℓ bits), and fairly schedules all other actions. We demonstrate Ad's behavior in Figure 5.2. Though this behavior may be unfair, in every infinite run of Ad, every correct client gets infinitely many opportunities to take steps. We use Ad to build an unfair run with no progress (no write returns), and then build an indistinguishable fair run to contradict lock-freedom. The following observation immediately follows from the adversary's freezing of base objects in F.

Observation 5. *Assume run r of algorithm A in which the environment behaves like Ad. For each base object bo, if $bo \in F(t)$ at some time t, then $bo \in F(t')$ for all $t' > t$ in r.*

Another consequence of Ad's behavior is captured by the following:

Lemma 10. *Consider a run r of algorithm A. If the adversary behaves like Ad, then for every time t and for every write operation w in r, $||(\mathcal{S} \setminus F(t))(t, w)|| < D$.*

Proof. Assume by way of contradiction that there is time t and write operation w performed by client c_j s.t. $||(\mathcal{S} \setminus F(t))(t, w)|| \geq D$. The definition of $(\mathcal{S} \setminus F(t))(t, w)$ takes into account only blocks returned by w's oracle that are stored outside of $c_j(t)$. Thus, w triggered at least one RMW that has a matching response before time t in r. Let $t' \leq t$ be the time when the last RMW triggered by w responded, and denote this RMW by rmw and the base object on which rmw was triggered by bo. By Ad, $w \in C^-(t' - 1)$, and therefore, by definition, $||(\mathcal{S} \setminus F(t' - 1))(t' - 1, w)|| \leq D - \ell$. Now consider two cases:

- First, rmw adds blocks (possibly overwriting other blocks) with less than ℓ bits to bo. In this case, since bo is the only storage component that changed at time t', $|(\mathcal{S} \setminus F(t'))(t', w)| < D$.
- Second, rmw adds blocks (possibly overwriting other blocks) with at least ℓ bits to bo. In this case, $bo \in F(t')$. Now since $||(\mathcal{S} \setminus F(t' - 1))(t' - 1, w)|| \leq D$, by Observation 5, $F(t' - 1) \subseteq F(t')$, and given $bo \in F(t')$ and it is the only storage component that changed at time t', we get $||(\mathcal{S} \setminus F(t'))(t', w)|| \leq ||(\mathcal{S} \setminus F(t' - 1))(t' - 1, w)|| < D$.

So far we showed that $||(\mathcal{S} \setminus F(t'))(t', w)|| < D$. By Observation 5, and since no RMW by w takes effect after time t', $(\mathcal{S} \setminus F(t''))(t'', w) \subseteq (\mathcal{S} \setminus F(t'))(t', w), \forall t'' \geq t'$. Therefore, we get $||(\mathcal{S} \setminus F(t))(t, w)|| < D$. A contradiction.

□

The next corollary uses Lemmas 9 and 10 in order to conclude that Ad can prevent progress of write operations.

Corollary 5. *Consider a run r of algorithm A that begins with the invocation of c concurrent write operations. If the adversary behaves like Ad and $|F(t)| \leq f$ for all t in r, then no write operation returns in r.*

Proof. By Lemma 10, for every time t for every write operation w in r, $||(\mathcal{S} \setminus F(t))(t, w)|| < D$. And since $B \subset \mathcal{S}$, for every time t for every write operation w in r, $||(B \setminus F(t))(t, w)|| < D$. Now since $|F(t)| \leq f$ for every time t in r, $|B \setminus F(t)| \geq n - f$. Therefore, by Lemma 9, no write operation returns in r.

□

We have shown that Ad can prevent completion of write operations in algorithms that store ℓ bits in less than $f + 1$ base objects. However, this does not directly imply a storage bound, since Ad is not fair. In the next lemma we close this gap by showing a fair run in which lock-freedom must be satisfied , i.e., operations invoked by correct clients must eventually complete, in order to blow up the storage. We show that for every algorithm,

we can build a run where at some point the algorithm either stores ℓ bits in each of $f+1$ base objects (namely, $\exists t : |F(t)| > f$), or there are c concurrent operations each of which adds at least $D - \ell + 1$ bits to the storage cost (i.e., $|C^+(t)| = c$).

Lemma 11. *There is a run r of A and a time t in r when $|C^+(t)| = c$ or $|F(t)| > f$.*

Proof. Assume by way of contradiction that there is no such run of algorithm A. We first build a run r of A with c clients that concurrently write different values, in which the environment behaves like adversary Ad. By the contradiction assumption, $|C^+(t)| < c$ and $|F(t)| \leq f$ for all t in r. We start with the invocation of c concurrent write operations, and allow the run to proceed indefinitely according to Ad. We say that a client c, which performs write operation w, is *stuck* in r if there is a time t in r s.t. for all $t' \geq t$, $w \in C^+(t')$ (and so no RMWs triggered by c take effect after time t). By Observation 5 and the assumption that $|F(t)| \leq f$ for all t, there is a time t_1 in r s.t. for every time $t_2 \geq t_1$, $F(t_1) = F(t_2)$.

Now we build a run r' that is identical to r but every base object $bo \in F(t_1)$ fails at time t_1 ($|F(t_1)| \leq f$), and every stuck client fails after its last RMW takes effect. Since by Ad, RMWs do not take effect on base objects in $F(t_1)$ after time t_1, runs r and r' are indistinguishable to all correct clients and base objects. Now notice that by Ad's behavior, each correct client in r' gets infinitely many opportunities to trigger RMWs. In addition, since (1) for every correct client c_i in r' there are infinitely many times t when $c_i \in C^-(t)$, (2) Ad picks responses from base objects not in $F(t)$ in the order they are triggered, and (3) there are no correct base objects in $F(t')$ for all $t' > t_1$, every RMW triggered by a correct client on a correct base object has a matching response in r'. Therefore, run r' is fair.

By the contradiction assumption $|C^+(t)| < c$ for all t in r. Therefore, there is at least one client that is not stuck in r, and thus, there is at least one client that is correct in r'. Hence, by lock-freedom, some client eventually completes its write operation in r'. Now since r and r' are indistinguishable to all clients that are correct in both, the same is true in r. However, by Corollary 5, no write operation completes in r. A contradiction.

□

So far we have shown that every algorithm has a run where at some point either ℓ bits are stored in $f+1$ base objects, or there are c concurrent operations each of which adds at least $D - \ell + 1$ bits to the storage cost. We now combine this result with Observation 4 to conclude our lower bound:

Proof (Theorem 5). Let $\ell = D/2$. By Lemma 11, there is a run r of A and a time t in r when $|C^+(t)| = c$ or $|F(t)| > f$. If $|F(t)| > f$, then the storage cost at time t in r is $(f+1)\ell = (f+1)D/2 = \Omega(fD)$. Otherwise, $|C^+(t)| = c$, and so by Observation 4, the storage cost at time t in r is at least $c(D - \ell) = cD/2 = \Omega(cD)$. The theorem follows.

□

By picking $\ell = D$, we get a second conclusion from Lemma 11 and Observation 4. The following corollary proves that any coding scheme short of full replication must exhibit storage growth linear in the concurrency.

Corollary 6. *The storage cost of any algorithm that uses a black-box coding scheme to simulate a weakly regular lock-free register, and does not store D bits (enough to represent a full replica) in $f+1$ base objects, grows linearly with the concurrency.*

3.4 Adaptive Regular Register

We present here a storage algorithm that combines full replication with erasure coding in order to achieve the advantages of both.

3.4.1 Algorithm

erasure codes. A k-of-n erasure code takes a value from $\mathbb{V}$ and produces a set S of n blocks from $\mathcal{E}$ s.t. the value can be restored from any subset of S that contains no less than k different blocks. We assume that the size of each block is D/k. $Oracle_{\mathbb{E}}$ and $Oracle_{\mathbb{D}}$ are encapsulated by two functions *encode* and *decode*, respectively: *encode* gets a value $v \in \mathbb{V}$ and returns a set of n ordered elements $W = \{\langle e_1, 1\rangle, \ldots, \langle e_n, n\rangle\}$, where $e_1, \ldots, e_n \in \mathcal{E}$, and *decode* gets a set $W' \subset \mathcal{E} \times \mathbb{N}$ and returns $v' \in \mathbb{V}$ s.t. if $|W'| \geq k$ and $W' \subseteq W$, then $v = v'$. We use $k = n - 2f$. Note that when $k = 1$, we get full replication.

The main idea behind our algorithm is to have base objects store blocks from at most k different *writes*, and then turn to store full replicas. Our algorithm satisfies strong regularity and FW-termination. In the next section we prove the following:

Theorem 6. *There is an FW-terminating algorithm that simulates a stringly regular register, whose storage cost is $min((c+1)(2f+k)D/k, (2f+k)2D)$ bits. Moreover, in a run with a finite number of writes, if all the writers are correct, the storage is eventually reduced to $(2f+k)D/k$ bits.*

Notice that k is a parameter of the algorithm, and if we pick $k = f$, then asymptotically the storage cost of our algorithm is $O(min(cD, fD)) = O(min(c, f) \cdot D)$.

The algorithm's pseudocode appears in Algorithms 3-5. The algorithm uses a set of n shared base objects $bo_1, \ldots, bo_n$ each of which holds three fields V_p, V_f, and $storedTS$.

Algorithm 3 Definitions.

1: $TimeStamps = \mathbb{N} \times \Pi$, with selectors num and c, ordered lexicographically.
2: $Pieces = (\mathcal{E} \times \mathbb{N})$
3: $Chunks = Pieces \times TimeStamps$, with selectors val, ts
4: $encode : \mathbb{V} \rightarrow 2^{\mathcal{E} \times \{1,2,\ldots,n\}}$, $decode : 2^{\mathcal{E} \times \{1,2,\ldots,n\}} \rightarrow \mathbb{V}$ s.t. $\forall v \in \mathbb{V}$, $encode(v) = \{\langle *, 1\rangle, \ldots, \langle *, n\rangle\} \wedge \forall W \in 2^{\mathcal{E} \times \mathbb{N}}$, if $W \subseteq encode(v) \wedge |W| \geq k$, then $decode(W) = v$

5: **base objects:**
6: $\forall i \in \{1, \ldots, n\}$, $bo_i = \langle storedTS, V_p, V_f\rangle$ s.t.
$V_f, V_p \subset Chunks$, and $storedTS \in TimeStamps$,
initially $\langle \langle 0, 0\rangle, \{\langle \langle 0, 0\rangle, \langle v_{0_i}, i\rangle\rangle\}, \{\}\rangle$.

The V_p field holds a set of timestamped code blocks so that the i^{th} block of a value can be stored in the V_p field of object bo_i. The V_f field stores a timestamped replica of a *single* value, (represented as a set of k code blocks). And $storedTS$ holds a timestamp, as explained below.

Write operation and storage efficiency The write operation (lines 3–14) consists of 3 sequentially executed rounds: *read timestamp*, *update*, and *garbage collection*; and, the read consists of one or more sequentially executed *read* rounds. At each round, the client invokes RMWs on all base objects in parallel, and awaits responses from at least $n - f$ base objects. The read rounds of both write and read rely on the readValue routine (lines 21–28) to collect the contents of the V_p and V_f, fields from $n - f$ base objects, as well as to determine the highest $storedTS$ known to these objects. The implementations of the update and garbage collection rounds are given by the update (lines 29–36) and GC (lines 37–42) routines, respectively.

The write implementation starts by encoding v into k code blocks (line 4) and invoking the read round where the client uses the combined contents of the V_p, V_f and $storedTS$ fields returned by readValue to determine the timestamp ts to be stored alongside v's code blocks on the base object; ts is set to be higher than all returned timestamps thus ensuring that the order of the timestamps associated with the stored values is compatible with the order of their corresponding writes, (which is essential for regularity).

The client then proceeds to the update round where it attempts to store the i^{th} code block $\langle e, i\rangle$ of v in $bo_i.V_p$ if the size of $bo_i.V_p$ is less than k (lines 33), or its full replica in $bo_i.V_f$ if ts is higher than the timestamp associated with the value currently stored in $bo_i.V_f$ (line 35). Storing $\langle e, i\rangle$ in $bo_i.V_p$ coincides with an attempt to reduce its size by removing stale code blocks of values whose timestamps are smaller than $storedTS$ (line 33). This guarantees that the size of V_p never exceeds the number of concurrent writes, which is a key for achieving our adaptive storage bound. Lastly, the client updates $bo_i.storedTS$ so as its new value is at least as high as the one returned by the readValue routine. This allows the timestamp associated with the latest complete update to propagate to the base object being written, in order to prevent future writes of old blocks into this base object.

In the write's garbage collection round, the client attempts to further reduce the stor-

age usage by (1) removing all code blocks associated with timestamps lower than ts from both $bo_i.V_p$ and $bo_i.V_f$ (lines 38–39), and (2) replacing a full replica (if it exists) of its written value v in $bo_i.V_f$ with its i^{th} code block $\langle e, i\rangle$ (line 41). It is safe to remove the full replica and values with older timestamps at this point, since once the update round has completed, it is ensured that the written value or a newer written value is restoreable from any $n - f$ base objects. This mechanism ensures that all code blocks except the ones comprising the value written with the highest timestamp are eventually removed from all objects' V_p and V_f sets, which reduces the storage to a minimum in runs with finitely many writes, which all complete. The garbage collection round also updates the $bo_i.storedTS$ field to ensure its value is at least as high as ts.

Algorithm 4 regular register emulation. Algorithm for client c_j.

```
1:  local variables:
2:      storedTS, ts ∈ TimeStamp, WriteSet ∈ Pieces

3:  operation Write(v)
4:      WriteSet ← encode(v)
        ▷ round 1: read timestamps
5:      ⟨storedTS, ReadSet⟩ ← readValue()
6:      tmp ← max(storedTS.num,
        max{tmp' | ⟨⟨tmp', ∗⟩, ∗⟩ ∈ ReadSet})
7:      ts ← ⟨n + 1, j⟩
        ▷ round 2: update
8:      || for i = 1 to n
9:          update(bo_i, WriteSet, ts, storedTS, i)
10:     wait for n − f responses
        ▷ round 3: garbage collect
11:     || for i = 1 to n
12:         GC(bo_i, WriteSet, ts, i)
13:     wait for n − f responses
14:     return "ok"

15: operation Read()
16:     ⟨storedTS, ReadSet⟩ ← readValue()
17:     while ∄ts ≥ storedTS s.t.
        |{⟨ts, v⟩ | ⟨ts, v⟩ ∈ ReadSet}| ≥ k do
18:         ⟨storedTS, ReadSet⟩ ← readValue()
19:     ts' ← max_{ts≥storedTS} (|{⟨ts, v⟩ | ⟨ts, v⟩ ∈ ReadSet}| ≥ k)
20:     return decode({v | ⟨ts', v⟩ ∈ ReadSet})
```

Key Invariant and read operation The write implementation described above guarantees the following key invariant: at all times, a value written by either the latest complete write or a newer write is available from every set consisting of at least $n - f$ base objects (either in the form of k code blocks in the objects' V_p fields, or in full from one of their V_f fields). Therefore, a read will always be able to reconstruct the latest completely written or a newer value provided it can successfully retrieve k matching blocks of this value. However, a read round may sample different base objects at different times (that

is, it does not necessarily obtain an atomic snapshot of the base objects), and the number of blocks stored in V_p is bounded. Thus, the read may be unable to see k matching blocks of any single new value, as long as new values continue to be written concurrently with the read.

Nevertheless, for FW-Termination, the reads are only required to return in runs where a finite number of writes are invoked. Our implementation of read (lines 15–20) proceeds by invoking consecutive rounds of RMWs on the base objects via the readValue routine. After each round, the reader examines the collection of returned values and timestamps to determine if any value has k code blocks and is also associated with a timestamp that is at least as high as $storedTS$ (line 17). If any such value is found, the one associated with the highest timestamp is returned (line 20). Otherwise, the reader proceeds to invoke another round of base object accesses. Note that returning values associated with older timestamps may violate regularity, since they may have been written earlier than the write with timestamp $storedTS$, which in turn may have completed before the read was invoked.

Algorithm 5 Functions used in regular register emulation.

```
21: procedure readValue()
22:     ReadSet ← {}, T ← {}
23:     || for i=1 to n
24:         tmp ← read(bo_i)
25:         ReadSet ← ReadSet ∪ tmp.V_f ∪ tmp.V_p
26:         T ← T ∪ {tmp.storedTS}
27:     wait for n − f responses
28:     return ⟨max(T), ReadSet⟩

29: update(bo, WriteSet, ts, storedTS, i) ≜
30:     if ts ≤ bo.storedTS
31:         return
32:     if |bo.V_p| < k
            ▷ write a piece and remove old pieces
33:         bo.V_p ← bo.V_p \ {⟨ts′, v⟩ ∈ bo.V_p | ts′ < storedTS}
            ∪{⟨ts, ⟨e, i⟩⟩ | ⟨e, i⟩ ∈ WriteSet}
34:     else if bo.V_f = {} ∨ ∃ts′ < ts : ⟨ts′, ∗⟩ ∈ bo.V_f
            ▷ write a piece and remove old pieces
35:         bo.V_f ← {⟨ts, ⟨e, j⟩⟩ | ⟨e, j⟩ ∈ WriteSet
            ∧j ∈ {1, ..., k}}
36:     bo.storedTS ← max(bo.storedTS, storedTS)

37: GC(bo, WriteSet, ts, i) ≜
        ▷ keep only new pieces
38:     bo.V_p ← {⟨ts′, v⟩ ∈ bo.V_p | ts′ ≥ ts}
39:     bo.V_f ← {⟨ts′, v⟩ ∈ bo.V_f | ts′ ≥ ts}
40:     if ⟨ts, ∗⟩ ∈ bo.V_f
            ▷ V_f holds a full replica of my write
            ▷ V_f keep only one piece of it
41:         bo.V_f ← {⟨ts, ⟨e, i⟩⟩ | ⟨e, i⟩ ∈ WriteSet}
42:     bo.storedTS ← max(bo.storedTS, ts)
```

3.4.2 Correctness Proofs

Note that we prove here that the algorithm satisfies strong regularity and FW-termination, which are stronger safety and liveness properties than the one used in our lower bound.

We start by proving the storage cost.

Observation 6. *For every run of the algorithm, for every base object bo_i, $bo_i.ts$ monotonically increasing.*

Lemma 12. *Consider a run r of the algorithm, and two* writes *w_1, w_2, where w_1 writes with timestamp ts_1. If $w_1 \prec_r w_2$, then w_2 sets its $\hat{ts}$, to a timestamp that is not smaller than ts_1.*

Proof. By Observation 6, for each base object bo, $bo.ts$ is monotonically increasing. Therefore, after w_1 finishes the garbage collection phase, there is a set S consisting of $n - f$ base objects s.t. for each $bo_i \in S$, $bo_i.ts \geq ts$. Recall that $n = 2f + k$, thus every two sets of $n - f$ base objects have at least one base object in common. Therefore, w_2 gets a response from at least one base object in S in its first phase, and thus sets $\hat{ts} = ts'$ s.t. $ts' \geq ts$.

□

Lemma 13. *For any run r of the algorithm, for any base object bo at any time t in r, $bo.V_p$ does not store more than one piece of the same write.*

Proof. The *writes* perform the second phase at most one time on each base object bo, and in each update they store at least one piece in $bo.V_p$. And since they does not store in $bo.V_p$ during the third phase, the lemma follows.

□

Lemma 14. *Consider a run r of the algorithm in which the maximum number of concurrent writes is $c < k - 1$. Then the storage at any time in r is not bigger than $(2f + k)(c + 1)D/k$ bits.*

Proof. Recall that we assume that $n = 2f + k$ and the size of each piece is D/k. Thus it suffices to show that there is no time t in r s.t. some base object stores more than $c + 1$ pieces at time t.

Assume by way of contradiction that the claim is false. Consider the time t when some $bo \in N$ stores $c + 2$ pieces for the first time. Notice that $|bo.V_p| \leq c + 1 < k$ till time t, and therefore, $bo.V_p$ does not contain more then one piece from the same write, and $bo.V_f = \perp$ till time t'. Now consider the write w that was invoked last among all the writes that store pieces in $bo.V_p$ at time t, denote its piece by p. Since bo stores $c + 2$ pieces at time t', by Lemma 14, there must be two writes w_1 and w_2 whose pieces p_1, p_2 are stored at time t in $bo.V_p$, and both returns before w is invoked. Denote their timestamps ts_1 and ts_2, and assume without loss of generality that $ts_1 > ts_2$. By Lemma 12, w sets its $\hat{ts}$ to ts' s.t. $ts' \geq ts_1 > ts_1$. Now consider two cases. First, if p was added before p_2, then $bo.ts > ts_2$ when p_2 was added. A contradiction. Otherwise, p was added after p_2. Thus, p_2 was deleted in line 33 of the update when p was added. A contradiction.

□

Lemma 15. *The storage is never more than $(2f + k)2D$ bits at any time t in any run r of the algorithm.*

Proof. Each base object stores no more than $2k$ pieces at any time t in r. The lemma follows.

□

Lemma 16. *Consider a run r of the algorithm with finite number of writes, in which all writes correct. Then the storage is eventually reduced to $(2f + k)D/k$ bits.*

Proof. Consider a *write* w with the biggest timestamp ts in r. Since w is correct, and since *writes* are wait-free, w returns, and eventually performs *free* on every base object. Consider a base object bo s.t. w performs *free* on bo at time t. Notice that w deletes all pieces with smaller timestamps than ts and set $bo.ts = ts$ at time t. Now recall that bo ignore all updates with timestamp less than $bo.ts$, and therefore, bo store only w's piece at any time after time t. The lemma follows.

□

From Lemmas 14, 15, and 16 we get:

Corollary 7. *The storage of the algorithm is bounded by $(2f + k)2D$ bits, and in runs with at most $c < k$ concurrent writes the storage is bounded by $(c + 1)D/k$ bits. Moreover, in a run with a finite number of writes, if all the writes are correct, the storage is eventually reduced to $(2f + k)D/k$ bits.*

We next prove the liveness property.

Lemma 17. *Consider a fair run r of the algorithm. Then every* write *w invoked by a correct client c_i eventually completes.*

Proof. Consider a correct client c_i. The *write* w is divided into three phase s.t. in each phase, c_i invokes operations on all the base objects, and waits for $n - f$ responses. The run r is fair, so every action invoked by c_i on a correct base object eventually returns, and no more than f base objects fail in r. Therefore, eventually c_i receives $n - f$ responses in each of the phases and returns.

□

Observation 7. *When a piece from $bo.V_p$ is deleted, $bo.ts$ is increased.*

Lemma 18. *If at time t, c_i completes the second phase of write with timestamp ts, then for every $t' > t$ for every $S \subseteq N$ s.t. $|S| \geq n - f$, exist write w with $ts' \geq ts$ s.t. at least k pieces of w are stored in S.*

Proof. Consider time t'. Let $\hat{ts}$ be the highest timestamp written by a write w that completed the second phase by time t. It is sufficient to show the lemma hold for $\hat{ts}$.

First note that $\forall bo$, $bo.ts \leq \hat{ts}$ before time t, because no write with a larger timestamp than $\hat{ts}$ started the third phase. This means that w's *update* left at lest one piece in which

bo it occurred. Now consider a set S of $n - f$ base objects, and since $n = 2f + k$, w's *update* occurred in set S' that contains at least k base objects in S.

If w wrote to V_p, it was not overwritten by time t, because (1) no other write began *free* with timestamp bigger than $\hat{fs}$, and (2) since there is no base object bo s.t. $bo.ts \geq \hat{fs}$, no write delete w's piece in the second phase. Therefore if w wrote to V_p in all base objects in S', the lemma holds.

Otherwise, w wrote k pieces to V_f in base objects in some set $S'' \subseteq S'$. Consider two cases: First, there is base object $bo' \in S''$ s.t. some write overwritten w's pieces in $bo'.V_f$ before time t. Since there is no write with timestamp bigger than $\hat{fs}$ that started the third phase before time t, it is guarantee that k pieces with timestamp $ts' > \hat{fs}$ stored in $bo'.V_f$ at time t, and the lemma holds. Else, since w's pieces stored in $S' \setminus S''$ does not overwritten before time t, the lemma holds (no matter if w performed the third phase or not).

□

Invariant 1. *For any run r of the algorithm, for any time t in r, for any set S of $n - f$ base objects. Let $\hat{ts}_s = max\{bo.ts \mid bo \in S\}$. Then there is a timestamp $ts' \geq \hat{ts}_s$ s.t. there are at least k different pieces associated with ts' in S.*

Proof. We prove by induction. **Base:** the invariant holds at time 0. **Induction:** Assume that the induction holds before the t^{th} action is scheduled, we show that it holds also at time t. Assume that the t^{th} action is RMW on a base object bo, and consider any set S of $n - f$ base objects. If $bo \notin S$ then the invariant holds. Else, consider the two possible RMW actions:

- The t^{th} action is *update*. If no pieces are deleted, the invariant holds. If $bo.ts$ is increased, then consider the write with timestamp ts that is the the biggest timestamp among all writes that complete the second phase before time t. Notice that $bo.ts \leq ts$ at time t, and by Lemma 18, the invariant holds. The third option is that a piece p with timestamp $ts' > bo.ts$ of a *write* w is deleted and $bo.ts$ is not increased. Note that by Observation 7, such piece can be deleted only from $bo.V_f$, and since p is overwritten by k pieces with bigger timestamp, the invariant holds.
- The t^{th} action is *free*. If $bo.ts$ is not changes, then the invariant holds. Else, Consider the write with the biggest timestamp ts among all writes that complete the second phase before time t. Note that $bo.ts$ is set to a timestamp $ts' \leq ts$, so by Lemma 18, the invariant holds.

□

Lemma 19. *Consider a fair run r of the algorithm. If there is a finite number of write invocations in r, then every read operation rd invoked by a client c_i eventually returns.*

Proof. Assume by way of contradiction that rd does not return in r. By Lemma 17, the *writes* are wait-free, and since the number of *write* invocations in r is finite, there is a

time t in r s.t. no *write* performs actions after time t. Therefore, any *read* that invokes $readValue()$ procedure after time t receives a set S of values that is stored in a set of $n - f$ base objects at time t. By invariant 1, there is a timestamp ts s.t. there is at least k different pieces in S associated with ts, and $ts > bo.ts$ for all $bo \in S$. Now since the every correct *read rd* invokes $readValue()$ infinitely many times in r, *rd* returns. A contradiction.

□

The next corollary follows from Lemmas 17, 19.

Corollary 8. *The algorithm satisfies the WF-termination property.*

We now prove that the algorithm satisfies strong regularity.

Definition 13. *For every run r, σ_r is a sequential run s.t. the* writes *in r are ordered in σ_r by their timestamp, and every* read *in r that returns a value associate with timestamp ts, is ordered in σ_r immediately after the* write *that is associate with timestamp ts.*

For simplicity we say the that v_0 was written by *write* w_0 that associated to timestamp 0 at time 0.

Lemma 20. *Consider a run r, and a read rd that returns a value v. Consider also the timestamp ts' that rd obtains in line 19 (Algorithm 4). Then v is the value written by a* write *associated with timestamp ts' or v_0 if $ts' = 0$.*

Proof. By the code, if $ts' = 0$, then rd returns v_0. Now notice that rd obtains at least k different pieces associated with timestamp ts', thus by decode definition, rd returns v.

□

Corollary 9. *For every run r, σ_r satisfies the sequential specification.*

Observation 8. *Consider a* write *w that obtains ts and $\hat{ts}$ in the first phase, then $ts > \hat{ts}$.*

Lemma 21. *For every run r, for every two writes w_1, w_2 with timestamp ts_1, ts_2. If w_2 was invoked after w_1 finished the second phase, then $ts_1 < ts_2$.*

Proof. First notice that for every base object bo, if a *write* w overwrites pieces of a *write* w' in bo, V_f, that w' timestamp is bigger than w''s. And by Observation 8, if w deletes w''s piece from $bo.V_p$, then it stores a piece with bigger timestamp than w''s timestamp. Therefore, the maximal timestamp in each base object is monotonically increasing. Now recall that in the second phase w_1 performed *update* on $n - f$ base object, and notice that after w_1 performs *update* on base object bo the maximal timestamp in bo is at lest as big as ts_1. Now since two sets of $n - f$ base object have at least one base object in common, w_2 picks $ts > ts_1$.

□

Lemma 22. *For every run r, for every two writes w_1, w_2 in r, if $w_1 \prec_r w_2$, then w_2 is not ordered before w_1 in σ_r.*

Proof. Follows immediately from Lemma 21.

□

Lemma 23. *For every run r, for every read rd and write w_1, if $rd \prec_r w_1$, then w_1 is not ordered before rd in σ_r.*

Proof. Assume that rd returns value that is associated with timestamp ts belonging to some *write* w, and w_1 is associated with timestamp ts_1. Since rd returns w's value, w begins the third phase before rd returns. And since w_1 was invoked after rd returns, w_1 was invoked after w's second phase. Therefore, by Lemma 21, $ts_1 > ts$, and thus w_1 is ordered after w in σ_r. Recall that by the construction of σ_r, rd is ordered immediately after w in σ_r, hence, rd is ordered before w_1 in σ_r.

□

Lemma 24. *For every run r, for every read rd and write w_1, if $w_1 \prec_r rd$, then rd is not ordered before w_1 in σ_r.*

Proof. Consider a *write* w_1 with timestamp ts_1 and a *read* rd s.t. $w_1 \prec_r rd$. Assume by way of contradiction that rd is ordered before w_1 in σ_r. Then rd returns a value with a timestamp ts that is associated with a *write* w that is ordered before w_1 in σ_r. By the construction of σ_r, $ts_1 > ts$. Now since w_1 completed the third phase before rd invoked, and since by Observation 6, for each bo, $bo.ts$ is monotonically increasing, when rd invoked, for every set S of $n - f$ base objects, the maximal $bo.ts$ of all $bo \in S$ is bigger than or equal to ts_1, and thus bigger than ts. Therefore rd set $\hat{ts}$, in the first phase, to timestamp bigger than ts, and thus does not return w's value. A contradiction.

□

The next corollary follows from Corollary 9, and Lemmas 22, 23, 24.

Corollary 10. *The algorithm simulates a strongly regular register.*

The following theorem stems from Corollaries 7, 8, and 10.

Theorem 7. *There is a FW-terminating algorithm that simulates a strongly regular register, which storage is bounded by $(2f + k)2D$ bits, and in runs with at most $c < k$ concurrent writes, the storage is bounded by $(c + 1)D/k$ bits. Moreover, in a run with a finite number of writes, if all the writes are correct, the storage is eventually reduced to $(2f + k)D/k$ bits.*

3.5 A (Simple) Safe and Wait-free Algorithm

We present here a simple storage-efficient algorithm that ensures *safe* semantics, but not *regularity*. Although this algorithm has no practical use, it shows that the impossibility result of Section 3.3 does not apply to a weaker safety property.

3.5.1 Algorithm

This algorithm simulates a wait-free and strongly safe MWMR register using erasure codes (see Section 3.4). It stores exactly n pieces of the data, one in each base object. The algorithm's definitions we use here are the same as in section 3.4 (Algorithm 3), and the pseudocode of client c_j can be found in Algorithm 6.

Since memory is fault-prone, actions are triggered in parallel on all base objects. This parallelism is denoted using $||$**for** in the code. Operations then wait for $n - f$ base objects to respond. Recall that $n = 2f + k$, so every two sets of $n - f$ base objects have at least k pieces in common. Thus, if a write completes after storing pieces on $n - f$ base objects, a subsequent read accessing any $n - f$ base objects finds k pieces of the written value (as needed for restoring the value), provided that they are not over-written by later writes.

A $write(v)$ operation (lines 1–8) first produces n pieces from v using *encode*, then reads from $n - f$ base objects to obtain a new timestamp, and finally, tries to store every piece together with the timestamp at a different base object. For every base object bo, c_j triggers the *update* RMW function, which overwrites bo only if c_j's timestamp is bigger than the timestamp stored in bo.

A *read* (lines 12–17) reads the values stored in $n - f$ base objects, and then tries to restore valid data as follows. If c_j reads at least k values with the same timestamp, it uses the *decode* function, and returns the restored value. Otherwise, it returns v_0. The latter occurs only if there are outstanding *writes*, that had updated fewer than $n - f$ base objects before the reader has accessed them. Therefore, these *writes* are concurrent with c_j's *read*, and by the safety property, any value can be returned in this case. The algorithm's correctness is formally in the next section.

Algorithm 6 Safe register emulation. Algorithm for client c_j.

```
1:  operation write(v)
2:      W ← encode(v)
3:      R ← readValue()
4:      ts ← ⟨max({ts | ⟨⟨ts, *⟩, *⟩ ∈ R}) + 1, j⟩
5:      || for all ⟨v, i⟩ ∈ W
6:          update(bo_i, ⟨v, i⟩, ts)      ▷ trigger RMW on bo_i
7:      wait for n − f responses
8:      return "ok"
9:  update(bo, w, ts) ≜
10:     if ts > bo.ts
11:         bo ← ⟨w, ts⟩
12: operation read()
13:     R ← readValue()
14:     if ∃ts s.t. |{v | ⟨ts, v⟩ ∈ R}| ≥ k
15:         ts′ ← ts s.t. |{v | ⟨ts, v⟩ ∈ R}| ≥ k
16:         return decode({v | ⟨ts′, v⟩ ∈ R})
17:     return v_0
18: procedure readValue()
19:     R ← {}
20:     || for i=1 to n
21:         R = R ∪ read(bo_i)
22:     wait until |R| ≥ n − f
23:     return R
```

3.5.2 Correctness proof

Lemma 25. *The storage of the algorithm is nD/k.*

Proof. The size of each piece is D/k. We have n base objects, and each base object stores exactly one piece.

□

Lemma 26. *The algorithm is wait-free.*

Proof. There are no loops in the algorithm, and the only blocking instructions are the waits in lines 7 and 22. In both cases, clients wait for no more than $n - f$ responses, and since no more than f base objects can fail, clients eventually continue. Therefore, a client that gets the opportunity to perform infinitely many actions completes its operations.

□

We now prove that the algorithm satisfies strongly safety. We relay on the following single observation.

Observation 9. *The timestamps in the base objects are monotonically increasing.*

Definition 14. *For every run r, we define the sequential run σ_{w_r} as follows: All the completed write operations in r are ordered in σ_{w_r} by their timestamp.*

Lemma 27. *For every run r, the sequential run σ_{w_r} is a linearization of r.*

Proof. Since σ_{w_r} has no read operations, the sequential specification is preserved in σ_{w_r}. Thus, we left to show the real time order: For every two completed *writes* w_i, w_j in r, we need to show that if $w_i \prec_r w_j$, then $w_i \prec_{\sigma_r} w_j$.

Denote w_i's timestamp by ts. By Observation 9, at any point after w_i's return, at least $n - f$ base objects store timestamps bigger than or equal to ts. When w_j picks a timestamp, it chooses a timestamp bigger than those it reads from $n - f$ base objects. Since, $n > 2f$, w_j picks a timestamp bigger than ts, and therefore w_j is ordered after w_i in σ_{rd}.

□

Definition 15. *For every run r, for every* read *rd that has no concurrent* write *operations in r, we define the sequential run $\sigma_{r_{rd}}$ by adding rd to σ_{w_r} after all the writes that precede it in r.*

In order to show that the algorithm simulates a safe register, we proof in Lemmas 28 and 29 that the real time order and sequential specification respectively, are preserved in $\sigma_{r_{rd}}$.

Lemma 28. *For every run r, for every read rd that has no concurrent write operations in r, $\sigma_{r_{rd}}$ preserves r's operation precedence relation (real time order).*

Proof. By Lemma 27, the order between the writes in $\sigma_{r_{rd}}$ are preserved, and by construction of σ_{rd} the order between rd and write operations is also preserved.

□

Lemma 29. *Consider a run r and any read rd that has no concurrent writes in r. Then rd returns the value written by the write with the biggest timestamp that precedes rd in r, or v_0 if there is no such* write.

Proof. In case there is no *write* before *rd* in *r*, since there are also no writes concurrent with *rd*, *rd* reads pieces with timestamp $\langle 0, 0 \rangle$ from all base objects, and thus, returns v_0. Otherwise, let w be the $write(v)$ associated with the biggest timestamp ts among all the *writes* invoked before *rd* in *r*. Let t be the time when *rd* is invoked. Recall that *rd* has no concurrent *writes*, so all the writes invoked before time t complete before time t and store there pieces in $n - f$ base objects unless the base objects already hold a higher timestamp. By Observation 9 and the fact that w has the highest timestamp by time t, we get that at time t there are at least $n - f$ base objects that store a piece of v. Since $n = 2f + k$, every two sets of $n - f$ base objects have at least k base objects in common. Therefore, *rd* reads at least k pieces of v, and thus, restores and returns v.

□

Corollary 11. *There exists an algorithm that simulates a safe wait-free MWMR register with a worst-case storage cost of $nD/k = (2f/k + 1)D$.*

3.6 Discussion

We studied the inherent space requirements of reliable storage in asynchronous distributed settings. We proved an asymptotic bound of $\Omega(min(f, c) \cdot D)$ for any storage algorithm using a symmetric black-box coding scheme, which produces code blocks of values independently of other values. We then presented an algorithm that combines replication and erasure codes, whose storage cost is $O(min(f, c) \cdot D)$.

Our work leaves open questions for future work. First, it is unclear whether the same lower bound still applies when stored bits are allowed to depend on multiple concurrent write values. The main requirement for extending our proof to general coding is a model that correctly accounts for the information stored in the base objects and clients when the clients code jointly. Our black-box assumption rules out such joint coding. Whereas in principle, [22] allows stored information to depend on multiple input values, their assumption that only one round of the protocol depends on the written value essentially forces clients to "forget" the value they used in such joint coding. For example, if the algorithm stores $v_1 + v_2$ instead of either v_1 or v_2, it cannot reproduce the original values, rendering such joint coding useless. Second, while asymptotically optimal, the constants in our bound are not tight, and it could be interesting to close this gap. Finally, we believe that our model and adversary definitions can yield additional lower bounds.

Chapter 4

On Liveness of Dynamic Storage

Many works in the last decade have dealt with *dynamic* reliable distributed storage emulation [66, 8, 40, 41, 24, 17, 53, 19, 50, 16, 15, 37, 49, 74]. The motivation behind such storage is to allow new processes (nodes) to be phased in and old or dysfunctional ones to be taken offline. From a fault-tolerance point of view, once a faulty process is removed, additional failures may be tolerated. For example, consider a system that can tolerate one failure: once a process fails, no additional processes are allowed to fail. However, once the faulty process is replaced by a correct one, the system can again tolerate one failure. Thus, while static systems become permanently unavailable after some constant number of failures, dynamic systems that allow infinitely many reconfigurations can survive forever.

Previous works can be categorized into two main types: Solutions of the first type assume a churn-based model [51, 59] in which processes are free to announce when they join the storage emulation [17, 15, 16, 11] via an auxiliary broadcast sub-system that allows a process to send a message to all the processes in the system, (which my be unknown to the sending processes). The second type solutions extend the register's API with a reconfiguration operation for changing the current configuration of participating processes [66, 8, 50, 37, 40, 41, 24], which can be only invoked by members of the current configuration. In this paper we consider the latter. Such an API allows administrators (running privileged processes), to remove old or faulty processes and add new ones without shutting down the service; once a process is removed from the current configuration, a system administrator may shut it down. Note that in the churn-based model, in contrast, if processes have to perform an explicit operation in order to leave the system (as in [17, 11]), a faulty process can never be removed. In addition, since in API-based models only processes that are already within the system invoke operations, it is possible to keep track of the processes in the system, and thus auxiliary broadcast is not required.

Though the literature is abundant with dynamic storage algorithms in both models, to the best of our knowledge, all previous solutions in asynchronous and eventually

synchronous models restrict reconfigurations in some way in order to ensure completion of all operations. Churn-based solutions assume a bounded churn rate [17, 15, 11], meaning that there is a finite number of joining and removing processes in a given time interval. Some of the API-based solutions [66, 8, 50, 37] provide liveness only when the number of reconfigurations is finite, whereas others discuss liveness only in synchronous runs [40, 41, 24]. Such restrictions may be problematic in emerging highly-dynamic large-scale settings.

Baldoni et al. [15] showed that it is impossible to emulate a dynamic register that ensures completion of all operations without restricting the churn rate in asynchronous churn-based models in which processes can freely abandon the computation without an explicit leave operation. Since a leave and a failure are indistinguishable in such models, the impossibility can be proven using a partition argument as in [10].

In this chapter we revisit this question in the API-based model. First, we prove a similar result for asynchronous API-based dynamic models, in which *one* unremoved process can fail and successfully removed ones can go offline. Specifically, we show that even the weakest type of storage, namely a *safe* register [52], cannot be implemented so as to guarantee liveness for all operations (i.e., wait-freedom) in asynchronous runs with an unrestricted reconfiguration rate. Note that this bound does not follow from the one in [15] since a process in our model can leave the system only after an operation that removes it successfully completes.

Second, to circumvent our impossibility result, we define a dynamic failure detector that can be easily implemented in eventually synchronous systems, and use it to implement dynamic storage. We present an algorithm, based on state machine replication, that emulates a strong shared object, namely a wait-free atomic dynamic multi-writer, multi-reader (MWMR) register, and ensures liveness for all operations without restricting the reconfiguration rate. Though a number of previous algorithms have been designed for eventually synchronous models [40, 41, 24, 17, 15, 53, 19], to the best of our knowledge, our algorithm is the first to ensure liveness of all operations without restricting the reconfigurations rate.

In particular, previous algorithms [40, 41, 24, 53, 19] that used failure detectors, only did so for reaching consensus on the new configuration. For example, reconfigurable Paxos variants [53, 19], which implement atomic storage via dynamic state machine replication, assume a failure detector that provides a leader in every configuration. However, a configuration may be changed, allowing the previous leader to be removed (and then fail) before another process p (with a pending operation) is able to communicate with it in the old configuration. Though a new leader is elected by the failure detector in the ensuing configuration, this scenario may repeat itself indefinitely, so that p's pending operation never completes.

We, in contrast, use the failure detector also to implement a helping mechanism, which ensures that eventually some process will help a slow one before completing its own reconfiguration operation even if the reconfiguration rate is unbounded. Such

mechanism is attainable in API-based models since only members of the current configuration invoke operations, and thus helping process can know which processes may need help. Note that in churn-based models in which processes announce their own join, implementing such a helping mechanism is impossible, since a helping process cannot possibly know which processes need help joining.

The remainder of this chapter is organized as follows: In Section 4.1 we present the model and define the dynamic storage object we seek to implement. Our impossibility proof appears in Section 4.2, and our algorithm in Section 4.3. Finally, we conclude the paper in Section 4.4.

4.1 Model

In Section 4.1.1, we present the preliminaries of our model, and in Section 4.1.2, we define the dynamic storage service.

4.1.1 Preliminaries

We consider an asynchronous message passing system consisting of an infinite set of processes Π. Processes may fail by crashing subject to restrictions given below. Process failure is modeled via an explicit fail action. Each pair of processes is connected by a communication link. A *service* exposes a set of *operations*. For example, a dynamic storage service exposes read, write, and reconfig operations. Operations are invoked and subsequently respond.

An *algorithm* A defines the behaviors of processes as deterministic state machines, where state transitions are associated with *actions*, such as send/receive messages, operation invoke/response, and process failures. A *global state* is a mapping to states from system components, i.e., processes and links. An *initial global state* is one where all processes are in initial states and all links are empty. A send action is *enabled* in state s if A has a transition from s in which the send occurs.

A *run* of algorithm A is a (finite or infinite) alternating sequence of global states and actions, beginning with some initial global state, such that state transitions occur according to A. We use the notion of *time* t *during a run* r to refer to the t^{th} action in r and the global state that ensues it. A *run fragment* is a contiguous subsequence of a run. An operation invoked before time t in run r is *complete* at time t if its response event occurs before time t in r; otherwise it is *pending* at time t. We assume that runs are *well-formed* [**?**], in that each process's first action is an invocation of some operation, and a process does not invoke an operation before receiving a response to its last invoked one.

We say that operation op_i *precedes* operation op_j in a run r, if op_i's response occurs before op_j's invocation in r. Operations op_i and op_j are *concurrent* in run r, if op_i does not precede op_j and op_j does not precede op_i in r. A *sequential run* is one with no concurrent operations. Two runs are *equivalent* if every process performs the same sequence of operations (with the same return values) in both, where operations that are pending in one can either be included in or excluded from the other.

4.1.2 Dynamic storage

The distributed storage service we consider is a *dynamic multi-writer, multi reader (MWMR) register* [8, 50, 37, 72, 56, 41], which stores a value v from a domain $\mathbb{V}$, and offers an interface for invoking *read*, *write*, and *reconfig* operations. Initially, the register holds some initial value $v_0 \in \mathbb{V}$. A *read* operation takes no parameters and returns a value from $\mathbb{V}$, and a *write* operation takes a value from $\mathbb{V}$ and returns "ok". We define *Changes* to be the set $\{remove, add\} \times \Pi$, and call any subset of Changes a *set of changes*. For example,

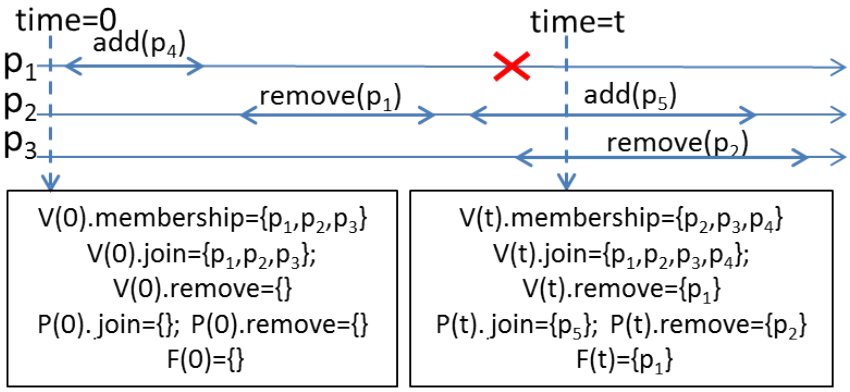


Figure 4.1: Notation illustration. $add(p)$ ($remove(p)$) represents $reconfig(\langle add, p\rangle)$ (respectively, $reconfig(\langle remove, p\rangle)$).

$\{\langle add, p_3\rangle, \langle remove, p_2\rangle\}$ is a set of changes. A *reconfig* operation takes as a parameter a set of changes and returns "ok". For simplicity, we assume that a process that has been removed is not added again.

Notation For every subset w of *Changes*, the *removal set* of w, denoted $w.remove$, is $\{p_i | \langle remove, p_i\rangle \in w\}$; the *join set* of w, denoted $w.join$, is $\{p_i | \langle add, p_i\rangle \in w\}$; and the *membership* of w, denoted $w.membership$, is $w.join \setminus w.remove$. For example, for a set $w = \{\langle add, p_1\rangle, \langle remove, p_1\rangle, \langle add, p_2\rangle\}$, $w.join = \{p_1, p_2\}$, $w.remove = \{p_1\}$, and $w.membership = \{p_2\}$. For a time t in a run r, we denote by $V(t)$ the union of all sets q s.t. *reconfig(q)* completes before time t in r. A *configuration* is a finite set of processes, and the *current configuration at time t* is $V(t).membership$. We assume that only processes in $V(t).membership$ invoke operations at time t. The initial set of processes $\Pi_0 \subset \Pi$ is known to all and we say, by convention, that $reconfig(\{\langle add, p\rangle | p \in \Pi_0\})$ completes at time 0, i.e., $V(0).membership = \Pi_0$.

We define $P(t)$ to be the set of *pending changes* at time t in run r, i.e., the set of all changes included in pending reconfig operations. We denote by $F(t)$ the set of processes that have failed before time t in r; initially, $F(0) = \{\}$. For a series of arbitrary sets $S(t)$, $t \in \mathbb{N}$, we define $S(*) \triangleq \bigcup_{t\in\mathbb{N}} S(t)$. The notation is illustrated in Figure 4.1.

Correct processes and fairness A process p is *correct* if $p \in V(*).join \setminus F(*)$. A run r is *fair* if every send action by a correct process that is enabled infinitely often eventually occurs, and every message sent by a correct process p_i to a correct process p_j is eventually received at p_j. Note that messages sent to a faulty process from a correct one may or may not be received. A process p is *active* if p is correct, and $p \notin P(*).remove$.

Service specification A *linearization* of a run r is an equivalent sequential run that preserves r's operation precedence relation and the service's sequential specification. The

sequential specification for a register is as follows: A read returns the latest written value, or v_0 if none was written. An MWMR register is *atomic*, also called *linearizable* [45], if every run has a linearization. Lamport [52] defines a *safe* single-writer register. Here, we generalize the definition to multi-writer registers in a weak way in order to strengthen the impossibility result. Intuitively, if a read is not concurrent with any write we require it to return a value that reflects some possible outcome of the writes that precede it; otherwise we allow it to return an arbitrary value. Formally: An MWMR register is *safe* if for every run r for every *read* operation rd that has no concurrent *writes* in r, there is a linearization of the subsequence of r consisting of rd and the *write* operations in r.

A *wait-free* service guarantees that every active process's operation completes regardless of the actions of other processes.

Failure model and reconfiguration The reconfig operations determine which processes are allowed to fail at any given time. Static storage algorithms [10] tolerate failures of a minority of their (static) universe. At a time t when no reconfig operations are ongoing, the dynamic failure condition may be simply defined to allow less than $|V(t)membership|/2$ failures of processes in $V(t).membership$. When there are pending additions and removals, the rule must be generalized to take them into account. For our algorithm in Section 4.3, we adopt a generalization presented in previous works [8, 50, 72, 7]:

Definition 16 (minority failures)**.** *A model allows* minority failures *if at all times t in r, fewer than $|V(t).membership \setminus P(t).remove|/2$ processes out of $V(t).membership \cup P(t).join$ are in $F(t)$.*

Note that this failure condition allows processes whose remove operations have completed to be (immediately) safely switched off as it only restricts failures out of the current membership and pending joins. We say that a service is *reconfigurable* if failures of processes in $V(t).remove$ are unrestricted.

In order to strengthen our lower bound in Section 4.2 we weaken the failure model. Like FLP [36], our lower bound applies as long as at least *one* process can fail. Formally, a failure is allowed whenever all failed processes have been removed and the current membership consists of at least three processes[1]. We call such a state "clean", captured by the following predicate: $clean(t) \triangleq (V(t).membership \cup P(t).join) \cap F(t) = \{\} \wedge |V(t).membership \setminus P(t).remove| \geq 3$. The minimal failure condition is thus defined as follows:

Definition 17 (minimal failure)**.** *A model allows* minimal failure *if in every run r ending at time t when $clean(t)$, for every process $p \in V(t).membership \cup P(t)$, there is an extension of r where p fails at time $t+1$.*

Notice that the minority failure condition allows minimal failure, and so all algorithms that assume minority failures [8, 50, 72, 7] are a fortiori subject to our lower bound, which is proven for minimal failures.

[1] Note that with fewer than three processes, even static systems cannot tolerate failures [10].

4.2 Impossibility of Wait-Free Dynamic Safe Storage

In this section we prove that there is no implementation of wait-free dynamic safe storage in a model that allows minimal failures. We construct a fair run with infinitely many reconfiguration operations in which a slow process p never completes its write operation. We do so by delaying all of p's messages. A message from p to a process p_i is delayed until p_i is removed, and we make sure that all processes except p are eventually removed and replaced.

Theorem 8. *There is no algorithm that emulates wait-free dynamic safe storage in an asynchronous system allowing minimal failures.*

Proof (Theorem 8). Assume by contradiction that such an algorithm A exists. We prove two lemmas about A.

Lemma 30. *Consider a run r of A ending at time t s.t. $clean(t)$, and two processes $p_i, p_j \in V(t).membership$. Extend r by having p_j invoke operation op at time $t+1$. Then there exists an extension of r where (1) op completes at some time $t' > t$,(2) no process receives a message from p_i between t and t', and (3) no process fails and no operations are invoked between t and t'.*

Lemma 30. By the minimal failure condition, p_i can fail at time $t+2$. Consider a fair extension σ_1 of r, in which p_i fails at time $t+2$ and all of its in-transit messages are lost, no other process fails, and no operations are invoked. By wait-freedom, op eventually completes at some time t_1 in σ_1. Since p_i fails and all its outstanding messages are lost, then from time t to t_1 in σ_1 no process receives any messages from p_i. Now let σ_2 be identical to σ_1 except that p_i does not fail, but all of its messages are delayed. Note that σ_1 and σ_2 are indistinguishable to all processes except p_i. Thus, op returns at time t_1 also in σ_2.

□

Lemma 31. *Consider a run r of A ending at time t s.t $clean(t)$. Let $v_1 \in \mathbb{V} \setminus \{v_0\}$ be a value s.t. no process invokes $write(v_1)$ in r. If we extend r fairly so that p_i invokes $w = write(v_1)$ at time $t+1$ which completes at some time $t_1 > t+1$ s.t. $clean(t')$ for all $t < t' \leq t_1$ then in the run fragment between $t+1$ and t_1, some process $p_k \neq p_i$ receives a message sent by p_i.*

Lemma 31. Assume by way of contradiction that in the run fragment between $t+1$ and t_1 no process $p_k \neq p_i$ receives a message sent by p_i, and consider a run r' that is identical to r until time t_1 except that p_i does not invoke w at time t. Now assume that some process $p_j \neq p_i$ invokes a *read* operation rd at time t_1+1 in r'. By the assumption, $clean(t_1)$ and therefore $clean(t_1+1)$. Thus, by Lemma 30, there is a run fragment σ beginning at the final state of r' (time t_1+1), where rd completes at some time t_2, s.t. between t_1+1 and t_2 no process receives a message from p_i. Since

no process invokes $write(v_1)$ in r', and no writes are concurrent with the read, by safety, rd returns some $v_2 \neq v_1$.

Now notice that all global states from time t to time t_1 in r and r' are indistinguishable to all processes except p_i. Thus, we can continue run r with an invocation of read operation rd' by p_j at time t_1, and append σ to it. Operation rd' hence completes and returns v_2. A contradiction to safety.

□

To prove the theorem, we construct an infinite fair run r in which a *write* operation of an active process never completes, in contradiction to wait-freedom.

Consider some initial global state c_0, s.t. $P(0) = F(0) = \{\}$ and $V(0).membership = \{p_1 \dots p_n\}$, where $n \geq 3$. An illustration of the run for $n = 4$ is presented in Figure 4.2. Now, let process p_1 invoke a write operation w at time $t_1 = 0$, and do the following:

Let process p_n invoke *reconfig(q)* where $q = \{\langle add, p_j \rangle | n+1 \leq j \leq 2n-2\}$ at time t_1. The state at the end of r is clean (i.e., $clean(t_1)$). So by Lemma 30, we can extend r with a run fragment σ_1 ending at some time t_2 when *reconfig(q)* completes, where no process $p_j \neq p_1$ receives a message from p_1 in σ_1, no other operations are invoked, and no process fails.

Then, at time $t_2 + 1$, p_n invokes *reconfig(q')*, where $q' = \{\langle remove, p_j \rangle | 2 \leq j \leq n-1\}$. Again, the state is clean and thus by Lemma 30 again, we can extend r with a run fragment σ_2 ending at some time t_3 when *reconfig(q')* completes s.t. no process $p_j \neq p_1$ receives a message from p_1 in σ_2, no other operations are invoked, and no process fails.

Recall that the minimal failures condition satisfies reconfigurability, i.e., all the processes in $V(t_3).remove$ can be in $F(t_3)$ (fail). Let the processes in $\{p_j \mid 2 \leq j \leq n-1\}$ fail at time t_3, and notice that the fairness condition does not mandate that they receive messages from p_1. Next, allow p_1 to perform all its enabled actions till some time t_4.

Now notice that at t_4, $|V(t_4).membership| = n$, $P(t_4) = \{\}$, $(V(t_4).membership \cup P(t_4).join) \cap F(t_4) = \{\}$, and $|V(t_4).membership \setminus P(t_4).removal| \geq 3$. We can rename the processes in $V(t_4).membership$ (except p_1) so that the process that performed the remove and add operations becomes p_2, and all others get names in the range $p_3 \dots p_n$. We can then repeat the construction above. By doing so infinitely many times, we get an infinite run r in which p_1 is active and no process ever receives a message from p_1. However, all of p_1's enabled actions eventually occur. Since no process except p_1 is correct in r, the run is fair. In addition, since $clean(t)$ for all t in r, by the contrapositive of Lemma 31, w does not complete in r, and we get a violation of wait-freedom.

□

4.3 Oracle-Based Dynamic Atomic Storage

We present an algorithm that circumvents the impossibility result of Section 4.2 using a failure detector. In this section we assume the minority failure condition. In Section 4.3.1,

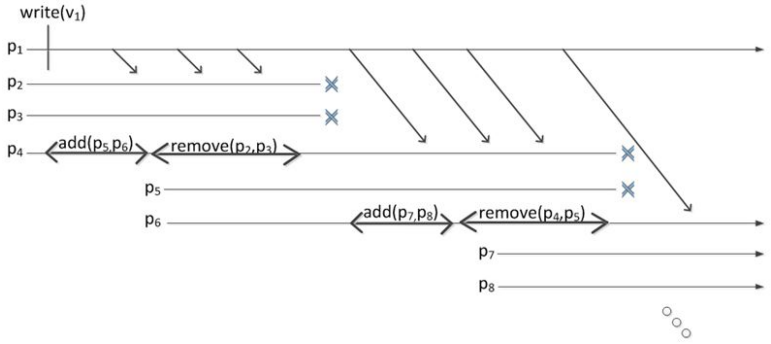


Figure 4.2: Illustration of the infinite run for $n = 4$.

we define a dynamic eventually perfect failure detector. In Section 4.3.2, we describe an algorithm, based on dynamic state machine replication, that uses the failure detector to implement a wait-free dynamic atomic MWMR register. The algorithm's correctness is proven in Section 4.3.3.

4.3.1 Dynamic failure detector

Since the set of processes is potentially infinite, we cannot have the failure detector report the status of all processes as static failure detectors typically do. Dynamic failure detectors addressing this issue have been defined in previous works, either providing a set of processes that have been excluded from or included into the group [54], or assuming that there is eventually a fixed set of participating processes [28]. In our model, we do not assume that there is eventually a fixed set of participating processes, as the number of reconfig operations can be infinite. And we do not want the failure detector to answer with a list of processes, because in dynamic systems, this gives additional information about participating processes that could have been unknown to the inquiring process, and thus it is not clear how such a failure detector can be implemented.

Instead, our dynamic failure detector is queried separately about each process. For each query, it answers either *fail* or *ok*. It can be wrong for an unbounded period, but for each process, it eventually returns a correct answer. Formally, a *dynamic eventually perfect* failure detector, $\Diamond P^D$, satisfies two properties:

- **Strong completeness:** For each process p_i that fails at time t_i, there is a time $t > t_i$ s.t. the failure detector answers *fail* to every query about p_i after time t.
- **Eventual strong accuracy:** There exists a time t, called the *stabilization time*, s.t. the failure detector answers *ok* to every query at any time $t' > t$ about a correct process in $V(t').join$.

Note that $\Diamond P^D$ can be implemented in a standard way in the eventually (partially) synchronous model by pinging the queried process and waiting for a response until a timeout.

4.3.2 Dynamic storage algorithm

We first give the overview of our algorithm and and then present the full description.

Algorithm overview

The key to achieving liveness with unbounded reconfig operations is a novel helping mechanism, which is based on our failure detector. Intuitively, the idea is that every process tries to help all other processes it believes are correct, (according to its failure detector), to complete their concurrent operations together with its own. At the beginning of an operation, a process p queries all other processes it knows about for the operations they currently perform. The failure detector is needed in order to make sure that (1) p does not wait forever for a reply from a faulty process (achieved by strong completeness), and (2) every slow correct process eventually gets help (achieved by eventual strong accuracy).

State machine emulation of a register We use a state machine *sm* to emulate a wait-free atomic dynamic register, *DynaReg*. Every process has a local replica of *sm*, and we use consensus to agree on *sm*'s state transitions. Notice that each process is equipped with a failure detector FD of class $\diamond P^D$, so consensus is solvable under the assumption of a correct majority in a given configuration [53].

Each instance of consensus runs in some static configuration *c* and is associated with a unique timestamp. A process participates in a consensus instance by invoking a *propose* operation with the appropriate configuration and timestamp, as well as its proposed decision value. Consensus then responds with a *decide* event, so that the following properties are satisfied: *Uniform Agreement* – every two decisions are the same. *Validity* – every decision was previously proposed by one of the processes in *c*. *Termination* – if a majority of *c* is correct, then eventually every correct process in *c* decides. We further assume that a consensus instance does not decide until a majority of the members of the configuration propose in it.

The *sm* (lines 2-5 in Algorithm 7) keeps track of dynaReg's value in a variable *val*, and the configuration in a variable *cng*, containing both a list of processes, *cng.mem*, and a set of removed processes, *cng.rem*. Write operations change *val*, and reconfig operations change *cng*. A consensus decision may bundle a number of operations to execute as a single state transition of *sm*. The number of state transitions executed by *sm* is stored in the variable *ts*. Finally, the array *lastOps* maps every process p in *cng.mem* to the sequence number (based on p's local count) of p's last operation that was performed on the emulated DynaReg together with its result.

Each process partakes in at most one consensus at a time; this consensus is associated with timestamp *sm.ts* and runs in *sm.cng.mem*. In every consensus, up to $|sm.cng.mem|$ ordered operations on the emulated DynaReg are agreed upon, and *sm*'s state changes according to the agreed operations. A process's *sm* may change either when consensus

decides or when the process receives a newer *sm* from another process, in which case it skips forward. So *sm* goes through the same states in all the processes, except when skipping forward. Thus, for every two processes p_k, p_l, if $sm_k.ts = sm_l.ts$, then $sm_k = sm_l$. (A subscript i indicates the variable is of process p_i.)

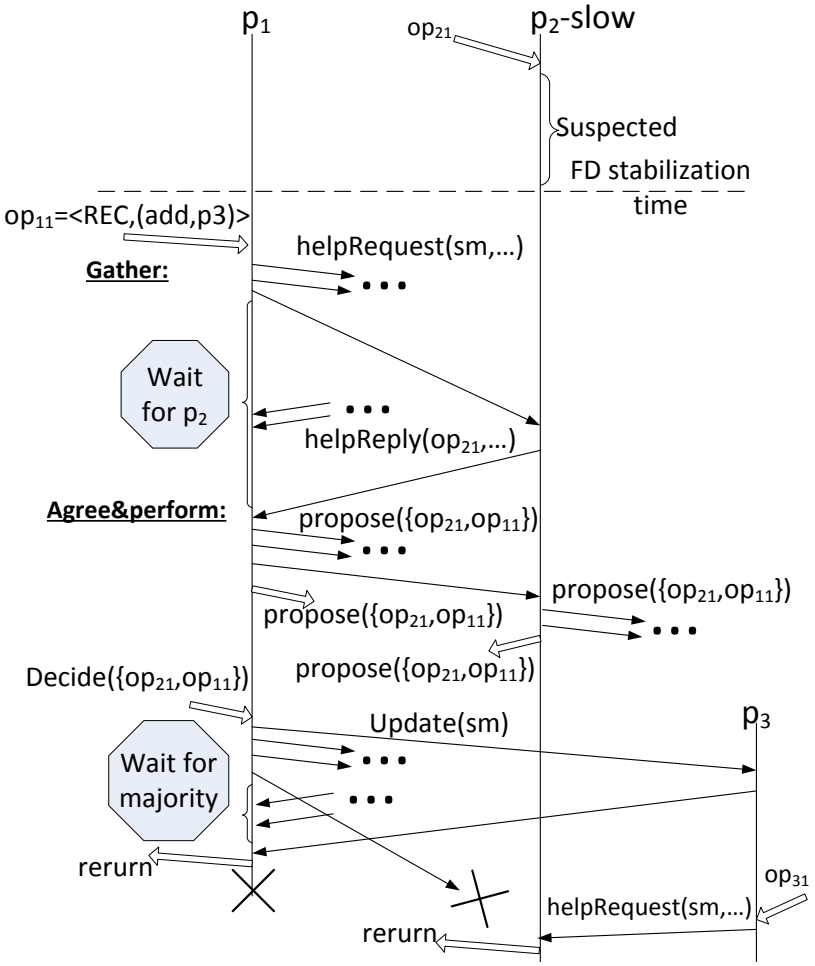


Figure 4.3: Flow illustration: process p_2 is slow. After stabilization time, process p_1 helps it by proposing its operation. Once p_2's operation is decided, it is reflected in every up-to-date *sm*. Therefore, even if p_1 fails before informing p_2, p_2 receives from the next process that performs an operation, namely p_3, an *sm* that reflects its operation, and thus returns. Line arrows represent messages, and block arrows represent operation or consensus invocations and responses.

Helping The problematic scenario in the impossibility proof of Section 4.2 occurs because of endless reconfig operations, where a slow process is never able to communicate with members of its configuration before they are removed. In order to circumvent this problem, we use FD to implement a helping mechanism. When proposing an op-

eration, process p_i tries to help other processes in two ways: first, it helps them complete operations they may have successfully proposed in previous rounds (consensuses) but have not learned about their outcomes; and second, it proposes their new operations. To achieve the first, it sends a helping request with its *sm* to all other processes in $sm_i.cng.mem$. For the second, it waits for each process to reply with a help reply containing its latest invoked operation, and then proposes all the operations together. Processes may fail or be removed, so p_i cannot wait for answers forever. To this end, we use FD. For every process in $sm_i.cng.mem$ that has not been removed, p_i repeatedly inquires FD and waits either for a reply from the process or for an answer from FD that the process has failed. Notice that the strong completeness property guarantees that p_i will eventually continue, and strong accuracy guarantees that every slow active process will eventually receive help in case of endless reconfig operations.

Nevertheless, if the number of reconfig operations is finite, it may be the case that some slow process is not familiar with any of the correct members in the current configuration, and no other process performs an operation (hence, no process is helping). To ensure progress in such cases, every correct process periodically sends its *sm* to all processes in its $sm.cng.mem$.

State survival Before the reconfig operation can complete, the new *sm* needs to propagate to a majority of the new configuration, in order to ensure its survival. Therefore, after executing the state transition, p_i sends sm_i to $sm_i.cng$ members and waits until it either receives acknowledgements from a majority or learns of a newer *sm*. Notice that in the latter case, consensus in $sm_i.cng.mem$ has decided, meaning that at least a majority of $sm_i.cng.mem$ has participated in it, and so have learned of it.

Flow example The algorithm flow is illustrated in Figure 4.3. In this example, a slow process p_2 invokes operation op_{21} before FD's stabilization time, ST. Process p_1 invokes operation $op_{11} = \langle add, p_3 \rangle$ after ST. It first sends *helpRequest* to p_2 and waits for it to reply with *helpReply*. Then it proposes op_{21} and op_{11} in a consensus. When *decide* occurs, p_1 updates its *sm*, sends it to all processes, and waits for majority. Then op_{11} returns and p_1 fails before p_2 receives its update message. Next, p_3 invokes a *reconfig* operation, but this time when p_2 receives *helpRequest* with the up-to-date *sm* from p_3, it notices that its operation has been performed, and op_{21} returns.

Detailed description

The data structure of process p_i is given in Algorithm 7. The type *Ops* defines the representation of operations. The emulated state machine, sm_i, is described above. Integer $opNum_i$ holds the sequence number of p_i's current operation; ops_i is a set that contains operations that need to be completed for helping; the flag $pend_i$ is a boolean that indicates whether or not p_i is participating in an ongoing consensus; and $myOp_i$ is the latest operation invoked at p_i.

Algorithm 7 Data structure of process p_i

1: $Ops \triangleq \{\langle RD, \perp\rangle\} \cup \{\langle WR, v\rangle \mid v \in \mathbb{V}\} \cup \{\langle REC, c\rangle \mid c \subset Changes\}$
2: $sm_i.ts \in \mathbb{N}$, initially 0
3: $sm_i.value \in \mathbb{V}$, initially v_0
4: $sm_i.cng = \langle mem, rem\rangle$, where $mem, rem \subset \Pi$, initially $\langle \Pi_0, \{\}\rangle$
5: $sm_i.lastOps$ is a vector of size $|sm_i.cng.mem|$, where $\forall p_j \in sm_i.cng.mem$, $sm_i.lastOps[j] = \langle num, res\rangle$, where $num \in \mathbb{N}$, $res \in \mathbb{V} \cup \{\text{“ok”}\}$, initially $\langle 0, \text{“ok”}\rangle$
6: $pend_i \in \{true, false\}$, initially $false$
7: $opNum_i \in \mathbb{N}$, initially 0
8: $ops_i \subset \Pi \times Ops \times \mathbb{N}$, initially $\{\}$
9: $myOp_i \in operation$, initially $\perp$

The algorithm of process p_i is presented in Algorithms 8 and 9. We execute every event handler, (operation invocation, message receiving, and consensus decision), atomically excluding wait instructions; that is, other event handlers may run after the handler completes or during a wait (lines 16,18,27 in Algorithm 8). The algorithm runs in two phases. The first, *gather*, is described in Algorithm 8 lines 11–16 and in Algorithm 9 lines 52–58. Process p_i first increases its operation number $opNum_i$, writes op together with $opNum_i$ to the set of operations ops_i, and sets $myOp_i$ to be op. Then it sends $\langle$“helpRequest”, $\ldots\rangle$ to every member of $A = sm_i.cng.mem$ (line 15), and waits for each process in A that is not suspected by the FD or removed to reply with $\langle$“helpReply”, $\ldots\rangle$. Notice that sm_i may change during the wait because messages are handled, and p_i may learn of processes that have been removed.

When $\langle$“helpRequest”, $num, sm\rangle$ is received by process $p_j \neq p_i$, if the received sm is newer than sm_j, then process p_j adopts sm and abandons any previous consensus. Either way, p_j sends $\langle$“helpReply”, $\ldots\rangle$ with its current operation $myOp_j$ in return.

Upon receiving $\langle$“helpReply”, $opNum_i, op, num\rangle$ that corresponds to the current operation number $opNum_i$, process p_i adds the received operation op, its number num, and the identity of the sender to the set ops_i.

At the end of this phase, process p_i holds a set of operations, including its own, that it tries to agree on in the second phase (the order among this set is chosen deterministically, as explained below). Note that p_i can participate in at most one consensus per timestamp, and its propose might end up not being the decided one, in which case it may need to propose the same operations again. Process p_i completes op when it discovers that op has been performed in sm_i, whether by itself or by another process.

The second phase appears in Algorithm 8 lines 17–28, and in Algorithm 9 lines 31–51. In line 17, p_i checks if its operation has not been completed yet. In line 18, it waits until it does not participate in any ongoing consensus ($pend_i$=$false$) or some other process helps it complete op. Recall that during a wait, other events can be handled. So if a message with an up-to-date sm is received during the wait, p_i adopts the sm. In case op has been completed in sm, p_i exits the main while (line 19). Otherwise, p_i waits until it does not participate in any ongoing consensus. This can be the case if (1) p_i has not proposed yet, (2) a message with a newer sm was received and a previous consensus was subsequently abandoned, or (3) a *decide* event has been handled. In all cases, p_i marks that it now

participates in consensus in line 20, prepares a new request Req with the operations in ops_i that have not been performed yet in sm_i in line 27, proposes Req in the consensus associated with $sm_i.ts$, and sends $\langle$"propose", ...$\rangle$ to all the members of $sm_i.cng.mem$.

Algorithm 8 Process p_i's algorithm: performing operations

```
10: upon invoke operation(op) do
                                                    ▷ phase 1: gather
11:     opNum_i ← opNum_i + 1
12:     ops_i ← {⟨p_i, op, opNum_i⟩}
13:     myOp_i ← op
14:     A ← sm_i.cng.mem
15:     for all p ∈ A send ⟨"helpRequest", opNum_i, sm_i⟩ to p
16:     for all p ∈ A wait for ⟨"helpReply", opNum_i, ...⟩ from p or p is suspected or p ∈ sm_i.cng.rem
                                                    ▷ phase 2: agree&perform
17:     while sm_i.lastOps[i].num ≠ opNum_i
18:         wait until ¬pend_i or sm_i.lastOps[i].num = opNum_i
19:         if sm_i.lastOps[i].num = opNum_i then break
20:         pend_i ← true
21:         Req ← {⟨p_j, op, num⟩ ∈ ops_i | num > sm_i.lastOps[j].num}
22:         propose(sm_i.cng, sm_i.ts, Req)
23:         for all p ∈ sm_i.cng.mem send ⟨"propose", sm_i, Req⟩ to p
24:     if op.type = REC
25:         ts ← sm_i.ts
26:         for all p ∈ sm_i.cng.mem send ⟨"update", sm_i, opNum_i⟩ to p
27:         wait for ⟨"ACK", opNum_i⟩ from majority of sm_i.cng.mem or sm_i.ts > ts
28:     return sm_i.lastOps[i].res

29: periodically:
30:     for all p ∈ sm_i.cng.mem send ⟨"update", sm_i, ⊥⟩ to p
```

When $\langle$"propose", $sm, Req \ldots\rangle$ is received by process $p_j \neq p_i$, if the received sm is more updated than sm_j, then process p_j adopts sm, abandons any previous consensus, proposes Req in the consensus associated with $sm.ts$, and forwards the message to all other members of $sm_j.cng.mem$. The same is done if sm is identical to sm_j and p_j has not proposed yet in the consensus associated with $sm_j.ts$. Otherwise, p_j ignores the message.

The event $decide_i(sm.cng, sm_i.ts, Req)$ indicates a decision in the consensus associated with $sm_i.ts$. When this occurs, p_i performs all the operations in Req and changes sm_i's state. It sets the value of the emulated DynaReg, $sm_i.value$, to be the value of the *write* operation of the process with the lowest id, and updates $sm_i.cng$ according to the *reconfig* operations. In addition, for every $\langle p_j, op, num\rangle \in Req$, p_i writes to $sm_i.lastOps[j]$, num and op's response, which is "ok" in case of a *write* or a *reconfig*, and $sm_i.value$ in case of a *read*. Next, p_i increases $sm_i.ts$ and sets $pend_i$ to false, indicating that it no longer participates in any ongoing consensus.

Finally, after op is performed, p_i exits the main while. If op is not a *reconfig* operation, then p_i returns the result, which is stored in $sm_i.lastOps[i].res$. Otherwise, before returning, p_i has to be sure that a majority of $sm_i.cng.mem$ receives sm_i. It sends $\langle$"update", $sm, \ldots\rangle$ to all the processes in $sm_i.cng.mem$ and waits for $\langle$"ACK", ...$\rangle$ from a majority of them. Notice that it may be the case that there is no such correct majority due to later reconfig operations and failures, so, p_i stops waiting when a more updated sm is

received, which implies that a majority of $sm_i.cng.mem$ has already received sm_i (since a majority is needed in order to solve consensus).

Upon receiving $\langle$"update", $sm, num\rangle$ with a new sm from process p_i, process p_j adopts sm and abandons any previous consensus. In addition, if $num \neq \perp$, p_j sends $\langle$"ACK", $num\rangle$ to p_i (Algorithm 9 lines 59–63).

Beyond handling operations, in order to ensure progress in case no operations are invoked from some point on, every correct process periodically sends $\langle$"update", $sm, \perp\rangle$ to all processes in its $sm.cng.mem$ (Algorithm 8 line 30).

Algorithm 9 Process p_i's algorithm: event handlers

```
31: upon decide_i(sm_i.cng, sm_i.ts, Req) do
32:     W ← {⟨p, value⟩ | ⟨p, ⟨WR, value⟩, num⟩ ∈ Req}
33:     if W ≠ {}                          ▷ deterministically choose one of the writes to be the last
34:         sm_i.value ← value with smallest p in W
35:     for all ⟨p_j, op, num⟩ ∈ Req                                          ▷ apply op to sm
36:         if op.type = WR
37:             sm_i.lastOps[j] ← ⟨num, "ok"⟩
38:         else if op.type = RD
39:             sm_i.lastOps[j] ← ⟨num, sm_i.value⟩
40:         else
41:             sm_i.cng.rem ← sm_i.cng.rem ∪ {p | ⟨remove, p⟩ ∈ op.changes}
42:             sm_i.cng.mem ← sm_i.cng.mem ∪ {p | ⟨add, p⟩ ∈ op.changes} \ sm_i.cng.rem
43:             sm_i.lastOps[j] ← ⟨num, "ok"⟩
44:     sm_i.ts ← sm_i.ts + 1
45:     pend_i ← false

46: upon receiving ⟨"propose", sm, Req⟩ from p_j do
47:     if (sm_i.ts > sm.ts) or (sm_i.ts = sm.ts ∧ pend_i = true) then return
48:     sm_i ← sm
49:     pend_i ← true
50:     propose(sm_i.cng, sm_i.ts, Req)
51:     for all p ∈ sm_i.cng.mem send ⟨"propose", sm_i, Req⟩ to p

52: upon receiving ⟨"helpRequest", num, sm⟩ from p_j do
53:     if sm_i.ts < sm.ts then                                               ▷ learn new sm
54:         sm_i ← sm
55:         pend_i ← false
56:     send ⟨"helpReply", num, myOp_i, opNum_i⟩

57: upon receiving ⟨"helpReply", opNum_i, op, num⟩ from p_j do
58:     ops_i ← ops_i ∪ ⟨p_j, op, num⟩

59: upon receiving ⟨"update", sm, num⟩ from p_j do
60:     if sm_i.ts < sm.ts then                                               ▷ learn new sm
61:         sm_i ← sm
62:         pend_i ← false
63:     if num ≠ ⊥ then send ⟨"ACK", num⟩ to p_j
```

4.3.3 Correctness proof

Atomicity

Every operation is uniquely defined by the process that invoked it and its local number. During the proof we refer to operation op invoked by process p_i with local number $opNum_i = n$ as the tuple $\langle p_i, op, n \rangle$. We begin the proof with three lemmas that link completed operations to *sm* states.

Lemma 32. *Consider operation op invoked by some process p_i in r with local number $opNum_i = n$. If op returns in r at time t, then there is at least one request* Req *that contains $\langle p_i, op, n \rangle$ and has been chosen in a consensus in r before time t.*

Proof. When operation op return, $sm_i.lastOps[i].num = n$ (line 17 or 18 in Algorithm 8). Processes update *sm* during a decide handler, or when a newer *sm* is received, and the first update occurs when some process p_j writes n to $sm_j.lastOps[i].num$ during a decide handler. In the decide handler, n is written to $sm.lastOps[i].num$ when the chosen request in the corresponding consensus contains $\langle p_i, op, n \rangle$.

□

Lemma 33. *For two processes p_i, p_j, let t be a time in a run r in which neither p_i or p_j is executing a decide handler. Then at time t, if $sm_i.ts = sm_j.ts$, then $sm_i = sm_j$.*

Proof. We prove by induction on timestamps. Initially, all correct processes have the same *sm* with timestamp 0. Now consider timestamp TS, and assume that for every two processes p_i, p_j at any time not during the execution of decide handlers, if $sm_i.ts = sm_j.ts = TS$, then $sm_i = sm_j$. Processes increase their $sm.ts$ to $TS + 1$ either at the end of a *decide* handler associated with TS or when they receive a message with *sm* s.t. $sm.ts = TS + 1$. By the agreement property of consensus and by the determinism of the algorithm, all the processes that perform the *decide* handler associated with TS perform the same operations, and therefore move *sm* (at the end of the handler) to the same state. It is easy to show by induction that all the processes that receive a message with *sm* s.t. $sm.ts = TS + 1$ receive the same *sm*. The lemma follows.

□

Observation 10. *For two process p_i, p_j, let sm_1 and sm_2 be the values of sm_j at two different times in a run r. If $sm_1.ts \geq sm_2.ts$, then $sm_1.lastOps[i].num \geq sm_2.lastOps[i].num$.*

Lemma 34. *Consider operation $\langle p_i, op, opNum_i \rangle$ invoked in r with $opNum_i = n$. Then $\langle p_i, op, n \rangle$ is part of at most one request that is chosen in a consensus in r.*

Proof. Assume by way of contradiction that $\langle p_i, op, n \rangle$ is part of more than one request that is chosen in a consensus in r. Now consider the earliest one, *Req*, and assume that it is chosen in a consensus associated with timestamp TS. At the end of the *decide* handler associated with timestamp TS, $sm.lastOps[i].num = n$ and the timestamp is increased to $TS + 1$. Thus, by Lemma 33 $sm.lastOps[i].num = n$ holds for every *sm* s.t. $sm.ts =$

$TS + 1$. Consider now the next request, Req_1, that contains $\langle p_i, op, n \rangle$, and is chosen in a consensus. Assume that this consensus associated with timestamp TS', and notice that $TS' > TS$. By the validity of consensus, this request is proposed by some process p_j, when $sm_j.ts$ is equal to TS'. By Observation 10, at this point $sm_j.lastOps[i].num \geq n$, and therefore p_j does not include $\langle p_i, op, n \rangle$ in Req_1 (line 27 in Algorithm 8). A contradiction. □

Based on the above lemmas, we can define, for each run r, a linearization σ_r, where operations are ordered as they are chosen for execution on sm's in r.

Definition 18. *For a run r, we define the sequential run σ_r to be the sequence of operations decided in consensus instances in r, ordered by the order of the chosen requests they are part of in r. The order among operations that are part of the same chosen request is the following: first all* writes*, then all* reads*, and finally, all* reconfig *operations. Among each type, operations are ordered by the process ids of the processes that invoked them, from the highest to the lowest.*

Note that for every run r, the sequential run σ_r is well defined. Moreover, σ_r contains every completed operation in r exactly once, and every invoked operation at most once.

In order to prove atomicity we show that (1) σ_r preserves r's real time order (lemma 35); and (2) every *read* operation rd in r returns the value that was written by the last *write* operation that precedes rd in σ_r, or $\perp$ if there is no such operation (lemma 36).

Lemma 35. *If operation op_1 returns before operation op_2 is invoked in r, then op_1 appears before op_2 in σ_r.*

Proof. By Lemma 32, op_1 is part of a request Req_1 that is chosen in a consensus before op_2 is invoked, and thus op_2 cannot be part of Req_1 or any other request that is chosen before Req_1. Hence op_1 appears before op_2 in σ_r. □

Lemma 36. *Consider read operation $rd = \langle p_i, RD, n \rangle$ in r, which returns a value v. Then v is written by the last write operation that precedes rd in σ_r, or $v = \perp$ if there is no such operation.*

Proof. By Lemmas 32 and 34, rd is part of exactly one request Req_1 that is chosen in a consensus, associated with some timestamp TS. Thus $sm.lastOps[i]$ is set to $\langle n, val \rangle$ in the *decide* handler associated with TS. By Lemma 33, $sm.lastOps[i] = \langle n, val \rangle$ for all sm s.t. $sm.ts = TS + 1$. By Lemma 34 and since we consider only well-formed runs, $sm_i.lastOps[i] = \langle n, val \rangle$ when rd returns, and therefore rd returns val. Now consider three cases:

- There is no *write* operation in Req_1 or in any request that was chosen before Req_1 in r. In this case, there is no *write* operation before rd in σ_r, and no process writes to $sm.value$ before $sm.lastOps[i]$ is set to $\langle n, val \rangle$, and therefore, rd returns $\perp$ as expected.

- There is a *write* operation in Req_1 in r. Consider the *write* operation w in Req_1 that is invoked by the process with the lowest id, and assume its argument is v'. Notice that w is the last *write* that precedes *rd* in σ_r. By the code of the *decide* handler, $sm.value$ equals v' at the time when $sm.lastOps[i]$ is set to $\langle n, val \rangle$. Therefore, $val = v'$, *rd* returns the value that is written by the last *write* operation that precedes it in σ_r.

- There is no *write* operation in Req_1, but there is a request that contains a *write* operation and is chosen before Req_1 in r. Consider the last such request Req_2, and consider the *write* operation w invoked by the process with the lowest id in Req_2. Assume that w's argument is v', and Req_2 was chosen in a consensus associated with timestamp TS' (notice that $TS' < TS$). By the code of the *decide* handler and Lemma 33, in all the sm's s.t. $sm.ts = TS' + 1$, the value of $sm.value$ is v'. Now, since there is no *write* operation in any chosen request between Req_2 and Req_1 in r, no process writes to $sm.value$ when $TS' < sm.ts < TS$. Hence, when $sm.lastOps[i]$ is set to $\langle n, val \rangle$, $sm.value$ equals v', and thus $val = v'$. Therefore, *rd* returns the value that is written by the last *write* operation that precedes *rd* in σ_r.

□

Corollary 12. *Algorithms 7–9 implement an atomic storage service.*

Liveness

Consider operation op_i invoked at time t by a correct process p_i in run r. Notice that r is a run with either infinitely or finitely many invocations. We show that, in both cases, if p_i is active in r, then op_i returns in r.

We associate the addition or removal of process p_j by a process p_i with the timestamp that equals $sm_i.ts$ at the time when the operation returns. The addition of all processes in P_0 is associated with timestamp 0.

First, we consider runs with infinitely many invocations. In Lemma 37, we show that for every process p, every sm associated with a larger timestamp than p's addition contains p in $sm.cng.mem$. In Observation 3, we show that in a run with infinitely many invocations, for every timestamp ts, there is a completed operation that has a bigger timestamp than ts at the time of the invocation. Moreover, after the stabilization time of the FD, operations must help all the slow active processes in order to complete. In Lemma 38, we use the observation to show that any operation invoked in a run with infinitely many invocations returns.

Next, we consider runs with finitely many invocations. We show Lemma 39 that eventually all the active members of the last sm adopt it. Then, in Lemma 40, we show that every operation invoked by an active process completes. Finally, Theorem 9, stipulates that the algorithm satisfies wait-freedom.

Lemma 37. *Assume the addition of p_i is associated with timestamp TS in run r. If p_i is active, then $p_i \in sm.cng.mem$ for every sm s.t. $sm.ts \geq TS$.*

Proof. The proof is by induction on $sm.ts$. **Base:** If $p_i \in P_0$, then $p_i \in sm.cng.mem$ for all sm s.t. $sm.ts = 0$. Otherwise, $\langle add, p_i \rangle$ is part of a request that is chosen in a consensus associated with timestamp $TS' = TS - 1$, and thus, by Lemma 33, $p_i \in sm.cng.mem$ for all sm s.t. $sm.ts = TS' + 1 = TS$. **Induction:** Process p_i is active, so no process invokes $\langle remove, p_i \rangle$, and therefore, together with the validity of consensus, no chosen request contains $\langle remove, p_i \rangle$. Hence, if $p_i \in sm.cng.mem$ for sm with $sm.ts = k$, then $p_i \in sm.cng.mem$ for every sm s.t. $sm.ts = k + 1$.

□

Claim 3. *Consider a run r of the algorithm with infinitely many invocations. Then for every time t and timestamp TS, there is a completed operation that is invoked after time t by a process with $sm.ts > TS$ at the time of the invocation.*

Proof. Recall that r is well-formed and only processes in $V(t).join$ can invoke operations at time t. Therefore, there are infinitely many completed operations in r. Since a finite number of operations are completed with each timestamp, the claim follows.

□

Lemma 38. *Consider an operation op_i invoked at time t by an active process p_i in a run r with infinitely many invocations. Then op_i completes in r.*

Proof. Assume by way of contradiction that p_i is active and op_i does not complete in r. Assume w.l.o.g. that p_i's addition is associated with timestamps TS and op_i is invoked with $opNum_i = n$. Consider a time $t' > t$ after p_i invokes op_i and the FD has stabilized. By Claim 3, there is a completed operation op_j in r, invoked by some process p_j at a time $t'' > t'$ when $sm_j.ts > TS$, whose completion is associated with timestamp TS'. By Lemma 37, $p_i \in sm_j.cng.mem$, at time t''. Now by the algorithm and by the eventual strong accuracy property of the FD, p_j proposes op_j and op_i in the same request, and continues to propose both of them until one is selected. Note that it is impossible for op_j to be selected without op_i since any process that helps p_j after stabilization also helps p_i. Hence, since op_j completes, they are both performed in the same *decide* handler. The run is well-formed, so p_i does not invoke operations that are associated with $opNum_i > n$. Hence, following the time when op_i is selected, for all sm s.t. $sm.ts > TS'$, $sm.lastOps[i].num = n$. Now, again by Claim 3, consider a completed operation op_k in r, that is invoked by some process p_k at time t''' after the stabilization time of the FD s.t. $sm_k.ts > TS'$ at time t'''. Operation op_k cannot complete until p_i receives p_k's sm. Therefore, p_i receives sm s.t. $sm.ts \geq TS'$, and thus $sm.lastOps[i].num = n$. Therefore, p_i learns that op_i was performed, and op_i completes. A contradiction.

□

We now proceed to prove liveness in runs with finitely many invocations.

Definition 19. *For every run r of the algorithm, and for any point t in r, let TS_t be the timestamp associated with the last consensus that made a decision in r before time t. Define sm^t, at any point t in r, to be the sm's state after the completion of the decide handler associated with timestamp TS_t at any process. By Lemma 33, sm^t is unique. Recall that sm^0 is the initial state.*

Claim 4. *For every run r of the algorithm, and for any point t in r, there is a majority of $sm^t.cng.mem$ M s.t. $M \subseteq (V(t).membership \cup P(t).join) \setminus F(t)$.*

Proof. By the code of the algorithm, for every run r and for any point t in r, $V(t).membership \subseteq sm^t.cng.mem$ and $sm^t.cng.mem \cap V(t).remove = \{\}$. The claim follows from failure condition.

□

Observation 11. *Consider a run r of the algorithm with finitely many invocations. Then there is a point t in r s.t. for every $t' > t$, $sm^t = sm^{t'}$. Denote this sm to be $\hat{sm}$.*

The following lemma follows from Lemma 33, Claim 4, and the periodic update messages; for space limitations, we omit its proof.

Lemma 39. *Consider a run r of the algorithm with finitely many invocations. Then eventually for every active process $p_i \in \hat{sm}.cng.mem$, $sm_i = \hat{sm}$.*

Lemma 40. *Consider an operation op_i invoked at time t by an active process p_i in a run r with finitely many invocations. Then op_i completes in r.*

Proof. By Lemma 37, $p_i \in \hat{sm}.cng.mem$, and by Lemma 39, there is a point t' in r s.t. $sm_i = \hat{sm}$ for all $t \geq t'$. Assume by way of contradiction that op_i does not complete in r. Therefore, op_i is either stuck in one of its waits or continuously iterates in a while loop. In each case, we show a contradiction. Denote by *con* the consensus associated with timestamp $\hat{sm}.ts$. By definition of $\hat{sm}$, no decision is made in *con* in r.

- Operation op_i waits in line 16 (Algorithm 8) forever. Notice that $\hat{sm}.cng.rem$ contains all the process that were removed in r, so, after time t', p_i does not wait for a reply from a removed process. By the strong completeness property of FD, p_i does not wait for faulty processes forever. A contradiction.
- Operation op_i waits in line 18 (Algorithm 8) forever. Notice that from time t' till p_i proposes in *con*, $pend_i$=*false*. Therefore, p_i proposes in *con* in line 22 (Algorithm 8), and waits in line 18 after the propose. By Observation 4, there is a majority M of $\hat{sm}.cng.em$ s.t. $M \subseteq V(t).membership \cup P(t).join \setminus F(t)$. Therefore, by the termination of consensus, eventually a decision is made in *con*. A contradiction to the definition of $\hat{sm}$.
- Operation op_i remains in the while loop in line 17 (Algorithm 8) forever. Since it does not waits in line 18 (Algorithm 8) forever, op_i proposes infinitely many times, and since each propose is made in a different consensus and p_i can propose in a consensus beyond the first one only once a decision is made in the previous one, infinitely many decisions are made in r. A contradiction to the definition of $\hat{sm}$.

- Operation op_i waits in line 27 (Algorithm 8) forever. Consider two cases. First, $sm_i \neq \hat{sm}$ when p_i performs line 26 (Algorithm 8). In this case, p_i continues at time t', when it adopts $\hat{sm}$, because $sm_i.ts > ts$ hold at time t'. In the second case ($sm_i = \hat{sm}$ when p_i performs line 26), p_i sends *update* message to all processes in $\hat{sm}.cng.mem$, and waits for a majority to reply. By Observation 4, there is a correct majority in $\hat{sm}.cng.mem$, and thus p_i eventually receives the replies and continues. In both cases we have contradiction.

Therefore, p_i completes in r.

□

We conclude with the following theorem:

Theorem 9. *Algorithms 7–9 implement wait-free atomic dynamic storage.*

4.4 Conclusion

We proved that in an asynchronous API-based reconfigurable model allowing at least one failure, without restricting the number of reconfigurations, there is no way to emulate dynamic safe wait-free storage. We further showed how to circumvent this result using a dynamic eventually perfect failure detector: we presented an algorithm that uses such a failure detector in order to emulate a wait-free dynamic atomic MWMR register.

Our dynamic failure detector is (1) sufficient for this problem, and (2) can be implemented in a dynamic eventually synchronous [32] setting with no restriction on reconfiguration rate. An interesting question is whether a weaker such failure detector exists. Note that when the reconfiguration rate is bounded, dynamic storage is attainable without consensus, thus such a failure detector does not necessarily have to be strong enough for consensus.

Chapter 5

Dynamic Reconfiguration: Abstraction and Optimal Asynchronous Solution

Our experience with building real-life distributed systems repeatedly surfaces reconfiguration as an important issue whose practice is less understood than desired. Providing clean and efficient foundations and tools for reconfiguration is therefore a crucial enabler for today's distributed system management. We make a step towards providing such foundations in this work.

The goal of this work is to take a static fault-tolerant object like an atomic read/write register and turn it into a dynamic fault-tolerant one. A static object exposes an API (e.g., read/write) to its clients, and is emulated on top of a set of fault-prone servers (sometimes called base objects) via protocols like ABD [10]. We refer to the underlying set of fault-prone servers as a *configuration*. To convert a static object into a dynamic one, we first extend the object's API to support *reconfiguration*. Such an API is essential for administrators, who should be able to remove old or faulty servers and add new ones without shutting down the service. One of the challenges in formalizing dynamic models is to define a precise fault condition, so that an administrator who requests to remove a server s via a reconfiguration operation will know when she can switch s off without risking losing the object's state (e.g., the last written value to a read/write register).

To this end , we first define a clean dynamic failure model, in which an administrator can immediately switch a server s off once a reconfiguration operation that removes s completes. Then, we provide an abstraction for consensus-less reconfiguration in this model. To demonstrate the power of our *Reconfiguration* abstraction we use it to implement two dynamic atomic objects. First, we focus on the basic building block of a read/write register; thus, other (static) objects that can be emulated from read/write registers (e.g., atomic snapshots) can be made dynamic by replacing the underlying reg-

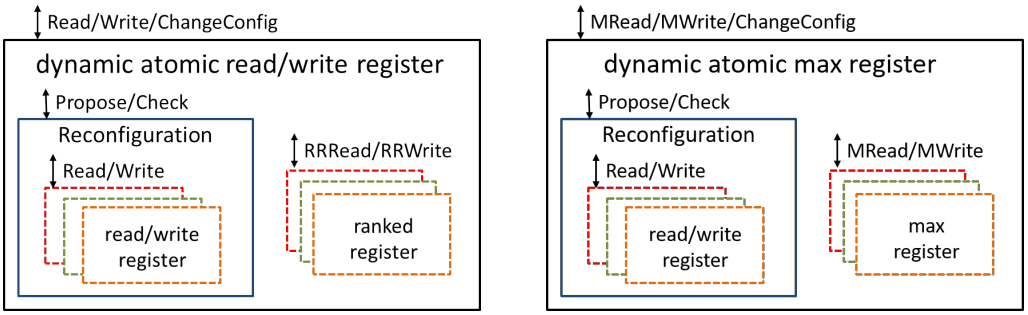
isters with dynamic ones. Second, we emulate a max-register [9], which on the one hand can be implemented asynchronously [10] (on top of fault-prone servers), and on the other hand cannot be emulated (for an unbounded number of clients) on top of a bounded set of read/write registers as we showed in Chapter 2[1]. Thus, a standalone implementation of dynamic max-registers is required.

Complexity. We present an optimal-complexity implementation of our Reconfiguration abstraction in asynchronous environments, which in turn leads to the first optimal implementation of a dynamic read/write register in this model. More concretely, faced with n administrator reconfiguration requests, the number of configurations that the dynamic object is implemented over is n; and the number of rounds (when the algorithm accesses underlying servers) per client operation is $O(n)$. A comparison with previous solutions appears in Section 5.1.

Dynamic fault model. In Section 5.2 we provide a succinct failure condition capturing a versatile set of faults under which the dynamic object's liveness is guaranteed. We define the dynamic fault model as an interplay between the object's implementation and its environment: New configurations are *introduced* by clients, (which are part of the object's environment). The object implementation then *activates* the requested configuration, at which point old configurations are *expired*. Between the time when a configuration is introduced and until it is expired, the environment can crash at most a minority of its servers. For example, when reconfiguring a register from configuration $\{A, B, C\}$ into $\{D, E, F\}$, initially a majority of $\{A, B, C\}$ must be available to allow read/write operations to complete. Then, when reconfiguration is triggered, $\{D, E, F\}$ is introduced, and subsequently, majorities of both configurations must be available, to allow state-transfer to occur. Finally, when the reconfiguration operation completes, leading to $\{D, E, F\}$'s activation, $\{A, B, C\}$ is expired, and every server in it may be immediately shutdown.

Reconfiguration abstraction. Since a configuration is a finite set of servers, we can use ABD [10] to emulate in each configuration a set of (static) atomic read/write registers (as well as max-registers), which are available as long as the configuration is not expired. The Reconfiguration abstraction, in contrast, is not tied to a specific configuration, but rather abstracts away the coordination among clients that wish to change the underlying set of servers (configuration) emulating the dynamic object. Its specification, which is formally defined in Section 5.3, exposes two API methods, *Propose* and *Check*. Clients use Propose to request changes to the configuration, and Check to learn of changes proposed by other clients. Both return a configuration and a set of *speculations*. The returned configuration reflects all previous proposals and possibly some ongoing ones. The less obvious return value of Reconfiguration is the speculation set. This set is required since there is no guarantee that all clients see the same sequence of configurations (indeed,

[1] A max-register for k clients requires at least k read/write registers.



(a) Dynamic atomic read/write register on top of the Reconfiguration abstraction.

(b) Dynamic atomic max-register on top of the Reconfiguration abstraction.

Figure 5.1: The Reconfiguration abstraction usage. Solid (dashed) blocks depict dynamic (resp. static) objects.

Reconfiguration is weaker than consensus). Therefore, a dynamic object implementation that uses Reconfiguration needs to read from every configuration that Check returned to any *other* client, and transfer the most up-to-date value read in any of these to the new configuration returned from Check. To this end, Reconfiguration returns a speculation set that includes all configurations previously returned to all clients (and possibly additional proposed ones).

In Section 5.4, we implement (1) a dynamic atomic read/write register on top of the Reconfiguration abstraction and static atomic ranked registers [26] (one in every configuration), and (2) a dynamic atomic max-register on top of Reconfiguration and static atomic max-registers. See Figure 5.1 for illustrations. In Section 5.5 we give an optimal consensus-less algorithm for Reconfiguration, which together with the read/write register emulation of Section 5.4 yields an optimal dynamic read/write register algorithm.

In summary, this chapter makes three contributions: it defines a failure condition that allows an administrator to shutdown removed servers; it introduces the Reconfiguration abstraction, which captures the essence of reconfiguration; and it presents an asynchronous optimal-complexity solution for dynamic atomic registers. Section 5.6 concludes the chapter.

5.1 Related Work

Model. The problem of object reconfiguration has gained growing attention in recent years [41, 56, 8, 66, 50, 40, 72, 37, 49, 70, 11, 17]. However, dynamic failure models do not always make it clear when exactly an administrator can shutdown a removed server. Early works supporting dynamic objects [56, 41, 24] simply assume that a configuration is available as long as some client may try to access it. SmartMerge [50] uses a shared non-reconfigurable auxiliary object (lattice agreement) that is forever available to all clients, meaning that a majority of the servers emulating this auxiliary object can never be switched off. DynaStore [8] was the only previous work to define dynamic failure conditions based on a reconfiguration API, but these conditions are complicated, and restrict reconfiguration attempts as well as failures. Moreover, DynaStore does not separate clients from servers as we do here. Following [37, 50], we formulate the problem in shared memory, which makes it easier to reason about and clearer.

Other works [11, 17] assume a broadcast mechanism for announcing joins instead of an API for adding and removing processes, and bound the rate of changes of the underlying set of servers; the latter is necessary if one wants to ensure liveness for all operations (as [11] does) – as we show in Chapter 4 no asynchronous reconfigurable service can ensure liveness unless the reconfiguration rate is limited in some way. Like many earlier works [8, 37, 50], in this chapter we do not explicitly bound the reconfiguration rate, and hence ensure liveness only if the number of reconfigurations is finite.

Abstractions. All previous works have considered reconfiguration in some specific context – state machine replication [53, 19, 20], read/write register emulation [8, 50, 37, 41], or atomic snapshot [70]. To the best of our knowledge, this work is the first to specify general dynamic objects as extensions of their static counterparts and to provide a general abstraction for dynamic reconfiguration. We note that although [37] define a reconfigure primitive intended to capture the core reconfiguration problem, that primitive is not sufficiently strong for implementing an atomic register, (in particular, since it does not require real-time order), and indeed, they do not implement their atomic register on top of it.

Dynamic register complexity. In a recent non-refereed tutorial [72], we give a generic formulation that allows us to compare the complexity of different algorithms [41, 56, 50, 37, 8], as follows: Given that n is the number of proposed configuration changes and m is the total number of operations (read/write/reconfig) invoked on the atomic register, DynaStore [8] goes through $O(min(mn, 2^n))$ configurations, and requires a constant number of operations in every configuration, so $O(min(mn, 2^n))$ is also DynaStore's operation complexity. Parsimonious SpSn [37] reduces the number of traversed configurations to $O(n)$, but since they invoke a linear number of operations in every configuration, their total operation complexity is $O(n^2)$.

Now notice that it is always possible to stagger reconfiguration proposals in a way that forces the system to go through $\Omega(n)$ configurations. The asymptotically optimal $O(n)$ operation complexity is straightforward to achieve in consensus-based solutions [41, 56, 24]. This complexity was also achieved by SmartMerge [50], but this was done using an auxiliary object that was assumed to be live indefinitely, i.e., was not reconfigurable in itself. Our algorithm is the first consensus-free and fully reconfigurable dynamic register algorithm with optimal complexity.

5.2 Dynamic Model

We consider a fault-prone shared memory model [48]: The system consists of an infinite set Π of *clients* (sometimes called processes), any number of which may fail by crashing, and an infinite set Φ of *servers* (sometimes called base objects) supporting arbitrary atomic low-level objects. Clients access servers via-low level operations (e.g., read/write), which may take arbitrarily long to arrive and complete, hence the system is asynchronous.

We address in the chapter two atomic objects: a classical fault tolerant read/write register and a max-register [9]. Both registers provide clients with two API methods: Read and Write in case of read/write register, and MRead and MWrite in case of max-register. In a well-formed execution, a client invokes API methods one at a time, though calls by different clients may be interleaved in real time. For a well-formed execution, there exists a serialization of all client operations that preserves the operations' real time order, such that (1) in case of read/write register a Read returns the value written in the latest Write preceding it, or $\perp$ if there is no preceding Write; and (2) in case of max-register an MRead returns the highest value written by an MWrite that precedes it, or $\perp$ if there is no preceding MWrite. (In case of max-registers, the values domain is ordered.)

Configurations. The universe of servers is infinite, but at any moment in time, a client chooses to interact with a subset of it. In our model, a *configuration* is a set of included and excluded servers, where configuration *membership* is the set of included and not excluded servers in the configuration. Formally:

Changes	$\triangleq$	$\{+s \mid s \in \Phi\} \cup \{-s \mid s \in \Phi\}$
Configuration	$\triangleq$	subset of *Changes*
C.membership	$\triangleq$	$\{s \mid +s \in C \wedge -s \notin C\}$

For example $C = \{+s_1, +s_2, -s_2, +s_3\}$ is a configuration representing the inclusion of servers s_1, s_2, and s_3, and the exclusion of s_2, and *C.membership* is $\{s_1, s_3\}$. Tracking excluded servers in addition to the configuration's membership is important in order to reconcile configurations suggested by different clients. The configuration size is the number of changes it includes– in this example, $|C| = 4$.

Dynamic fault model. A dynamic fault model is an interplay between the adversary's power and the following events, which are invoked as part of client operations:

introduce(*C*): indicates that *C* is going into use; and

activate(*C*): indicates that the state transfer to *C* is complete.

By convention we say that the initial configuration C_{init} is introduced and activated at time 0.

The above events govern the life-cycle of configurations. A configuration *C* becomes *activated* once an activate(*C*) event occurs. Note that not all introduced configurations are necessarily activated at some point. A configuration *C* becomes *expired* once activate(*D*) occurs s.t. *C* does not contain *D*. Intuitively, *D* reflects events (inclusions or exclusions) that are not reflected in *C*, and hence *C* has become "outdated". Our algorithm will enforce a containment order among activated configurations, and will thus ensure that the latest activated one is not expired.

The following two conditions constrain the power of the adversary:

Definition 20. *(liveness conditions)*

Availability: *The adversary can crash at most a minority of C.membership between the time when introduce(C) occurs and until C is expired.*

Weak Oracle: *When a client interacts with an expired configuration C, it either receives responses to calls from a majority of C.membership, or returns an exception notification* $\langle error, D \rangle$ *for some activated D, where* $C \not\supseteq D$.

Note that such an oracle (sometimes called directory service) is inherently required in order to allow slow clients to find non-expired configurations in an asynchronous system where old configurations may become unavailable [7, 70]. Our oracle definition is weak– in particular, the activated configuration it returns may itself be expired, and different clients may get different responses; it can be trivially implemented using a broadcast mechanism as assumed in some previous works [11, 17], and trivially holds if configurations must remain available as long as some client may access them, as in other previous works [41, 56, 24].

Static versus dynamic objects. A *static object* is one in which clients interact with a fixed configuration. In order to disambiguate a static object, scoped within a configuration C, from a dynamic one, we will label the methods of a static object with a "C.". For every configuration *C*, as long as a majority of *C.membership* is alive, clients can use ABD [10] to emulate (static) atomic registers on top of the servers in *C.membership*. We denote:

$C.x \leftarrow value$	A Write(*value*) operation to register x in configuration C
$C.x$	A Read of x
$C.collect(array)$	A bulk Read of all the registers in *array*

Since a complete array can be collected from servers using ABD in the same number of rounds as reading a single variable, we count a collect as a single operation for complexity purposes. Note that each register in the array is atomic in itself, but the collect is not atomic.

The methods of a dynamic object are not scoped with any configuration; it can start in some configuration and continue in a different one. A dynamic object's API includes a ChangeConfig operation that allows clients to change the set of servers implementing the object. The implementation of ChangeConfig is object-specific, because it needs to transfer the state of the object across configurations, e.g., the last written value in case of an atomic register.

Clients pass to ChangeConfig a parameter $Proposal \subset Changes$ containing a proposed set of configuration changes. ChangeConfig returns a configuration C s.t. (1) C is activated, (2) $C \supseteq Proposal$, and (3) every configuration introduced or activated by ChangeConfig consists of C_{init} plus a subset of changes proposed by clients before the operation returns.

The liveness guarantee of a dynamic object is that, assuming the number of ChangeConfig proposals is finite, every correct client's operation eventually completes. Note that if the number of ChangeConfig proposals is infinite, it is impossible to ensure liveness for all operations as we show in Chapter 4.

Usage example. Consider an administrator (a privileged client) who wants to switch server s off and invokes ChangeConfig$(\{-s\})$. By liveness, ChangeConfig completes, and by properties (1) and (2), it returns an activated configuration $C \supseteq \{-s\}$. The activation of C expires all configurations that do not contain C, and in particular, those that do not include $-s$. Hence, s is not part of the membership of any unexpired configuration, and by the availability condition, the administrator can safely switch s off immediately once ChangeConfig$(\{-s\})$ returns.

5.3 Reconfiguration Abstraction

We introduce a generic reconfiguration abstraction, which can be used for implementing dynamic objects as we illustrate in the next section. A Reconfiguration abstraction has two operations:

Propose(C, P**)** for a configuration C and a proposed set of changes P; and

Check(C**)** for a configuration C.

Propose is used to reconfigure the system, whereas Check is used in order to learn about other clients' reconfiguration attempts. Propose and Check invoke the introduce and activate events. Both Check and Propose return a pair of values $\langle D, S \rangle$, where D is a configuration and S is a *speculation set* containing configurations; when $\langle D, S \rangle$ is returned we

say that D is *nominated* by the operation that returns it. Intuitively, a nominated configuration is one that has been introduced and is a candidate for activation. By convention, we say that C_{init} is nominated at time 0. We assume that the first argument passed to both operations is a nominated configuration.

The first propert of Reconfiguration is validity, which (i) requires Propose(C, P) to include P in the returned nominated configuration; and (ii) does not allow configurations to include spurious changes not proposed by any client. Formally:

D_1 (Validity) (i) If Propose(C, P) returns $\langle D, S\rangle$ for some S, then $D \supseteq P$, and (ii) for every configuration D that is introduced or nominated by an operation op, for every $e \in D \setminus C_{init}$, there is a Propose$(C', P')$ for some C' that is invoked before op returns s.t. $e \in P'$.

The second property ensures that nominated configuration sizes monotonically increase over time, which is essential for real-time order of operations invoked on objects that use this abstraction:

D_2 (Real-time Order) A configuration D nominated by operation op is larger than or equal to every configuration nominated by an operation that strictly precedes op.

Since Reconfiguration is weaker than consensus, clients do not agree on a sequence of nominated configurations. Hence, in case some client c_1 proceeds to a configuration C', we want to ensure that if another client c_2 "skips" C', c_2 has C' in its speculation set, and can thus transfer any state that c_1 may have written there to the newer configuration c_2 nominates. This is captured by property S_1(ii) below. Property S_1(i) stipulates that these configurations are also introduced, ensuring a live majority in these configurations in order to allow state transfer.

S_1 (Speculation) If Check(C) or Propose(C, P) returns $\langle D, S\rangle$, then every $C' \in S$ is (i) introduced and (ii) S includes all nominated configurations C' s.t. $|C| \leq |C'| \leq |D|$. As a practical matter, if any C' between C and D has been activated, any C'' s.t. $|C''| < |C'|$ may be omitted.

In addition, we have to define when configurations are activated. Note that an activation of a new configuration leads to expiration of old ones, and thus to possible loss of information stored in them. Therefore, a configuration D is not immediately activated when a Propose returns $\langle D, S\rangle$ for some S. Instead, a configuration C is activated if Check(C) does not report any newer configuration:

A_1 (Activation) If Check(C) returns $\langle C, S\rangle$ for some S, then C is activated.

The liveness property of Reconfiguration is the same as in other dynamic objects [8, 50, 37, 70], namely, if the number of Propose operations is finite, then every operation by a correct client completes.

5.4 Building Dynamic Objects Using Reconfiguration

We first present a dynamic atomic read/write register emulation using Reconfiguration, then explain the modifications needed for supporting a dynamic atomic max-register [9], and finally provide a formal proof.

5.4.1 Dynamic atomic read/write register

Besides the Reconfiguration abstraction, our dynamic register implementation uses a (static) *ranked register* [26] emulation in every configuration, as illustrated in Figure 5.1a. A ranked register stores a tuple, called *version*, that consists of a value v and a monotonically increasing timestamp ts, and supports *RRRead*() and *RRWrite*(*version*) operations. The sequential specification of a *ranked register* is following: An *RRRead*() returns the version with the highest ts written by an *RRWrite* that precedes it, or $\perp$ if there is no preceding *RRWrite*. Like all static objects in our model, if the configuration where the ranked register is emulated expires, the oracle returns an error.

The basic framework for implementing the Read, Write, and ChangeConfig operations is a loop: (i) Check, (ii) read (using RRread) the highest version from all speculated configurations returned by Check, (iii) write (with RRWrite) the highest version to the configuration nominated by Check, (iv) repeat. The loop terminates when Check does not nominate a new configuration. The specific action of each of the three operations is as follows. A Read simply returns the value of the highest version at the end of the loop. A Write increments the timestamp and writes it with a new value at the beginning of the loop. ChangeConfig proposes a configuration change via Propose instead of Check in the first iteration.

Algorithm 10 Dynamic atomic read/write register using Reconfiguration.

```
   Client local variables:
1:    configuration C_curr, initially C_init
2:    TS = ℕ × Π with selectors num and id
3:    version ∈ 𝕍 × TS with selectors v and ts, initially ⟨v_0, ⟨0,client's id⟩⟩
4:    pickTS ∈ {true,false}, initially true.

   Code for client c_i ∈ Π:

5:  Read()
6:    transferState(Check(C_curr), ⊥)
7:    checkConfig()
8:    return version.v

9:  Write(v)
10:   transferState(Check(C_curr), v)
11:   checkConfig()
12:   pickTS ← true
13:   return ok

14: ChangeConfig(P)
15:   transferState(Propose(C_curr, P), ⊥)
16:   checkConfig()
17:   return C_curr

18: On ⟨error, D⟩ do
19:   C_curr ← D
20:   restart operation

21: procedure checkConfig()
22:   ⟨D, S⟩ ← Check(C_curr)
23:   while D! = C_curr do
24:     transferState(⟨D, S⟩, ⊥)
25:     ⟨D, S⟩ ← Check(C_curr)

26: procedure transferState(⟨D, S⟩, value)
27:   for each C ∈ S do
28:     tmp ← C.RRRead()
29:     if tmp.ts > version.ts then
30:       version ← tmp
31:   if value ≠ ⊥ ∨ pickTS = true then
32:     version ← ⟨value, ⟨version.ts.num +
      1, i⟩⟩
33:     pickTS ← false
34:   D.RRWrite(version)
35:   C_curr ← D
```

The pseudocode appears in Algorithm 10. The *transferState* method reads the register's version from the entire speculation set S and writes the latest version to the new configuration D. The *checkConfig* method repeatedly calls *transferState* until the configuration returned by *Check* stops changing. During the loop execution, an operation on an expired configuration may incur an exception, with a notification of the form $\langle error, D\rangle$ (see line 18). In this case, the loop is aborted and the operation starts over at configuration D. In case write is restarted after it has chosen a new timestamp, it skips the timestamp selection step.

We say that a configuration C becomes *stable* when some version is written to C in step (iii). We refer to the first version written to C as the *opening* version of C. Consider a completed operation (Read, Write, or ChangeConfig) op and let C be the last configuration in which op writes some version v, we say that op commits v in C when it completes. The correctness of the register emulation is based on the following key invariant:

Invariant 2. *For every stable configuration C, the opening version of C is higher than or equal to the highest version committed in any configuration C' s.t. $|C'| < |C|$.*

In other words, a larger stable configuration always holds a newer (or equal) version of

the register's value than that committed in a smaller activated one.

Algorithm 11 Dynamic atomic max-register using Reconfiguration.

Client local variables:
1: configuration C_{curr}, initially C_{init}
2: $value \in \mathbb{V}$, initially v_0

Code for client $c_i \in \Pi$:

3: **MRead()**
4: *transferState*$(Check(C_{curr}), \perp)$
5: *checkConfig*$()$
6: return $value$

7: **MWrite(v)**
8: *transferState*$(Check(C_{curr}), v)$
9: *checkConfig*$()$
10: return ok

11: **ChangeConfig(P)**
12: *transferState*$(Propose(C_{curr}, P), \perp)$
13: *checkConfig*$()$
14: return C_{curr}

15: **On** $\langle error, D \rangle$ **do**
16: $C_{curr} \leftarrow D$
17: restart operation

18: **procedure** *checkConfig*$()$
19: $\langle D, S \rangle \leftarrow Check(C_{curr})$
20: **while** $D! = C_{curr}$ **do**
21: *transferState*$(\langle D, S \rangle, \perp)$
22: $\langle D, S \rangle \leftarrow Check(C_{curr})$

23: **procedure** *transferState*$(\langle D, S \rangle, v)$
24: **if** $v \neq \perp$ **then**
25: $value \leftarrow v$
26: **for each** $C \in S$ **do**
27: $tmp \leftarrow C.MRead()$
28: **if** $tmp > value$ **then**
29: $value \leftarrow tmp$
30: *D.MWrite(value)*
31: $C_{curr} \leftarrow D$

Complexity. We measure complexity in terms of the number of accesses to low level objects, namely static atomic registers. Note that Read/Write/collect operations on static registers are emulated in a constant number of rounds using ABD. The complexity of the dynamic register's operations is determined by (1) the complexity of the operations inside the Checks invoked during the loop (plus possibly one Propose); and (2) the sum of the sizes of all speculation sets returned by Propose/Check operations in this loop (where the register's implementation performs Reads).

In a run with n ChangeConfig proposals, clearly, the best complexity we can hope for is $O(n)$. In the next section we present our algorithm for Reconfiguration, which achieves the asymptotically optimal $O(n)$ complexity.

5.4.2 Dynamic atomic max-register

The emulation of a max-register on top of Reconfiguration is similar to the read/write register emulation. It differs in how we keep and transfer the state, i.e., the register's value. First, instead of a (static) ranked register in each configuration, we use a (static) max-register. Second, instead of timestamps, we use the actual written values, that is, a writer writes its value in step (iii) only if it is higher than all the values read in step

(ii) (Otherwise, it transfers the highest value it read). The pseudocode appears in Algorithm 11.

5.4.3 Read/write register correctness proof

We start with notations:

Notation. We say that a version v is *higher* than a version v' if $v.ts > v'.ts$. An operation (Read, Write, or ChangeConfig) *stores* a version v in configuration C when it performs *C.RRWrite(v)* (line 34). A configuration C becomes *stable* when some operation stores a version in C. We say that the *opening* version of a stable configuration C is the first version that is stored in C. A configuration C *holds* a version v at time t if every *C.RRRead()* performed after time t returns a version that is higher than or equal to v. Consider a completed operation (Read, Write, or ChangeConfig) op and let C be the last configuration in which op stores some version v, we say that op *commits* v in C when it completes.

The following observation follows immediately from the specification of *ranked register*:

Observation 12. *A stable configuration holds at time t the highest version that was stored in it before time t.*

The following observation follows immediately from the definitions of activated, stable and nominated configurations, and the code:

Observation 13. *Every activated configuration is Stable and every stable configuration is nominated.*

The following observations follow immediately from the code:

Observation 14. *Every completed operation commits a version.*

Observation 15. *A Check and Propose are always called with a stable configuration.*

We now ready to prove the correctness of our register, which rely on the following invariant:

Invariant 2 (restated). *For every stable configuration C, the opening version of C is higher than or equal to the highest version committed in any configuration C′ s.t.* $|C'| < |C|$.

Proof. Assume in a way of contradiction that there is a time t at which the invariant does not hold. Let C_{st} be the smallest stable configuration that violate the invariant at time t, let vst be the opening version of C_{st}, and let op_{st} be the operation that stores v_{st} in C_{st}. Now let t_1^{st} be the time when a *Check* or a *Propose* performed by op_{st} returns $\langle C_{st}, S\rangle$ for some S for the first time. Denote this *Check* or *Propose* by CP_{st}, and let t_2^{st} be the time when op_{st} invokes procedure $transferState(\langle C_{st}, S\rangle, \perp)$ (line 26). Note that $t_1^{st} < t_2^{st} < t$.

By the contradiction assumption there is at least one version, committed in a smaller configuration than C until time t, that is higher than v_{st}. Let op_c be an operation that commits version $v_c < v_{st}$ in configuration C_c until time t s.t. $|C_c| < C_{st}$. Now let t_1^c be the time when op_c stores v in C_c and let $t_2^c > t_1^c$ be the time when op_c invokes $Check(C_c)$ for the last time. Note that this $Check(C_c)$ returns $\langle C_c, S\rangle$ for some S. Now consider two case:

- First, $t_1^{st} < t_2^c$. Meaning that op_c invokes a $Check(C)$ that nominates C after op_{st} nominates C_{st}. A contradiction to property D_2 (Real time order) of the Reconfiguration abstraction.
- Second, $t_1^{st} > t_2^c$. Therefore, $t_2^{st} > t_1^c$, meaning that op_{st} performs $transferState(\langle C_{st}, S\rangle, \perp)$ after op_c stores v in C_c. Let C_{in} be the input configuration to CP_{st}. Now consider two case:
  - Fist, $C_c \in S$. Since op_{st} performs $C_c.RRRead()$ (in procedure *transferState*) after t_1^c, and since *transferState* stores the highest version it reads, we get that v_{st} is higher than or equal to v_c. A contradiction.
  - Second, $C_c \notin S$. By property S_1 (Speculation) of the Reconfiguration abstraction, this case is possible only if (1) $|C_c| < |C_{in}| \leq |C_{st}|$ and $C_{in} \in S$ or (2) S includes an activated configuration C_a s.t. $|C_c| < |C_a| \leq |C_{st}|$. By Observations 13 and 15, C_a and C_{in} are stable. Therefore, in both cases, S includes a stable configuration C' s.t. $|C_c| < |C'| \leq |C_{st}|$. Now consider two case:
    1. First, $|C'| = |C_{st}|$, meaning that C_{st} is stable at time t_1^{st}. Therefore, some operation stores a version in C_{st} before op_{st}. A contradiction to v_{st} being the opening version of C_{st}.
    2. Second, $|C'| < |C_{st}|$. Since C_{st} is the smallest stable configuration that violates the invariant at time t, C_a's opening version is higher than or equal to v_c. By Observation 12, and since *transferState* stores the highest version it reads from configurations in S, we get that v_{st} is higher than or equal to v_c. A contradiction.

□

From Invariant 2 and Observation 12 we get the following corollary:

Corollary 13. *At every time t a stable configuration C holds a version that is higher than or equal to the highest version committed in any configuration C' s.t. $|C'| \leq |C|$ before time t.*

The following lemma follows from Corollary 13 and the specification of the reconfiguration abstraction:

Lemma 41. *Consider an operation op_1 that commits a version v_1 in configuration C_1, and an operation op_2 that begins after op_1 returns. Let $Check_2$ be a Check performed by op_2, and let*

$\langle C_2, S_2 \rangle$ be the valued it returns. Then, S_2 includes a stable configuration that holds a version higher than or equal to v_1.

Proof. Note that op_1 nominates C_1, and since op_1 precedes op_2, op_1 performs a Check that nominates C_1 before op_2 invokes $Check_2$. Thus, by property D_2 (real time order) of the reconfiguration abstraction, $|C_1| \leq |C_2|$. Moreover, by Observation 15 and property S_1 (speculation) of the reconfiguration abstraction, S_2 includes a stable configuration C_s s.t. $|C_s| \geq |C_1|$. By Corollary 13, C_s holds a version with a timestamp $ts_s \geq ts_1$.

□

Notice that every completed Write operation picks exactly one timestamp (line 32), every picked timestamp is unique (ties are broken by clients ids), and there are no two different versions with the same timestamp. Thus, we say that Write operations are *associated* with the timestamp they pick. Note also that a Read operation returns the value in the version is commits. Therefore, we say that Read operations are associated with the timestamps of the versions they commits. The following Observation follows immediately from the code:

Observation 16. *A completed Write operation that is associated with timestamp ts commits a version with timestamp $ts' \geq ts$.*

Lemma 42. *Consider two completed Write operations w_1, w_2 that are associated with timestamps ts_1, ts_2, respectively. If w_1 precedes w_2, then $ts_1 < ts_2$.*

Proof. Let ts_1^c be the timestamp of the version committed by w_1. By Observation 16, $ts_1^c \geq ts_1$. Let $Cheak_2$ be the last check w_2 performs before picking a timestamp, and let $\langle C_2, S_2 \rangle$ be its return value. By Lemma 41, S_2 includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1^c \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 > ts_s \geq ts_1^c \geq ts_1$.

□

Lemma 43. *Consider two completed Read operations rd_1, rd_2 that are associated with timestamps ts_1, ts_2, respectively. If rd_1 precedes rd_2, then $ts_1 \leq ts_2$.*

Proof. Let $Cheak_2$ be the Check operation that rd_2 performs before storing a version with timestamp ts_2, and let $\langle C_2, S_2 \rangle$ be the return value of $check_2$. By Lemma 41, S_2 includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 \geq ts_s \geq ts_1$.

□

Lemma 44. *Consider a Write operation w_1 associated with timestamp ts_1, and a Read operation rd_2 associated with timestamp ts_2. If w_1 precedes rd_2, then $ts_1 \leq ts_2$.*

Proof. Let ts_1^c be the timestamp of the version committed by w_1. By Observation 16, $ts_1^c \geq ts_1$. Let $Cheak_2$ be the Check operation that rd_2 performs before storing a version with timestamp ts_2, and let $\langle C_2, S_2 \rangle$ be the return value of $check_2$. By Lemma 41, S_2

includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1^c \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 \geq ts_s \geq ts_1^c \geq ts_1$.

□

Lemma 45. *Consider a Read operation rd_1 associated with timestamp ts_1, and a Write operation w_2 associated with timestamp ts_2. If rd_1 precedes w_2, then $ts_1 < ts_2$.*

Proof. Let $Cheak_2$ be the last check w_2 performs before picking a timestamp, and let $\langle C_2, S_2 \rangle$ be its return value. By Lemma 41, S_2 includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 > ts_s \geq ts_1$.

□

Definition 21 (lineraization)**.** *For every run r we define the sequential run σ_r as follows: All the Write operations in r are ordered in σ_r by the timestamp they are associated with, and every Read operation associated with a timestamp ts in r is ordered in σ_r immediately after the Write that is associated with ts. Read operations that are associated with the same timestamps are ordered (among themselves) by the time they return.*

Theorem 10. *The algorithm emulates an atomic register.*

Proof. We need to show that for every run r, σ_r is a linearization of r. The sequential specification is satisfied by construction, and the real time order follows from Lemmas 42, 43, 44, and 45.

□

5.5 The Reconfiguration Abstraction Implementation

In this section we present an optimal and modular Reconfiguration implementation. In Section 5.5.1 we introduce the *Common Set (CoS)* building block, which is used by the Reconfiguration abstraction in every configuration. In Section 5.5.2 we show how CoS is used for non-optimal Reconfiguration and in Section 5.5.3 we optimize the algorithm. Formal proofs for correctness and complexity are given in Sections 5.5.4 and 5.5.5, respectively.

5.5.1 CoS building block

The *Common Set (CoS)* building clock is a static shared object, emulated in every configuration C over a set of (static) registers. Its API consists of a single operation, denoted C.CoS(P), where P is a set of arbitrary values. C.CoS returns an output set of sets satisfying the following:

Definition 22 (*Common Set* in configuration C)**.**

(CoS_1) Each set in the output is the union of some of the inputs and strictly contains C;

Algorithm 12 Efficient CoS; algorithm of client p_i in configuration C; optimization code shaded.

```
1:  Local variables:                                          ▷ flags accessible outside CoS
2:      firstTime set by reconfig and read by CoS
3:      drop set by CoS and read by reconfig

4:  Shared variables (emulated in configuration C):
5:      Boolean startingPoint, initially false                ▷ Is C a starting point for some client
6:      Mapping from client to registers Warr and Sarr , initially {}.

7:  procedure CoS(P)
8:      P ← PreCompute(P)                                     ▷ optimization
9:      if P ⊃ C then
            ▷ Something new to propose
10:         C.Warr[i] ← P
11:     ret ← C.collect(Warr)
12:     if ret = {} then
13:         return ret
14:     else
15:         return C.collect(Warr)

16: procedure PreCompute(P)
17:     if firstTime then
18:         C.startingPoint ← true
19:     C.Sarr[i] ← P
20:     drop ← false
21:     if ¬C.startingPoint then
22:         return P
        ▷ repeat collect until P stops changing.
23:     drop ← true
24:     tmp ← ⋃C.collect(Sarr)
25:     while tmp ≠ P do
26:         P ← tmp
27:         tmp ← ⋃C.collect(Sarr)
28:     return P
```

(CoS_2) if a client's input strictly contains C, then its output is not empty;

(CoS_3) there is a common non-empty set in all non-empty outputs; and

(CoS_4) every C.CoS invocation that strictly follows a C.CoS call that returns a non-empty output returns a non-empty output.

For example, consider three concurrent clients that input to $C.CoS$ the sets P1, P2, and P3, all of which contain C. A possible outcome is for their outputs to be $\{P1\}$, $\{P1, P1 \cup P2\}$, and $\{P1, P2, P3\}$, respectively. The intuitive explanation behind using CoS is that it builds a *common sequence* of configurations inductively: The first configuration in the sequence is C_{init}, the next is the common configuration returned by $C_{init}.CoS$ (property CoS_3), and so on. Although this sequence is not known to the clients themselves, every client observes this sequence starting with some activated configuration. Every configuration in this sequence contains the previous one.

CoS can be implemented directly using consensus or atomic snapshot, as illustrated in [72]. In Algorithm 12, (without the PreCompute function, which is an optimization and will be discussed later), we give an implementation based on DynaStore's weak snapshot [8]. In the pseudocode, we denote by $\bigcup S$ the union of all sets in a set of sets S. If the proposal P strictly contains C, p_i has something new to propose and it writes P into its cell in the "weak" snapshot array *Warr* (lines 9-10). (Note that *Warr* is a static array emulated in the configuration where CoS is implemented). Either way, it collects *Warr* (line 11). In case the collect is not empty, p_i collects *Warr* again and returns the set of collected proposals (lines 12-15). The second collect ensures that the intersection of non-

empty outputs includes the first written input, implying CoS_3; the remaining properties are satisfied by a single collect.

5.5.2 Simple Reconfiguration

Given CoS, we can solve Reconfiguration in a generic way as shown in Algorithm 13 (ignore the shaded areas for now). Both Check and Propose use the auxiliary procedure *reconfig*. Propose(C, P) first sets a local variable *proposal* to the union of C and P, whereas Check(C) initiates *proposal* to be C. Both then execute the loop in line 40. Each iteration selects the smallest configuration in *ToTrack*; we say that the iteration *tracks* this configuration. The loop tracks all configurations returned by CoS, smallest to largest, starting with C. In each tracked configuration C', the client introduces C', invokes $C'.CoS(proposal)$ and adds to *proposal* the union of the configurations returned from $C'.CoS$. This repeats for every configuration C' returned from CoS until there are no more configurations to track. Recall that by the liveness condition, if some configuration C' is expired and no longer supports $C'.CoS$, then the client gets in return to $C'.CoS$ an exception with some newer activated configuration C_a. In this case, *reconfig* starts over from C_a. At the end, Propose and Check return *proposal* and the set of all tracked configurations.

The common sequence starts with C_{init}, and is inductively defined as follows: If $C_k.CoS$ has a non-empty output, then C_{k+1} is the smallest common configuration returned by all non-empty $C_k.CoS$s. By CoS_3, all non-empty return values have at least one configuration in common, and if there is more than one such configuration, then we pick the smallest, breaking ties using lexicographic order. By CoS_1, each configuration in the common sequence strictly contains the previous one.

Algorithm 13 Generic Reconfiguration algorithm; optimization code shaded.

```
29: Propose(C, P)
30:     return reconfig(C, P)

31: Check(C)
32:     ret ← reconfig(C, {})
33:     if ret = ⟨C, ∗⟩ then activate(C)
34:     return ret

35: procedure reconfig(C, P)
36:     proposal ← P ∪ C
37:     ToTrack ← {C}
38:     speculation ← {}
39:     firstTime ← true
40:     while ToTrack ≠ {} do
41:         C′ ← argmin_{C″∈ToTrack} |C″|                ▷ smallest configuration
42:         introduce(C′)
43:         speculation ← speculation ∪ {C′}
44:         ret ← C′.CoS(proposal)
45:         if ret = ⟨"error", C_a⟩ then                 ▷ C′ is expired - restart from C_a
46:             return reconfig(C_a, proposal)
47:         ToTrack ← (ToTrack ∪ ret) \ {C′}
48:         firstTime ← false
49:         if drop = true then                          ▷ drop old configurations in ToTrack
50:             ToTrack ← ret
51:         proposal ← proposal ∪ ⋃ ToTrack
52:     C_curr ← proposal
53:     return ⟨proposal, speculation⟩
```

Correctness. The validity property (D_1) immediately follows from CoS property CoS_1 and the observation that *proposal* is set to include P at beginning of *reconfig* and never decreases.

To provide intuition for the remaining properties, we discuss the case in which all operations start in C_{init} and no exceptions occur; the proof for the general case appears in Section 5.5.4 Observe that since *proposal* always contains $\bigcup ToTrack$ and configurations are traversed from smallest to largest, we get from property CoS_2 that $C.CoS$ returns an empty set only if C includes *ToTrack*, i.e., C is the last traversed configuration. The key correctness argument is that all nominated configurations belong to the common sequence, and are thus related by containment:

Lemma 46. *For every reconfig that returns $\langle D, S\rangle$, D belongs to the common sequence.*

Proof - sketch for the special case (starting in C_{init}, no exceptions). Assume by way of contradiction that D_j is returned by reconfig operation rec_j but does not belong to the common sequence. Note that C_{init} is in the common sequence and is tracked by rec_j. Let $\tilde{C}_j$ be the last configuration tracked by rec_j that belongs to the common sequence. By assumption, $\tilde{C}_j \neq D_j$, and thus, rec_j gets a non-empty output from $\tilde{C}_j.CoS$ (it gets an output since we assume that there are no exceptions). But, this output includes some configuration in the common sequence, so rec_j tracks a configuration in the common sequence after $\tilde{C}_j$. A contradiction.

Liveness follows since (i) every call to CoS returns, either successfully or with an exception; and (ii) tracked configurations are monotonically increasing, and, provided that the number of reconfigurations is finite, they are bounded.

5.5.3 Optimal Reconfiguration

The key to the efficiency of our new algorithm is in its thrifty CoS implementation, and the signals it conveys to the reconfiguration algorithm, which minimize the number of tracked configurations. To this end, the efficient solution for CoS shares (local) state variables *firstTime* and *drop* with the Reconfiguration implementation.

To explain the intuition behind our algorithm, let us first consider a scenario in which all clients invoke register operations (Read, Write, or ChangeConfig) in the same starting configuration C_0 (e.g., C_0 may be C_{init}), and no exceptions occur. If n of the clients invoke Propose, then there are n sets $P_1, \ldots, P_n$ proposed by $reconfig(C, P_i)$ operations. The unoptimized (weak snapshot-based) CoS may return up to 2^n different subsets in CoS responses (assuming many clients invoke Read/Write operations), inducing high complexity.

Our algorithm reduces this complexity by running a pre-computation phase in *PreCompute*, which imposes a containment order on all configurations passed to, and hence returned from, CoS. This is done by running a variant of (strong) atomic snapshot [**?**] on all client proposals in configuration C_0. Specifically, each process writes its own proposal P (line 19) to the "strong" array *Sarr*, and then (lines 24-27) repeatedly collects the union of all *Sarr* cells into P, until P stops changing. Like an atomic snapshot, this ensures that all results of PreCompute are related by containment. Note, however, that unlike an atomic snapshot, the complexity of this pre-computation is linear in the number of *different* proposals written, rather than in the number of participating processes; if collect encounters a newly written value that does not change the union of written values, PreCompute returns. In case all operations start in C_0, there are no new proposals in other configurations, and so the containment order is preserved throughout the computation. This ensures that the number of different configurations tracked by all clients is at most n.

Next, we account for the case that clients invoke (or restart due to exceptions) their operations in different starting configurations. We have to identify configurations where some client starts, and run PreCompute in them too. To this end, we have clients signal (by raising the startingPoint flag) if a configuration is their starting point. Every client that later runs *C.CoS* sees this flag true, and executes the pre-computation. If a client p_i sees the flag false in *C.CoS*, p_i does not run the pre-computation. Nevertheless, since p_i checks the flag after writing its value to *Sarr*, p_i's proposal is already in the array before new clients that start in this configuration perform their collects, and so p_i's proposal is contained in theirs. Thus, at this new starting point, all clients obtain proposals that are related by containment among themselves.

The tricky part is that old proposals that were included in *ToTrack* before the new

starting point are not necessarily ordered relative to ensuing proposals, as in the following scenario:

- Clients p_1 and p_2 start in C_0 and propose $C_0 \cup \{+a\}$ and $C_0 \cup \{+b\}$, respectively; p_1 gets $\{C_1\}$, where $C_1 = C_0 \cup \{+a\}$, from $C_0.CoS$ and p_2 gets $\{C_1, C_2\}$, where $C_2 = C_0 \cup \{+a, +b\}$.
- Client p_1 tracks C_1, gets an empty set from $C_1.CoS$, and activates it. Client p_3 starts in C_1, (which is now activated), proposes $C_3 = C_1 \cup \{+c\}$ in $C_1.CoS$, and gets $\{C_3\}$.
- Later, p_2 tracks C_1, and gets C_3 in $C_1.CoS$'s output. At this point p_2's *ToTrack* contains C_2 and C_3, neither of which contains the other.

To achieve linear complexity, we have clients *drop* all configurations previously returned from CoS at all the starting points they encounter. One subtle point is ensuring safety in the presence of such drops, and our proof of the general case of Lemma 46 addresses this issue.

Intuitively, since the purpose of tracking all configurations is to ensure that clients traverse the common sequence, once we know C is in the common sequence, there is no need to continue to track any configuration older than C. So, the drop is safe.

A second subtle point is preserving linear complexity despite executing PreCompute in multiple starting points. But since (i) the worst-case complexity of a single pre-computation is linear in the number of different proposals written to it, (ii) each CoS begins with a proposal that reflects all those seen in previous CoSs, and (iii) there are n new proposals overall, the combined complexity of *all* pre-computations is $O(n)$.

Finally, we provide intuition for the complexity of the high-level dynamic atomic register given in Section 5.4. The full proof, which wraps this intuition into a technical induction, appears in the next sections. Recall that the register emulation performs a loop in which it repeatedly calls Check(C), where C is the configuration returned from the previous Check/Propose, until some Check(C') returns $\langle C', S \rangle$ for some C' and S. The loop performs a constant number of operations in every configuration returned in a speculated set S from Check. Therefore, we want the Checks in this loop to return the optimal number of configurations, and have optimal complexity themselves.

Since all the configurations introduced (and returned in speculation sets) by our algorithm are related by containment, we immediately conclude that the number of configurations returned in speculated sets S of all Checks together is bounded by n. Now we show that the complexity of all Checks combined is $O(n)$. First observe that all Checks combined invoke at most n CoSs. Second, each CoS writes at most three times to shared registers (lines 10, 18, and 19), reads once (in line 21), and performs each of the collects in lines 11, 15, and 24 at most once. Now observe that CoS performs the collect in line 27 only if the previous collect (in line 24 or 27) contained a proposal $P_1 \not\subseteq P$, which means that none of the CoSs collected P_1 before. Since there are at most n proposals, all CoSs together perform the collect in line 27 at most n times. All in all, we get that the complexity of all Checks is $O(n)$.

5.5.4 Reconfiguration Correctness Proof

The proof makes use of the following simple observation:

Observation 17. *The output of* PreCompute *contains its input.*

Lemma 47. *Algorithm 12 implements CoS.*

Proof. We show the four properties of Definition 22:

CoS_1. Each CoS's output is a set of proposals all of which were returned from PreCompute and, by line 9, strictly contain C. In PreCompute, P is always a subset of the union of inputs to CoS.

CoS_2. Consider a configuration C and a client p_i that invokes $C.CoS(P)$ s.t. $P \supset C$. By Observation 17, $P \supset C$ also in line 9. Therefore, p_i writes P into $Warr[i]$ (line 10), and since no other client writes into the same cell, p_i collects a non-empty set in its collects (lines 11 and 15). Thus, p_i returns a non-empty set.

CoS_3. If there are no non-empty outputs, then we are done. Otherwise, there is at least one client that writes its P to $Warr$. Let p_i be the first such client (because $Warr$ consists of atomic registers and atomicity is composable, the first is well-defined), and denote its P as P_i. We next show that every returned non-empty set contains P_i.

Consider a client p_j that returns a non-empty set. Then p_j collects $Warr$ twice, and the first collect is not empty. Therefore, p_j completes its first collect after p_i writes P_i into $Warr[i]$, and thus, it is guaranteed that p_j reads P_i from $Warr[i]$ during its second collect, and returns a set that contains P_i.

CoS_4. Consider two complete C.CoS calls op_i and op_j s.t. op_i strictly follows op_j, and op_j returns a non-empty set. There is a non-empty cell in $Warr$ before op_j completes, and since nothing is erased from $Warr$, op_i's collects are not empty.

□

We continue with the following notations:

Notation. We start with some notation that we use throughput the proof. The *common sequence* of configurations $C_{init}, C_1, C_2, ...$, which only an outside observer can view, is inductively defined as follows: It starts with C_{init}, and if $C_k.CoS$ has a non-empty output, then C_{k+1} is the smallest common configuration returned in all non-empty $C_k.CoS$ outputs (by CoS property CoS_3, the intersection is not empty), with ties broken in lexicographic order.

We say that a set is *monotonic* if all its elements are related by containment. A sequence is monotonic if every element $C^k \neq C^0$ contains C^{k-1}. By CoS property CoS_1, the

common sequence is monotonic. For a *reconfig*(C, P) operation rec_j, we say a configuration is *tracked* by rec_j if it is selected as C' in line 41. We define $tracked(j) = C_j^0, C_j^1, \ldots, C_j^m$ to be the sequence of configurations tracked by rec_j in the last recursive call of *reconfig* in the order they are tracked. Note that $tracked(j)$ does not include configurations tracked before an exception is received. We further denote rec_j's return value by $\langle D_j, S_j \rangle$.

The following observations follow immediately from the code:

Observation 18. *Consider a reconfig operation rec_j, then $D_j = C_j^m \supseteq C_j^0 \cup \cdots \cup C_j^m$.*

Observation 19. *If some reconfig reads drop=true in C.CoS at time t, then there is a reconfig rec_j s.t. $C_j^0 = C$ that starts before time t.*

The next claim stipulates that *reconfig* returns once CoS returns it an empty set.

Claim 5. *Consider a reconfig operation rec_j and a configuration $C' \in tracked(j)$. If $C'.CoS$ called during rec_j returns an empty set, then $C' = D_j$.*

Proof. By CoS property CoS_2, since $C'.CoS$ returns an empty set, when $C'.CoS$ is called (line 44), $proposal \not\supset C'$. Moreover, whenever CoS is called during a *reconfig* operation, $proposal \supseteq \bigcup ToTrack$. Together, we get that C' is not strictly contained in $\bigcup ToTrack$. Now notice that C' is selected in line 41 as $\underset{C \in ToTrack}{\operatorname{argmin}} |C|$. Therefore, we get that $ToTrack = \{C'\}$ when $C'.CoS$ is called (line 44). By the assumption, $C'.CoS$ returns $\{\}$. Hence, in line line:ToTrackUpdate1 50/47, *ToTrack* becomes $\{\}$, and thus the *reconfig* exits the while loop, and $C' = D_j$.

□

The following observation follows from the usage of *reconfig* and the oracle behavior:

Observation 20. *For every reconfig operation rec_j, C_j^0 is a nominated configuration that is returned by some reconfig before rec_j is invoked.*

The next claim shows that our algorithm drops old configurations only upon tracking a configuration in the common sequence.

Claim 6. *Assume that for every reconfig operation rec_j that returns before some time t, D_j belongs to the common sequence. Now consider a configuration C that belongs to the common sequence, and a reconfig operation rec_i that tracks C and returns at time t. If rec_i gets a non-empty output from C.CoS, then rec_i tracks another configuration belonging to the common sequence after C.*

Proof. First observe that rec_i gets a configuration C' that belongs to the common sequence from $C.CoS$, and rec_i's *ToTrack* contains C' at the end of the corresponding while loop. If rec_i tracks C', we are done. Otherwise, rec_i drops C' after calling some $C''.CoS$ (while tracking C''). By Observation 19, there is a *reconfig* rec_j s.t. $C_j^0 = C''$ that starts before time t. By our assumption and by Observation 20, C'' belongs to the common sequence, and we are done.

□

We now show that every nominated configuration belongs to the common sequence.

Theorem 46 (restated). *For every reconfig that returns $\langle D, S \rangle$, D belongs to the common sequence.*

Proof. We prove by induction on time $t \geq 0$ that for every *reconfig* operation rec_j that returns at time t, D_j belongs to the common sequence.

Base: Since no operation returns at time 0 or earlier, the lemma holds for $t = 0$.

Step: We now assume that the lemma holds for some $t \geq 0$, and prove for $t + 1$. Let rec_j be a *reconfig* operation that returns at time t. By Observation 20, C_j^0 was returned by some *reconfig* before time t. Therefore, by the induction assumption C_j^0 belongs to the common sequence. Now assume in a way of contradiction that D_j does not belong to the common sequence. Let C be the last configuration tracked by rec_j that belongs to the common sequence (there is such a configuration since C_j^0 belongs to the common sequence). By the contradiction assumption, $C \neq D_j$, and thus by Claim 5, rec_j gets a non-empty output from $C.CoS$. Therefore, by Claim 6, rec_j tracks a configuration belonging to the common sequence after C. A get a contradiction.

□

The following is an immediate conclusion from Lemma 46 and the monotonicity of the common sequence.

Corollary 14. *The set of nominated configurations is monotonic.*

Claim 7. *Consider a reconfig(C, P) operation rec_j that returns D_j. Then rec_j tracks every configuration in the common sequence between C and D_j.*

Proof. Assume by way of contradiction that there is a configuration C' in the common sequence between C and D_j that rec_j does not track. Let C'' be the last configuration before C' in the common sequence that is tracked by rec_j. (There must be such a configuration because $C = C_j^0$ is in the common sequence.) By Claim 5, rec_j gets a non-empty output from $C''.CoS$, and by the common sequence definition, this output contains the next configuration C^{next} in the common sequence. Now recall that rec_j tracks configurations from smallest to largest, and it does not track C^{next}. Therefore, rec_j drops C^{next} after calling CoS in some configuration $\tilde{C}$ not in the common sequence. By Lemma 46, $\tilde{C}$ is not nominated, and thus not activated. Therefore, by the oracle definition no *reconfig* starts in $\tilde{C}$, and thus the *drop* flag in $\tilde{C}.CoS$ is always false. Hence, rec_j does not drop configurations after calling $\tilde{C}.CoS$. A contradiction.

□

Theorem 11. *Algorithms 13 and 12 implement the Reconfiguration abstraction.*

Proof. We now show that all Reconfiguration properties are satisfied:

D_1 (i) Consider a Propose(C, P) operation that calls *reconfig*(C, P) and returns $\langle D, S\rangle$. We have to show that $D \supseteq P$. Now observe that *proposal* is set to contain P at the beginning of the *reconfig*, and never decreases, including recursive calls to *reconfig*. The property follows from line 53. (ii) Consider a Check or a Propose operation op that introduces, activates, or return configuration C', and consider $e \in C' \setminus C \cup P$. Observe that op gets e by some CoS output. The property follows by inductively using CoS_1.

D_2 Consider an operation (Propose or Check) op_j that strictly precedes another operation op_i. Let rec_j be the *reconfig* operation that is called during op_j, and let rec_i be the *reconfig* operation that is called during op_i. Notice that rec_j strictly precedes rec_i. By Lemma 46, D_j and D_i are in the common sequence. From CoS_1, either $D_i \supseteq D_j$ or $D_j \supseteq D_i$. Assume by contradiction that $|D_i| < |D_j|$. Thus, $D_j \supset D_i$, and D_i precedes D_j in the common sequence. This means that $D_i.CoS$ returned a non-empty output to some *reconfig* operation before D_j was added to the common sequence, and so before rec_j returned D_j. Since rec_j strictly precedes rec_i, we get that $D_i.CoS$ returned a non-empty output before rec_i invoked $D_i.CoS$, and so by CoS_4, $D_i.CoS$ returns a non-empty output also to rec_i, a contradiction to the assumption that rec_i returns D_i.

S_1 Consider a Check(C) or Propose(C, P) operation op that returns $\langle D, S\rangle$. Let $rec_i =$ *reconfig*(C', P') be the last *reconfig* operation that is called during op. By the oracle definition, by Lemma 46, and since every activate configuration is also nominated, *reconfig*(C, P) recursively calls *reconfig*$(C_a, *)$ only if C_a is activated and nominated, and $|C_a| > |C|$. Thus, if $C' \neq C$, then C' is activated and belongs to the common sequence. By Lemma 46, all the nominated configurations are in the common sequence. Therefore, by Claim 7 and by the observation that the *speculation* set of rec_i is $tracked(i)$, S includes all nominated configurations C'' s.t. $|C'| \leq |C''| \leq |D|$.

□

5.5.5 Reconfiguration Complexity Proof

We use the following additional notations:

Notation. For every introduced configuration C:

1. We define *OldProps(C)* to be the proposals suggested in $C.CoS$ by *reconfig* operations starting their traversals before C. That is,

$$OldProps(C) \triangleq \{P \mid \exists \textit{reconfig} \text{ that calls C.CoS(P) while its flag } \textit{firstTime} = \textit{false}\}$$

2. Since by Corollary 14, the set of nominated configurations is monotonic, there is at most one nominated configuration of a given size. Thus, we can define *Pred(C)*

to be the biggest nominated configuration C' s.t. $|C'| < |C|$ for $C \neq C_{init}$. For completeness, we define $Pred(C_{init}) \triangleq C_{init}$.

For every nominated configuration C':

1. All introduced configurations that have C' as their Pred are its successors:

$$Successors(C') \triangleq \{C'\} \cup \{C \text{ is introduced} \mid C' = Pred(C)\}, \text{ and}$$

$$successors^i(C) \triangleq \{C' \in successors(C) \mid |C'| \leq i\}.$$

2. All configurations that are in *ToTrack* of *reconfig* operations after tracking C' are potential successors of C':

$$PotentialSuccessors(C') \triangleq \{C \mid \text{some } reconfig \text{ calls } C'.CoS \text{ in line 44 and}$$

$$C \in ToTrack \text{ in line 51 in the same iteration}\}$$

An illustration of *Successors* and *PotentialSuccessors* appears in Figure 5.2.

We denote $introducedSet^i \triangleq \{C \text{ is introduced} \mid |C| \leq i\}$, and $nominatedSet^i \triangleq \{C \text{ is nominated} \mid |C| \leq i\}$

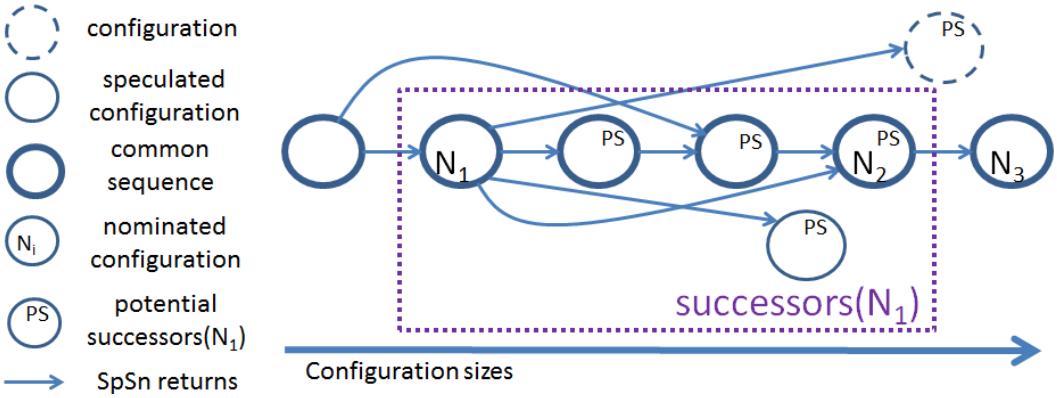


Figure 5.2: Example run of client p_i of our algorithm. The dashed configuration is in p_i's *ToTrack* after calling $N_1.CoS$, and thus it is in $potentialSuccessors(N_1)$. But it is dropped after p_i calls $N_2.CoS$, and thus it is never introduced, and so it is not in $successors(N_2)$.

We now state some observations that follow immediately from the code.

Observation 21. *Whenever CoS is called during a reconfig operation, proposal* $\supseteq \bigcup ToTrack$.

Observation 22. *Consider a reconfig operation rec that calls* $C.CoS(P)$ *s.t.* $P \supseteq C$ *and returns ret. Then* $\exists C' \in ret$ *s.t.* $P \subseteq C'$.

Observation 23. *The return value of every PreCompute that reads startPoint=false in C.CoS is in OldProps(C).*

Observation 24. *If a configuration C′ is returned by C.CoS, then there is a C.CoS invocation in which PreCompute returns C′.*

Corollary 15. *If all reconfig operations that call C.CoS read startPoint=false, then all C.CoS's return values in OldProps(C).*

Observation 25. *Let P_1 and P_2 be two proposals returned by PreCompute executions during C.CoS pc_1 and pc_2, respectively. If pc_1 and pc_2 execute lines 24 to 27 (repeat collecting* Sarr *until P stops changing), then P_1 and P_2 are related by containment.*

Claim 8. *Consider configuration C. If $OldProps(C)$ is monotonic, then all configurations returned from C.CoS invocations are related by containment.*

Proof. By Observation 24, we need to show that all proposals returned from PreCompute invoked during *C.CoS* are related by containment. Let P_1 and P_2 be two proposals returned by PreCompute executions during *C.CoS* pc_1 and pc_2, respectively. We show that P_1 and P_2 are related by containment. Consider three cases:

1. First, pc_1 and pc_2 read *startPoint=false* in line 21 executions during *C.CoS*. By Observation 23, $P_1, P_2 \in OldProps(C)$. Since by the assumption $OldProps(C)$ is monotonic, we are done.

2. Second, pc_1 reads *startPoint=false*, and pc_2 reads *startPoint=true*. Note that since pc_1 reads *startPoint=false*, it is called with P_1. Now since *startPoint* never changes from *true* to *false*, pc_1 reads *startPoint* before pc_2. Thus, pc_1 writes P_1 into *Sarr* in line 19 before pc_2 reads *startPoint=true*. Therefore, pc_2 sees P_1 in all the collects in lines 24 to 27, and thus $P_2 \supseteq P_1$.

3. Third, pc_1 and pc_2 read *startPoint=true*. Therefore, both execute lines 24 to 27, and thus by Observation 25, P_1 and P_2 are related by containment.

□

Claim 9. *If a configuration C is nominated, then C is introduced.*

Proof. By Lemma 46, *C* belongs to the common sequence, and so by the common sequence definition *C* is introduced.

□

Claim 10. *Consider a reconfig operation rec_j that introduces configuration C′ with firstTime=true and later introduces C. Then, $C' \subset C$.*

Proof. Note that when C' is introduced, *ToTrack* = $\{C'\}$. Recall that if an error is returned, then rec_j is aborted, and no later configurations are introduced by rec_j. Thus, no error is returned when rec_j introduces C', and by by CoS property CoS_1, at the end of the iteration

all the configurations in *ToTrack* are strictly contain C'. Therefore, the claim follows by inductively repeating this argument.

□

Corollary 16. *Consider an introduced configuration C. Then, $C \supseteq C_{init}$.*

Proof. Let rec_j be a *reconfig*(C', P') that introduces C. Observe that rec_j introduces C' with *firstTime=true*. Thus, by Claim 10, $C \supseteq C'$. By Lemma 46, C' belongs to the common sequence, and by monotonicity of the common sequence, $C' \supseteq C_{init}$. Therefore, $C \supseteq C_{init}$.
□

Claim 11. *Consider a* reconfig *operation rec_j that introduces configuration C in iteration iter. If firstTime=false at the beginning of iter, then there is at least one nominated configuration that is smaller than C, and rec_j tracks Pred(C) before iter.*

Proof. Let C' be the configuration that rec_j is called with. Since C' and $Pred(C)$ are nominated, by Corollary 14, C' and $Pred(C)$ belong to the common sequence. Now observe that rec_j introduces C' with *firstTime=true* before it introduces C. Thus, by Claim 10, $|C'| < |C|$. If $C' = Pred(C)$, we are done, Otherwise, $|C'| < |Pred(C)| < |C|$. The Claim follows from Claim 7, and the observation that rec_j tracks configurations from the smallest to the biggest.

□

Corollary 17. *Consider an introduced configuration $C \neq C_{init}$. Then there is at least one nominated configuration that is smaller than $|C|$ and a reconfig operation that tracks $Pred(C)$ and introduces C.*

Proof. Let rec_j be a *reconfig* operation that introduces C in an iteration *iter*, and consider two cases:

1. First, flag *firstTime=false* at the beginning of *iter* during rec_j. The Corollary follows by Claim 11.
2. Otherwise, C is the parameter rec_j is called with, and thus C nominated. Let rec_i be the first *reconfig* that tracks C, and let t be that time. Therefore, no *reconfig* returns C before time t. hence, rec_i is not called with C, and thus *firstTime=false* when rec_i tracks C. The Corollary follows by Claim 11.

□

Claim 12. *Consider a nominated configuration C, and an introduced configuration $C^{i+1} \in successors(C)$ of size $i+1 > |C|$. Assume that for every $C' \in successors^i(C)$, $PotentialSuccessors(C') \subseteq PotentialSuccessors(C)$, and $PotentialSuccessors(C)$ is monotonic. Then:*

(a) $C^{i+1} \in PotentialSuccessors(C)$

(b) $OldProps(C^{i+1}) \subseteq PotentialSuccessors(C)$

Proof. (a) Since $C^{i+1} \neq C_{init}$, by Corollary 17, there is a *reconfig* operation rec_j that tracks $C = Pred(C^{i+1})$ and introduces C^{i+1}. Now let C'^{i+1} be the biggest configuration in $successors^i(C)$ that is tracked by rec_j. By the assumptions $PotentialSuccessors(C'^{i+1}) \subseteq PotentialSuccessors(C)$, and thus monotonic. Therefore, there is at most one configuration in $PotentialSuccessors(C'^{i+1})$ whose size is $i+1$. And since a *reconfig* operation tracks configurations by the order of their sizes, there is at least one configuration in $PotentialSuccessors(C'^{i+1})$ whose size is $i+1$. Therefore, there is exactly one configuration C''^{i+1} in $PotentialSuccessors(C'^{i+1})$ of size is $i+1$, and rec_j tracks C''^{i+1} immediately after C'^{i+2}. By CoS property CoS_1, $C''^{i+2} = C^{i+2}$, and we are done.

(b) Consider some $P \in OldProps_{C^{i+1}}$, we need to show that $P \in PotentialSuccessors(C)$. Let rec_k be a *reconfig* operation that calls $C^{i+1}.CoS$ while its flag *firstTime= false*. By Claim 11, rec_k tracks C. Let C'''^{i+1} be the biggest configuration in $successors^i(C)$ that is tracked by rec_k. Since rec_k tracks configurations according to their sizes and since by (a) C^{i+1} is the only configuration in $successors(C)$ of size is $i+1$, rec_k tracks C^{i+1} immediately after it tracks C'''^{i+1}. By Observation 22, rec_k's proposal in $C'''^{i+1}.CoS$ is included by some configuration in rec_k's *ToTrack* before rec_k introduces C^{i+1}, and by definition, rec_k's *ToTrack* is included in $PotentialSuccessors(C'''^{i+1})$ before rec_k introduces C^{i+1}. Now By the assumptions, $PotentialSuccessors(C'''^{i+1}) \subseteq PotentialSuccessors(C)$, and thus monotonic. Therefore, $P \in PotentialSuccessors(C'''^{i+1}) \subseteq PotentialSuccessors(C)$, and we are done.

□

Claim 13. *Consider a nominated configuration C. Assume that no more configurations of size $|C|$ are introduced and $\{C\} \cup PotentialSuccessors(C)$ is monotonic. Denote $x \triangleq max(\{j \mid \exists C' \in successors(C) : |C'| = j\})$. Then for every $|C| \leq i < x$:*

- *For every $C' \in successors^i(C)$, $PotentialSuccessors(C') \subseteq PotentialSuccessors(C)$*

Proof. We prove by induction on i.

Base: $i = |C|$. If $i = x$, we are done. Otherwise, by the assumption, C is the only introduced configuration of size $|C|$, and the lemma follows.

Step: Now assume that the lemma holds for some $|C| \leq i < x$, we show that it holds for $i+1$. If $i+1 \geq x$, we are done. If there is no configuration in $successors(C)$ whose size is $i+1$, then (1) follows by induction. Otherwise, by the Claim 12 (a) and since $PotentialSuccessors(C)$ is monotonic, there is exactly one configuration $C^{i+1} \in successors^{i+1}(C)$ of size is $i+1$. Since $i+1 < x$, C^{i+1} is not nominated. Therefore, whenever $C^{i+1}.CoS$ is called by a *reconfig* operation, its *firstTime = false*. Thus, all *reconfig* operations that call $C^{i+1}.CoS$ read *startingPoint=false*. Now let $C'^{i+1} \in PotentialSuccessors(C^{i+1})$, we show that $C'^{i+1} \in PotentialSuccessors(C)$. If C'^{i+1} is returned by $C^{i+1}.CoS$, then by Corollary 15, $C'^{i+1} \in OldProps(C^{i+1})$. By Claim 12 (b), $OldProps(C^{i+1}) \subseteq PotentialSuccessors(C)$. Therefore, $C'^{i+1} \in PotentialSuccessors(C)$, and we are done. Otherwise, there is a *reconfig* operation *rec* that has C'^{i+1} in its *ToTrack* before it invokes $C^{i+1}.CoS$. Thus *rec*'s *first-*

Time=false when it introduces $C'^{i+1}.CoS$, and thus by Claim 11, *rec* track C. Now let C''^{i+1} be the last configuration *rec* tracks before C'^{i+1}, and note that $C''^{i+1} \in successors^i(C)$ and $C'^{i+1} \in PotentialSuccessors(C''^{i+1})$. By the induction assumption, $PotentialSuccessors(C''^{i+1}) \subseteq PotentialSuccessors(C)$. Therefore, $C'^{i+1} \in PotentialSuccessors(C)$, and we are done.

□

The following corollary immediately follows from Claims 12 and 13:

Corollary 18. *Consider two nominated configurations C, C' s.t. $C = Pred(C')$, and assume that no more configurations with size $|C|$ are introduced. If $\{C\} \cup$ PotentialSuccessors(C) is monotonic, then (1) $successors(C)$ is monotonic, (2) $OldProps(C') \subseteq PotentialSuccessors(C)$, and (3) for every $C'' \in successors^{|C'|-1}(C)$, $PotentialSuccessors(C'') \subseteq PotentialSuccessors(C)$.*

Claim 14. *Consider two nominated configurations C, C' s.t. $C = Pred(C')$, and let S be a monotonic set. If $OldProps(C') \subseteq S$ and $\forall C'' \in successors^{|C'|-1}(C)$, $PotentialSuccessors(C'') \subseteq S$, then $\{C'\} \cup PotentialSuccessors(C')$ is monotonic.*

Proof. Consider a configuration $C_1 \in PotentialSuccessors(C')$, we start by showing that $C_1 \supset C'$. By definition, there is a *reconfig* rec_1 that calls $C'.CoS$ in line 44 and $C_1 \in ToTrack$ in line 51 in the same iteration. Now consider two cases:

1. $C'.CoS$ in rec_1 returns C_1. Therefore, by CoS property CoS_1, $C_1 \supset C'$.
2. C_1 is in rec_1's *ToTrack* before it invokes $C'.CoS$. Thus, rec_1 calls $C'.CoS$ while its *firstTime = false*, and so by Claim 11, rec_1 tracks C. Now let C'' be the last configuration rec_1 tracks before C', and note that $C', C_1 \in PotentialSuccessors(C'')$. By the assumption, $PotentialSuccessors(C'')$ is monotonic, and by the observation that rec_j tracks configurations from smallest to biggest, $C_1 \supset C'$.

Consider another configuration $C_2 \in PotentialSuccessors(C')$. By definition, there is a *reconfig* rec_2 that calls $C'.CoS$ in line 44 and $C_2 \in ToTrack$ in line 51 in the same iteration. We now show that C_1 and C_2 are related by containment. there are three cases:

1. Both C_1 and C_2 are returned by $C'.CoS$. Therefore, by Claim 8, C_1 and C_2 are related by containment.
2. Neither C_1 nor C_2 is returned by $C'.CoS$. Thus C_1 (C_2) is in rec_1's (respectively, rec_2's) *ToTrack* before it invokes $C'.CoS$. Thus, rec_1 and rec_2 call $C'.CoS$ while their *firstTime = false*, and so by Claim 11, rec_1 and rec_2 track C. Now let C'_1 (C'_2) be the last configuration rec_1 (respectively, rec_2) tracks before C', and note that $C_1 \in PotentialSuccessors(C'_1)$ (and $C_2 \in PotentialSuccessors(C'_2)$). By the assumption, $PotentialSuccessors(C'_1)$, $PotentialSuccessors(C'_2) \subseteq S$, and thus $C_1, C_2 \in S$. Now since S is monotonic, we are done.
3. One of the configurations, w.l.o.g. C_1 is returned by $C'.CoS$, and C_2 is not. In this case, C_2 is in rec_2's *ToTrack* before it invokes $C'.CoS(P_2)$ and thus, by Observation

21, $P_2 \supseteq C_2$, and as in above $C_2 \in S$. In addition, since rec_2 does not drop C_2 after $C'.CoS$ returns, it reads *startingPoint=false* during $C'.CoS$. By Observation 24, there is a *reconfig* operation rec'_1 that gets C_1 from PreCompute during $C'.CoS$. Now consider two cases:

(a) rec'_1 reads *startingPoint=false*. Therefore, rec'_1 calls $C'.CoS$ while its *firstTime=false*, and PreCompute returns its input which is therefore C_1. Thus, by definition, $C_1 \in OldProps(C')$. By assumption, $C_1 \in S$, and we are done.

(b) rec'_1 reads *startingPoint=true*. Since rec_2 reads *startingPoint=false* during $C'.CoS(P_2)$ and writes its proposal to *Sarr* (line 19) before reading *startingPoint* 21, and since *startingPoint* never changes from *true false* and rec'_1 finds it true, rec'_1 collects P_2 in *Sarr* in line 24. Therefore, $C_1 \supseteq P_2 \supseteq C_2$.

□

Claim 15. *The set* $PotentialSuccessors(C_{init}) \cup \{C_{init}\}$ *is monotonic.*

Proof. By Corollary 16 Claim 11, all *reconfig* operations that calls $C_{init}.CoS$ do so with *firstTime=true*. Therefore, $OldProps(C_{init}) = \{\}$ and all configurations in $PotentialSuccessors(C_{init})$ are returned from $C_{init}.CoS$. Thus, by CoS property CoS_1, all configurations in $PotentialSuccessors(C_{init})$ contain C_{init}, and by Claim 8, they are related by containment. The claim follows.

□

Lemma 48. *The set of introduced configurations is monotonic.*

Proof. Let $x = |C_{init}|$. We will show by induction on $i \geq x$ that the following are satisfied:

(a) The set $introducedSet^i$ is monotonic.

(b) For every configuration $C \in nominatedSet^i$, $\{C\} \cup PotentialSuccessors(C)$ is monotonic.

The lemma will follow from (a). **Base:** we prove for $i = x$. By Corollary 16, there is no introduced configuration other than C_{init} whose size is smaller than or equal to x. Hence, (a) is satisfied; (b) follows from Claim 15.

Step: assume by induction that (a) and (b) hold for $i \geq x$, we prove for $i+1$. By Claim 9, every nominated configuration is also introduced, so if there is no introduced configuration whose size is $i+1$, then we are done. Otherwise, let C be an introduced configuration s.t. $|C| = i+1$, and let $C_p = Pred(C)$. Note that by definition, $C \in successors(C_p)$. Since $|C| = i+1 > x = |C_{init}|$, $|C_p| < |C| = i+1$. Therefore, by the induction assumption (b), $\{C_p\} \cup PotentialSuccessors(C_p)$ is monotonic, and by the induction assumption (a), C_p is the only introduced configuration of size $|C_p|$.

(a): By the first induction assumption $introducedSet^{|C_p|}$ is monotonic, thus it is enough to show that C contains or equals every configuration $C' \in introducedSet$ s.t. $|C_p| \leq |C'| \leq i+1$. Note that, by definition, $C' \in successors(C_p)$. By Corollary 18 (1), $successors(C_p)$

is monotonic. Therefore, C and C' are related by containment, and since $|C| = i + 1 \geq |C'|$, C contains or equals C', as requested.

(b): By (a), C is the only introduced configuration whose size is $i + 1$. If C is not nominated, then we are done. Otherwise, let $S = PotentialSuccessors(C_p)$ by Corollary 18 (2), $OldProps(C) \subseteq S$ and by 18 (3), for every $C'' \in successors^{|C|-1}(C_p)$, $PotentialSuccessors(C'') \subseteq S$. Therefore, by Claim 14, $\{C\} \cup PotentialSuccessors(C)$ is monotonic, as needed.

□

We are now ready to conclude the complexity of the dynamic objects that uses our algorithm, which is captured by the following lemma:

Lemma 49. *Consider an execution of a dynamic objects that uses our algorithm, and consider a loop in which Check(C) is repeatedly called, s.t. C is the configuration returned from the previous Check, until some Check(C') returns $\langle C', * \rangle$. Let n be the number of Propose(P) operations in the execution. Then:*

1. *All the Checks in the loop (together) return $O(n)$ configurations in the speculations sets.*
2. *The complexity of all Checks in the loop combined is $O(n)$.*

Proof. Since every Check in the loop starts where the previous returns, the Checks in the loop introduce different configurations. Thus by Lemma 48, we immediately conclude that the number of configurations returned in speculated sets of all Checks in the loop together is bounded by n. Moreover, by CoS_1, no configuration is returned more than once in the speculation sets. It is left to show that the complexity of all Checks combined is $O(n)$. First observe (again by Lemma 48) that all Checks combined invokes at most n CoSs. Second, each CoS writes at most three times to shared registers (lines 10, 18, and 19), reads once (in line 21), and performs each of the collects in lines 11, 15, and 24 at most once.

Now observe that CoS performs the collect in line 27 only if the previous collect (in line 24 or 27) contained a proposal $P_1 \not\subseteq P$, which means that none of the CoSs collected P_1 before. Since there are at most n proposals, all CoSs together perform the collect in line 27 at most n times. All in all, we get that the complexity of all Checks in the loop is $O(n)$.

□

5.6 Conclusions

In this chapter we defined a dynamic model with a clean failure condition that allows an administrator to reconfigure an object and switch a removed server off once the reconfiguration operation completes. In this model, we have captured a succinct abstraction for consensus-less reconfiguration, which dynamic objects like atomic read/write register

and max-register may use. We demonstrated the power of our abstraction by providing an optimal implementation of a dynamic register, which has better complexity than previous solutions in the same model.

Chapter 6

Conclusion

In this thesis we studied asynchronous distributed reliable storage and focused on two fundamental aspects that are essential in every distributed storage system: space cost and dynamic reconfiguration.

Storage space cost. The most common approach to achieve reliable distributed storage is via replication, i.e., storing multiple copies of each data block (on different servers). In this thesis we first compared the fundamental space cost of such algorithms as a function of the basic primitives they use (e.g., read/write registers versus max-registers). We introduced a new hierarchy, which classifies primitive types by the number of base objects of a given primitive required to emulate an f-tolerant register, as a function of the number of writers k and the number of available servers n.

Then, we considered algorithms that try to mitigate the significant cost of replication, which results from the immense size of the data, by using a symmetric black-box coding scheme. Given three problem parameters: f, c, and D, where f is the number of storage node failures tolerated, c is the concurrency allowed by the algorithm, and D is the data size, we proved that the storage cost is $\Theta(min(f, c) \cdot D)$. Asymptotically, this means either a storage cost as high as that of replication, or as high as keeping as many versions of the data as the concurrency level. Then, equipped with the insights from the lower bound, we also presented an algorithm that combines replication and erasure codes, whose storage cost is $O(min(f, c) \cdot D)$.

Dynamic reconfiguration. In this thesis we contributed to the understanding of reconfiguration of distributed storage by showing both negative (impossibility) and positive (algorithms) results. We first proved that in an asynchronous API-based reconfigurable model allowing at least one failure, without restricting the number of reconfigurations, there is no way to emulate dynamic safe wait-free storage. We further showed how to circumvent this result using a dynamic eventually perfect failure detector the we defined: we presented an algorithm that uses such a failure detector in order to emulate a

wait-free dynamic atomic MWMR register.

Then, we defined a dynamic API-based model with a clean failure condition that allows an administrator to reconfigure an object and switch a removed server off once the reconfiguration operation completes. In this model, we have captured a succinct abstraction for consensus-less reconfiguration, which dynamic objects like atomic read/write register and max-register may use. We demonstrated the power of our abstraction by providing an optimal implementation of a dynamic register, which has better complexity than previous solutions in the same model.

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.1 Additional results of Chapter 2

We prove here an extended version of Lemma 2 and use it to show additional lower bounds (see Theorems 12, 13, 14, and 15).

Lemma 2 (extended). *For all $k > 0$, $f > 0$, let A be an f-tolerant algorithm that emulates a WS-Safe obstruction-free k-register using a collection $\mathcal{S}$ of servers storing a collection $\mathcal{B}$ of wait-free MWMR atomic registers. Then, for every $F \subseteq S$ such that $|F| = f + 1$, there exist k failure-free runs r_i, $0 \leq i \leq k$, of A such that (1) r_0 is a run consisting of an initial configuration and $t_0 = 0$ steps, and (2) for all $i \in [k]$, r_i is a write-only sequential extension of r_{i-1} ending at time $t_i > 0$ that consists of i complete high-level writes of i distinct values $v_1, \ldots, v_i$ by i distinct clients $c_1, \ldots, c_i$ such that:*

a) $|Cov(t_i)| \geq if$

b) $\delta(Cov(t_i)) \cap F = \emptyset$

c) $|\delta(Tr(t_i) \setminus Cov(t_{i-1}))| > 2f$

d) $|\delta(\ Cov(t_i) \setminus Cov(t_{i-1})\ )| \geq f$

e) $Cov(t_i) \supseteq Cov(t_{i-1})$

Proof of c) – e). : Fix arbitrary $k > 0$, $f > 0$, and a set F of servers such that $|F| = f + 1$. We proceed by induction on i, $0 \leq i \leq k$. **Base:** Trivially holds for the run r_0 of A consisting of $t_0 = 0$ steps. **Step:** Assume that r_{i-1} exists for all $i \in [k-1]$. We use the extension r' constructed in the proof of Lemma 2 in Section 2.3 to show the implications c) – e) of the extended version are true:

c) $|\delta(Tr(t') \setminus Cov(t_{i-1}))| > 2f$: Follows immediately from Lemma 5 and Definition 2.1.

d) $|\delta(\ Cov(t') \setminus Cov(t_{i-1})\ )| \geq f$: Since by Corollary 2, $|Q_i(t_r)| = f$, and by Lemma 3.2, $Q_i(t_r) \subseteq Q_i(t')$, we get $|Q_i(t')| = f$. Hence, by Definition 2.4, $|\delta(Cov_i(t'))| \geq |Q_i(t')| \geq f$, and by Definition 2.3, $|\delta(\ Cov(t') \setminus Cov(t_{i-1})\ )| = |\delta(Cov_i(t'))| \geq f$.

e) $Cov(t_i) \supseteq Cov(t_{i-1})$: Follows immediately from Definition 4.

□

Theorem 12. *For all $k > 0$, and $f > 0$, let A be an f-tolerant algorithm emulating a WS-Safe obstruction-free k-register using a collection $\mathcal{S}$ of servers. Then, $|\mathcal{S}| \geq 2f + 1$.*

Proof. By Lemma 2.c), there exists a run r_1 of A consisting of t_1 steps such that $|\delta(Tr(t_1) \setminus Cov(0))| > 2f$. Therefore, $|\mathcal{S}| \geq |\delta(Tr(t_1))| \geq 2f + 1$. □

Number of registers per server. Theorem 2 implies that in case $|\mathcal{S}| = 2f + 1$, at least $(2f + 1)k$ registers are required. The following theorem further refines this result by showing that in this case, each server must store at least k registers:

Theorem 13. *Let $|\mathcal{S}| = 2f + 1$. For all $k > 0$, $f > 0$, every f-tolerant algorithm emulating an obstruction-free WS-Safe k-register stores at least k registers on each server in $\mathcal{S}$.*

Proof. Pick an arbitrary $f > 0, k > 0$, and suppose toward a contradiction that there is an f-tolerant algorithm A emulating an obstruction-free WS-Safe k-register that stores less than k registers on some server $s \in \mathcal{S}$ (i.e., $|\delta^{-1}(\{s\})| < k$).

Pick an arbitrary set $F \subset \mathcal{S}$ of size $|F| = f + 1$ such that $s \notin F$. By Lemma 2, there exist a sequential write-only run r_k consisting of k high-level write invocations $W_1, \ldots, W_k$ by k distinct clients, and k distinct times $t_1 < \cdots < t_k$ such that: $|\delta(Cov(t_1))| \geq f$ and $\delta(Cov(t_1)) \cap F = \emptyset$; and for all $i \in [k] \setminus \{1\}$, $|\delta(Cov(t_i) \setminus Cov(t_{i-1}))| \geq f$, $Cov(t_i) \supseteq Cov(t_{i-1})$, and $\delta(Cov(t_i)) \cap F = \emptyset$. By induction on $i \in [k]$, it is easy to see that all sets in the collection consisting of $Cov(t_1)$ and $Cov(t_i) \setminus Cov(t_{i-1})$ where $i \in [k] \setminus \{1\}$ are pairwise disjoint. Thus, at least f new registers become covered at each t_i, $i \in [k]$. Moreover, since no registers on the servers in F are covered at t_i, all registers that become covered at t_i must be located on the servers in $\mathcal{S} \setminus F$. Therefore, since $|\mathcal{S} \setminus F| = f$ and $s \in \mathcal{S} \setminus F$, we conclude that at least k distinct registers on s must be covered at time t_k, that is, $|\delta^{-1}(\{s\}) \cap Cov(t_k)| \geq k$. Therefore, $|\delta^{-1}(\{s\})| \geq k$. A contradiction. □

Servers with bounded storage. The following result provides a lower bound on the number of servers for the case the storage available on each server is bounded (as it is often the case in practice) by a known constant:

Theorem 14. *Let $m > 0$ be an upper bound on the number of registers mapped to each server in $\mathcal{S}$ (i.e., $\forall s \in \mathcal{S}, |\delta^{-1}(\{s\})| \leq m$). For all $f > 0$ and $k > 0$, every f-tolerant algorithm emulating an obstruction-free WS-Safe k-register from a collection $\mathcal{S}$ of servers such that $|\mathcal{S}| > 2f + 1$ uses at least $\lceil kf/m \rceil + f + 1$ servers (i.e., $|\mathcal{S}| \geq \lceil kf/m \rceil + f + 1$).*

Proof. Fix $F \subset S$ such that $|F| = f + 1$. By Lemma 2(a), there exists an extension r_k of r_{k-1} ending at time t_k such that $|\delta(T_r(t_k) \setminus Cov(t_{k-1}))| > 2f$. Since $|F| = f + 1$, $|\delta(T_r(t_k) \setminus Cov(t_{k-1})) \cap F| \leq f + 1$. Hence, we receive

$$|\delta(T_r(t_k) \setminus Cov(t_{k-1})) \setminus F| = \\ |\delta(T_r(t_k) \setminus Cov(t_{k-1}))| - |\delta(T_r(t_k) \setminus Cov(t_{k-1})) \cap F| \geq 2f + 1 - f - 1 = f$$

Thus,

$$|(T_r(t_k) \setminus Cov(t_{k-1})) \setminus \delta^{-1}(F)| \geq |\delta(T_r(t_k) \setminus Cov(t_{k-1})) \setminus F| \geq f \quad (1)$$

On the other hand,

$$|(T_r(t_k) \setminus Cov(t_{k-1})) \setminus \delta^{-1}(F)| = |(T_r(t_k) \setminus Cov(t_{k-1})) \cap \delta^{-1}(\mathcal{S} \setminus F)| = \\ |(T_r(t_k) \cap \delta^{-1}(\mathcal{S} \setminus F)) \setminus Cov(t_{k-1})| \leq |\delta^{-1}(\mathcal{S} \setminus F) \setminus Cov(t_{k-1})| \quad (2)$$

Since by Lemma 2(b) and (d), $|Cov(t_{k-1}| \geq (k-1)f$ and $\delta^{-1}(\mathcal{S} \setminus F) \supseteq Cov(t_{k-1})$, we get

$$|\delta^{-1}(\mathcal{S} \setminus F) \setminus Cov(t_{k-1})| = |\delta^{-1}(\mathcal{S} \setminus F)| - |Cov(t_{k-1})| \leq (|\mathcal{S} \setminus F|)m - (k-1)f \quad (3)$$

Combining (3) with (1) and (2), we get

$$(|\mathcal{S} \setminus F|)m - (k-1)f \geq |(T_r(t_k) \setminus Cov(t_{k-1})) \setminus \delta^{-1}(F)| \geq f$$

Since $\mathcal{S} \supset F$ and $|F| = f+1$, we obtain $(|\mathcal{S} \setminus F|)m - (k-1)f = |\mathcal{S}|m - (f+1)m - (k-1)f \geq f$. Therefore, $|\mathcal{S}|m \geq fm + m + kf$, which implies that $|\mathcal{S}| \geq kf/m + f + 1$. Since $|\mathcal{S}|$ is an integer, we conclude that $|\mathcal{S}| \geq \lceil kf/m \rceil + f + 1$. □

Adaptivity to Contention Given a run fragment r of an emulation algorithm, the *point contention* [2, 13] of r, PntCont(r), is the maximum number of clients that have an incomplete high-level invocation after some finite prefix of r. Similarly, we use PntCont(op) to denote PntCont(r_{op}), where r_{op} is the run fragment including all events between the op's invocation and response.

The resource complexity of A is *adaptive to point contention* if there exists a function M such that after all finite runs r of A, the resource consumption of A in r is bounded by $M(\text{PntCont}(r))$. Likewise, the time complexity of A is *adaptive to point contention* if there exists a function T such that for each client c_i, and operation op, the time to complete the invocation of op by c_i is bounded by $T(\text{PntCont}(op))$.

We show that no WS-Safe obstruction-free MWSR register can have a fault-tolerant emulation adaptive to point contention:

Theorem 15. *For any $f > 0$, there is no f-tolerant algorithm that emulates an WS-Safe obstruction-free k-register with resource complexity adaptive to point contention.*

Proof. By Lemma 2, there exists a run r of A consisting of k high-level writes by k distinct clients such that the resource complexity grows by f for each consecutive write that completes in r whereas the point contention remains equal 1 for the entire r. We conclude that no function mapping point contention to resource consumption can exist, and therefore, A's resource complexity is not adaptive to point contention. □

Hebrew Section

תקציר

בשנים האחרונות אנו רואים עלייה אקספוננציאלית בדרישות לאחסון מידע, מה שכמובן יוצר צורך בפתרונות לאחסון נתונים גדולים. בנוסף, הכלכלה של היום מדגישה מיזוגים, מה שהוליד מרכזי נתונים מסיבית וענני מחשוב. בעידן זה, אחסון מבוזר משחק תפקיד מפתח. זה מסומן על ידי שתי מגמות ברורות: ראשית, שוק האחסון הולך יותר ויותר לקראת פתרונות אחסון מבוזרים בין אם בתוך מרכזי נתונים ובין אם על פני מרכזי נתונים מרובים; מערכות כאלה בדרך כלל מורכבות מהרבה יחידות אחסון זולות בעלי אמינות נמוכה שמשיגות אמינות על ידי שמירה של עותקים של מידע במספר יחידות אחסון שונות. שנית, אנו רואים יותר ויותר משתמשים שמאחסנים נתונים בעננים וניגשים אליהם מרחוק דרך האינטרנט. בתזה זו אנו לומדים שני היבטים בסיסיים של אחסון אמין ומבוזר: שטח אחסון כולל ושיוני דינמי של יחידות האחסון הבסיסיות.

שטח אחסון. בהתחשב בנטייה הטבעית של יחידות אחסון זולות ורכיבי הרשת לכשול, אלגוריתם לאחסון אמין ומבוזר חייב לאחסן מידע עם יתירות על מנת לאפשר שחזור נתונים כאשר יש כשל בחלק מהמערכת. הגישה הנפוצה ביותר להשיג את זה היא באמצעות שכפול המידע, כלומר, אחסון עותקים מרובים של כל המידע ביחדות אחסון שונות. התוצאה הידועה של בר נוי, דולב, ועטייה מראה שאלגוריתם אמין לאחסון שמתמודד עם f נפילות (כשלים של יחידות אחסון) יכול להיות ממוש באמצעות אוסף של 2f+1 יחידות אחסון, כאשר כל יחידת אחסון שומרת אובייקט יחיד שתומך בפעולת קריאה-שינוי-כתיבה (RMW). תוצאה זו ידועה כאופטימלית במספר יחדות האחסון והאובייקטים. במחקר הזה אנחנו קודם כל מכלילים את החסם הזה וחוקרים את שטח האחסון הדרוש למימוש אלגוריתמים לאחסון מבוזר כפונקציה של סוג הפעולה (לדוגמה קריאה\כתיבה או קריאה-שינוי-כתיבה) הנתמכת על ידי האובייקטים ביחידות האחסון הבסיסיות. לאחר מכן, אנו מתמקדים בסוג האובייקט קריאה-שינוי-כתיבה וחוקרים האם קודים לתיקון שגיאות יכולים לעזור למזער את העלות המשמעות של שכפול מידע הנובעת מגודלו העצום. מספר עבודות קודמות ניסו לעשות זאת, אבל מבט מקרוב על פתרונות קיימים מגלה כי הם יוצרים עלויות אחסון במקומות אחרים. באופן ספציפי, השימוש בקידוד יוצר תרחישים שבהם מידע חדש לא יכול לדרוס מידע ישן, דבר המוביל לגדילה לא חוסומה של שטח האחסון כשאר המקביליות במערכת גדלה. במחקר זה אנו שופכים אור על שכלול התמורות בין יתירות נמוכה לכל בלוק מידע (כפי שמושג על ידי קודים), מקביליות, והיכולות לדרוס מידע ישן במידע חדש (כמו שמושג על ידי שכפול).

שיוני דינמי של יחידות האחסון הבסיסיות. אתגר נוסף של אחסון מבוזר הוא הבעיה של שינוי דינמי של יחידות האחסון הבסיסיות בהן משתמש האלגוריתם. כל מערכת אחסון המספקת אמינות וזמינות לתקופות זמן ארוכות חייבת להיות מסוגלת לשנות את סט יחידות האחסון הבסיסיות עליהן היא בנויה על מנת להוציא יחידות ישנות וכאלו שכשלו ולהחליפן ביחידות חדשות ומעודכנות. בנוסף, שיוני דינמי של יחידות האחסון הבסיסיות חיוני על מנת לממש את היתרון הגדול ביותר של אחסון מבוזר על פני אחסון מונוליטי מסורתי שהוא לאפשר תמיכה בגמישות וצמיחה - המוטיבציה העיקרית למודל הכלכלי של ענני המחשוב. בתזה זו אנו תורמים להבנה של שינוי שיוני דינמי של יחידות האחסון במערכות אחסון מבוזרות על ידי הוכחת תוצאות אי היתכנות, הבנתם התבונות והמגבלות הפונדמנטליות, והצגת אלגוריתמים שמתבססים על תובנות אלו. בפרט, אנחנו מראים ששינוי דינמי המקיים את תכונות האי המתנה אינו אפשרי כאשר מספר השינויים לא חסום, ואנחנו נותנים אלגוריתם אסינכרוני אופטימלי במספר הצעדים כתלות במספר השינויים במקרה החסום.

המחקר נעשה בהנחיית פרופ' עדית קידר בפקולטה להנדסת חשמל על שם ויטרבי בטכניון – מכון טכנולוגי לישראל.

תודתי נתונה לטכניון – מכון טכנולוגי לישראל ולקרן עזריאלי על התמיכה הכספית הנדיבה בהשתלמותי.

אלגוריתמים וחסמים במערכות אחסון מבוזרות

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר

דוקטור לפילוסופיה

אלכסנדר שפיגלמן

הוגש לסנט הטכניון – מכון טכנולוגי לישראל

סיון התשע"ח חיפה יוני 2018