Dynamic Reconfiguration: Abstraction and Optimal Asynchronous Solution

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Abstract

Providing clean and efficient foundations and tools for reconfiguration is a crucial enabler for distributed system management today. This work takes a step towards developing such foundations. It considers classic fault-tolerant atomic objects emulated on top of a static set of fault-prone servers, and turns them into dynamic ones. The specification of a dynamic object extends the corresponding static (non-dynamic) one with an API for changing the underlying set of fault-prone servers. Thus, in a dynamic model, an object can start in some configuration and continue in a different one. Its liveness is preserved through the reconfigurations it undergoes, tolerating a versatile set of faults as it shifts from one configuration to another.

In this paper we present a general abstraction for asynchronous reconfiguration, and exemplify its usefulness for building two dynamic objects: a read/write register and a max-register. We first define a dynamic model with a clean failure condition that allows an administrator to reconfigure the system and switch off a server once the reconfiguration operation removing it completes. We then define the Reconfiguration abstraction and show how it can be used to build dynamic registers and max-registers. Finally, we give an optimal asynchronous algorithm implementing the Reconfiguration abstraction, which in turn leads to the first asynchronous (consensus-free) dynamic register emulation with optimal complexity. More concretely, faced with n requests for configuration changes, the number of configurations that the dynamic register is implemented over is n; and the complexity of each client operation is $O(n)$.

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1 Introduction

Our experience with building real-life distributed systems repeatedly surfaces reconfiguration as an important issue whose practice is less understood than desired. Providing clean and efficient foundations and tools for reconfiguration is therefore a crucial enabler for today's distributed system management. We make a step towards providing such foundations in this paper.

The goal of this work is to take a static fault-tolerant object like an atomic read/write register and turn it into a dynamic fault-tolerant one. A static object exposes an API (e.g., read/write) to its clients, and is emulated on top of a set of fault-prone servers (sometimes called base objects) via protocols like ABD [5]. We refer to the underlying set of fault-prone servers as a *configuration*. To convert a static object into a dynamic one, we first extend the object's API to support *reconfiguration*. Such an API is essential for administrators, who should be able to remove old or faulty servers and add new ones without shutting down the service. One of the challenges in formalizing dynamic models is to define a precise fault condition, so that an administrator who requests to remove a server s via a reconfiguration

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operation will know when she can switch s off without risking losing the object's state (e.g., the last written value to a read/write register).

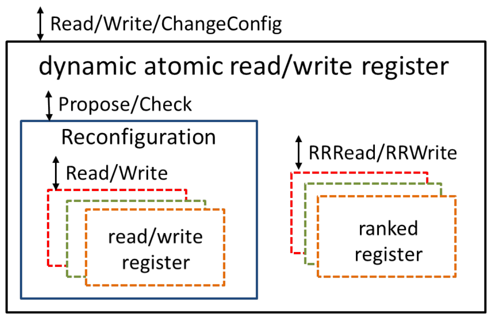
To this end , we first define a clean dynamic failure model, in which an administrator can immediately switch a server s off once a reconfiguration operation that removes s completes. Then, we provide an abstraction for consensus-less reconfiguration in this model. To demonstrate the power of our *Reconfiguration* abstraction we use it to implement two dynamic atomic objects. First, we focus on the basic building block of a read/write register; thus, other (static) objects that can be emulated from read/write registers (e.g., atomic snapshots) can be made dynamic by replacing the underlying registers with dynamic ones. Second, we emulate a max-register [4], which on the one hand can be implemented asynchronously [5, 12] (on top of fault-prone servers), and on the other hand cannot be emulated (for an unbounded number of clients) on top of a bounded set of read/write registers[1] [12, 4]. Thus, a standalone implementation of dynamic max-registers is required.

Complexity. We present an optimal-complexity implementation of our Reconfiguration abstraction in asynchronous environments, which in turn leads to the first optimal implementation of a dynamic read/write register in this model. More concretely, faced with n administrator reconfiguration requests, the number of configurations that the dynamic object is implemented over is n; and the number of rounds (when the algorithm accesses underlying servers) per client operation is $O(n)$. A comparison with previous solutions appears in Section 2.

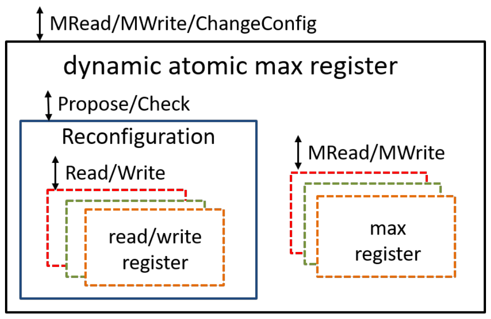
Dynamic fault model. In Section 3 we provide a succinct failure condition capturing a versatile set of faults under which the dynamic object's liveness is guaranteed. We define the dynamic fault model as an interplay between the object's implementation and its environment: New configurations are *introduced* by clients, (which are part of the object's environment). The object implementation then *activates* the requested configuration, at which point old configurations are *expired*. Between the time when a configuration is introduced and until it is expired, the environment can crash at most a minority of its servers. For example, when reconfiguring a register from configuration $\{A, B, C\}$ into $\{D, E, F\}$, initially a majority of $\{A, B, C\}$ must be available to allow read/write operations to complete. Then, when reconfiguration is triggered, $\{D, E, F\}$ is introduced, and subsequently, majorities of both configurations must be available, to allow state-transfer to occur. Finally, when the reconfiguration operation completes, leading to $\{D, E, F\}$'s activation, $\{A, B, C\}$ is expired, and every server in it may be immediately shutdown.

Reconfiguration abstraction. Since a configuration is a finite set of servers, we can use ABD [5] to emulate in each configuration a set of (static) atomic read/write registers (as well as max-registers), which are available as long as the configuration is not expired. The Reconfiguration abstraction, in contrast, is not tied to a specific configuration, but rather abstracts away the coordination among clients that wish to change the underlying set of servers (configuration) emulating the dynamic object. Its specification, which is formally defined in Section 4, exposes two API methods, *Propose* and *Check*. Clients use Propose to request changes to the configuration, and Check to learn of changes proposed by other clients. Both return a configuration and a set of *speculations*. The returned configuration

[1] A max-register for k clients requires at least k read/write registers [12].



(a) Dynamic atomic read/write register on top of the Reconfiguration abstraction.



(b) Dynamic atomic max-register on top of the Reconfiguration abstraction.

Figure 1 The Reconfiguration abstraction usage. Solid (dashed) blocks depict dynamic (resp. static) objects.

reflects all previous proposals and possibly some ongoing ones. The less obvious return value of Reconfiguration is the speculation set. This set is required since there is no guarantee that all clients see the same sequence of configurations (indeed, Reconfiguration is weaker than consensus). Therefore, a dynamic object implementation that uses Reconfiguration needs to read from every configuration that Check returned to any *other* client, and transfer the most up-to-date value read in any of these to the new configuration returned from Check. To this end, Reconfiguration returns a speculation set that includes all configurations previously returned to all clients (and possibly additional proposed ones).

In Section 5, we implement (1) a dynamic atomic read/write register on top of the Reconfiguration abstraction and static atomic ranked registers [11] (one in every configuration), and (2) a dynamic atomic max-register on top of Reconfiguration and static atomic max-registers. See Figure 1 for illustrations. In Section 6 we give an optimal consensus-less algorithm for Reconfiguration, which together with the read/write register emulation of Section 5 yields an optimal dynamic read/write register algorithm.

In summary, this paper makes three contributions: it defines a failure condition that allows an administrator to shutdown removed servers; it introduces the Reconfiguration abstraction, which captures the essence of reconfiguration; and it presents an asynchronous optimal-complexity solution for dynamic atomic registers. Section 7 concludes the paper.

2 Related Work

Model. The problem of object reconfiguration has gained growing attention in recent years [15, 20, 3, 21, 18, 14, 24, 13, 23, 17, 22, 6, 7]. However, dynamic failure models do not always make it clear when exactly an administrator can shutdown a removed server. Early works supporting dynamic objects [20, 15, 10] simply assume that a configuration is available as long as some client may try to access it. SmartMerge [18] uses a shared non-reconfigurable auxiliary object (lattice agreement) that is forever available to all clients, meaning that a majority of the servers emulating this auxiliary object can never be switched off. DynaStore [3] was the only previous work to define dynamic failure conditions based on a reconfiguration API, but these conditions are complicated, and restrict reconfiguration attempts as well as failures. Moreover, DynaStore does not separate clients from servers as we do here. Following [13, 18], we formulate the problem in shared memory, which makes it easier to reason about and clearer.

Other works [6, 7] assume a broadcast mechanism for announcing joins instead of an API for adding and removing processes, and bound the rate of changes of the underlying set of servers; the latter is necessary if one wants to ensure liveness for all operations (as [6] does) – no asynchronous reconfigurable service can ensure liveness unless the reconfiguration rate is limited in some way [23]. Like many earlier works [3, 13, 18], we do not explicitly bound the reconfiguration rate, and hence ensure liveness only if the number of reconfigurations is finite.

Abstractions. All previous works have considered reconfiguration in some specific context – state machine replication [19, 8, 9], read/write register emulation [3, 18, 13, 15], or atomic snapshot [22]. To the best of our knowledge, this work is the first to specify general dynamic objects as extensions of their static counterparts and to provide a general abstraction for dynamic reconfiguration. We note that although [13] define a reconfigure primitive intended to capture the core reconfiguration problem, that primitive is not sufficiently strong for implementing an atomic register, (in particular, since it does not require real-time order), and indeed, they do not implement their atomic register on top of it.

Dynamic register complexity. In a recent non-refereed tutorial [24], we give a generic formulation that allows us to compare the complexity of different algorithms [15, 20, 18, 13, 3], as follows: Given that n is the number of proposed configuration changes and m is the total number of operations (read/write/reconfig) invoked on the atomic register, DynaStore [3] goes through O$(min(mn, 2^n))$ configurations, and requires a constant number of operations in every configuration, so O$(min(mn, 2^n))$ is also DynaStore's operation complexity. Parsimonious SpSn [13] reduces the number of traversed configurations to $O(n)$, but since they invoke a linear number of operations in every configuration, their total operation complexity is $O(n^2)$.

Now notice that it is always possible to stagger reconfiguration proposals in a way that forces the system to go through $\Omega(n)$ configurations. The asymptotically optimal $O(n)$ operation complexity is straightforward to achieve in consensus-based solutions [15, 20, 10]. This complexity was also achieved by SmartMerge [18], but this was done using an auxiliary object that was assumed to be live indefinitely, i.e., was not reconfigurable in itself. Our algorithm is the first consensus-free and fully reconfigurable dynamic register algorithm with optimal complexity.

3 Dynamic Model

We consider a fault-prone shared memory model [16]: The system consists of an infinite set Π of *clients* (sometimes called processes), any number of which may fail by crashing, and an infinite set Φ of *servers* (sometimes called base objects) supporting arbitrary atomic low-level objects. Clients access servers via-low level operations (e.g., read/write), which may take arbitrarily long to arrive and complete, hence the system is asynchronous.

We address in the paper two atomic objects: a classical fault tolerant read/write register and a max-register [4]. Both registers provide clients with two API methods: Read and Write in case of read/write register, and MRead and MWrite in case of max-register. In a well-formed execution, a client invokes API methods one at a time, though calls by different clients may be interleaved in real time. For a well-formed execution, there exists a serialization of all client operations that preserves the operations' real time order, such that (1) in case of read/write register a Read returns the value written in the latest Write preceding it, or $\perp$ if there is no preceding Write; and (2) in case of max-register an MRead returns the highest value written by an MWrite that precedes it, or $\perp$ if there is no preceding MWrite. (In case of max-registers, the values domain is ordered.)

Configurations. The universe of servers is infinite, but at any moment in time, a client chooses to interact with a subset of it. In our model, a *configuration* is a set of included and excluded servers, where configuration *membership* is the set of included and not excluded servers in the configuration. Formally:

Changes	$\triangleq$	$\{+s \mid s \in \Phi\} \cup \{-s \mid s \in \Phi\}$
Configuration	$\triangleq$	subset of *Changes*
C.*membership*	$\triangleq$	$\{s \mid +s \in C \wedge -s \notin C\}$

For example $C = \{+s_1, +s_2, -s_2, +s_3\}$ is a configuration representing the inclusion of servers s_1, s_2, and s_3, and the exclusion of s_2, and $C.membership$ is $\{s_1, s_3\}$. Tracking excluded servers in addition to the configuration's membership is important in order to reconcile configurations suggested by different clients. The configuration size is the number of changes it includes– in this example, $|C| = 4$.

Dynamic fault model. A dynamic fault model is an interplay between the adversary's power and the following events, which are invoked as part of client operations:

introduce(C): indicates that C is going into use; and
activate(C): indicates that the state transfer to C is complete.

By convention we say that the initial configuration C_{init} is introduced and activated at time 0.

The above events govern the life-cycle of configurations. A configuration C becomes *activated* once an activate(C) event occurs. Note that not all introduced configurations are necessarily activated at some point. A configuration C becomes *expired* once activate(D) occurs s.t. C does not contain D. Intuitively, D reflects events (inclusions or exclusions) that are not reflected in C, and hence C has become "outdated". Our algorithm will enforce a containment order among activated configurations, and will thus ensure that the latest activated one is not expired.

The following two conditions constrain the power of the adversary:

▶ **Definition 1.** (liveness conditions)

Availability: The adversary can crash at most a minority of C.membership between the time when introduce(C) occurs and until C is expired.

Weak Oracle: When a client interacts with an expired configuration C, it either receives responses to calls from a majority of $C.membership$, or returns an exception notification $\langle error, D\rangle$ for some activated D, where $C \not\supseteq D$.

Note that such an oracle (sometimes called directory service) is inherently required in order to allow slow clients to find non-expired configurations in an asynchronous system where old configurations may become unavailable [2, 22]. Our oracle definition is weak– in particular, the activated configuration it returns may itself be expired, and different clients may get different responses; it can be trivially implemented using a broadcast mechanism as assumed in some previous works [6, 7], and trivially holds if configurations must remain available as long as some client may access them, as in other previous works [15, 20, 10].

Static versus dynamic objects. A *static object* is one in which clients interact with a fixed configuration. In order to disambiguate a static object, scoped within a configuration C, from a dynamic one, we will label the methods of a static object with a "C.". For every configuration C, as long as a majority of $C.membership$ is alive, clients can use ABD [5] to emulate (static) atomic registers on top of the servers in $C.membership$. We denote:

$C.x \leftarrow value$	A Write($value$) operation to register x in configuration C
$C.x$	A Read of x
$C.collect(array)$	A bulk Read of all the registers in $array$

Since a complete array can be collected from servers using ABD in the same number of rounds as reading a single variable, we count a collect as a single operation for complexity purposes. Note that each register in the array is atomic in itself, but the collect is not atomic.

The methods of a dynamic object are not scoped with any configuration; it can start in some configuration and continue in a different one. A dynamic object's API includes a ChangeConfig operation that allows clients to change the set of servers implementing the object. The implementation of ChangeConfig is object-specific, because it needs to transfer the state of the object across configurations, e.g., the last written value in case of an atomic register.

Clients pass to ChangeConfig a parameter $Proposal \subset Changes$ containing a proposed set of configuration changes. ChangeConfig returns a configuration C s.t. (1) C is activated, (2) $C \supseteq Proposal$, and (3) every configuration introduced or activated by ChangeConfig consists of C_{init} plus a subset of changes proposed by clients before the operation returns.

The liveness guarantee of a dynamic object is that, assuming the number of ChangeConfig proposals is finite, every correct client's operation eventually completes. Note that if the number of ChangeConfig proposals is infinite, it is impossible to ensure liveness for all operations [23].

Usage example. Consider an administrator (a privileged client) who wants to switch server s off and invokes ChangeConfig($\{-s\}$). By liveness, ChangeConfig completes, and by properties (1) and (2), it returns an activated configuration $C \supseteq \{-s\}$. The activation of C expires all configurations that do not contain C, and in particular, those that do not include $-s$. Hence, s is not part of the membership of any unexpired configuration, and by the availability condition, the administrator can safely switch s off immediately once ChangeConfig($\{-s\}$) returns.

4 Reconfiguration Abstraction

We introduce a generic reconfiguration abstraction, which can be used for implementing dynamic objects as we illustrate in the next section. A Reconfiguration abstraction has two operations:

Propose(C, P) for a configuration C and a proposed set of changes P; and
Check(C) for a configuration C.

Propose is used to reconfigure the system, whereas Check is used in order to learn about other clients' reconfiguration attempts. Propose and Check invoke the introduce and activate events. Both Check and Propose return a pair of values $\langle D, S\rangle$, where D is a configuration and S is a *speculation set* containing configurations; when $\langle D, S\rangle$ is returned we say that D is *nominated* by the operation that returns it. Intuitively, a nominated configuration is one that has been introduced and is a candidate for activation. By convention, we say that C_{init}

is nominated at time 0. We assume that the first argument passed to both operations is a nominated configuration.

The first propert of Reconfiguration is validity, which (i) requires Propose(C, P) to include P in the returned nominated configuration; and (ii) does not allow configurations to include spurious changes not proposed by any client. Formally:

D_1 (Validity) (i) If Propose(C, P) returns $\langle D, S \rangle$ for some S, then $D \supseteq P$, and (ii) for every configuration D that is introduced or nominated by an operation *op*, for every $e \in D \setminus C_{init}$, there is a Propose($C', P'$) for some C' that is invoked before *op* returns s.t. $e \in P'$.

The second property ensures that nominated configuration sizes monotonically increase over time, which is essential for real-time order of operations invoked on objects that use this abstraction:

D_2 (Real-time Order) A configuration D nominated by operation *op* is larger than or equal to every configuration nominated by an operation that strictly precedes *op*.

Since Reconfiguration is weaker than consensus, clients do not agree on a sequence of nominated configurations. Hence, in case some client c_1 proceeds to a configuration C', we want to ensure that if another client c_2 "skips" C', c_2 has C' in its speculation set, and can thus transfer any state that c_1 may have written there to the newer configuration c_2 nominates. This is captured by property S_1(ii) below. Property S_1(i) stipulates that these configurations are also introduced, ensuring a live majority in these configurations in order to allow state transfer.

S_1 (Speculation) If Check(C) or Propose(C, P) returns $\langle D, S \rangle$, then every $C' \in S$ is (i) introduced and (ii) S includes all nominated configurations C' s.t. $|C| \leq |C'| \leq |D|$. As a practical matter, if any C' between C and D has been activated, any C'' s.t. $|C''| < |C'|$ may be omitted.

In addition, we have to define when configurations are activated. Note that an activation of a new configuration leads to expiration of old ones, and thus to possible loss of information stored in them. Therefore, a configuration D is not immediately activated when a Propose returns $\langle D, S \rangle$ for some S. Instead, a configuration C is activated if Check(C) does not report any newer configuration:

A_1 (Activation) If Check(C) returns $\langle C, S \rangle$ for some S, then C is activated.

The liveness property of Reconfiguration is the same as in other dynamic objects [3, 18, 13, 22], namely, if the number of Propose operations is finite, then every operation by a correct client completes.

5 Building Dynamic Objects Using Reconfiguration

We first present a dynamic atomic read/write register emulation using Reconfiguration, then explain the modifications needed for supporting a dynamic atomic max-register [4], and finally provide a formal proof.

5.1 Dynamic atomic read/write register

Besides the Reconfiguration abstraction, our dynamic register implementation uses a (static) *ranked register* [11] emulation in every configuration, as illustrated in Figure 1a. A ranked register stores a tuple, called $version$, that consists of a value v and a monotonically increasing timestamp ts, and supports $RRRead()$ and $RRWrite(version)$ operations. The sequential specification of a *ranked register* is following: An $RRRead()$ returns the version with the highest ts written by an *RRWrite* that precedes it, or $\perp$ if there is no preceding *RRWrite*. Like all static objects in our model, if the configuration where the ranked register is emulated expires, the oracle returns an error.

The basic framework for implementing the Read, Write, and ChangeConfig operations is a loop: (i) Check, (ii) read (using RRread) the highest version from all speculated configurations returned by Check, (iii) write (with RRWrite) the highest version to the configuration nominated by Check, (iv) repeat. The loop terminates when Check does not nominate a new configuration. The specific action of each of the three operations is as follows. A Read simply returns the value of the highest version at the end of the loop. A Write increments the timestamp and writes it with a new value at the beginning of the loop. ChangeConfig proposes a configuration change via Propose instead of Check in the first iteration.

Algorithm 1 Dynamic atomic read/write register using Reconfiguration.

```
Client local variables:
    configuration $C_{curr}$, initially $C_{init}$
    $TS = \mathbb{N} \times \Pi$ with selectors $num$ and $id$
    $version \in \mathbb{V} \times TS$ with selectors $v$ and $ts$, initially $\langle v_0, \langle 0,\text{client's id}\rangle\rangle$
    $pickTS \in \{true, false\}$, initially $true$.

Code for client $c_i \in \Pi$:

Read()
    $transferState(Check(C_{curr}), \perp)$
    $checkConfig()$
    return $version.v$

Write(v)
    $transferState(Check(C_{curr}), v)$
    $checkConfig()$
    $pickTS \leftarrow true$
    return ok

ChangeConfig(P)
    $transferState(Propose(C_{curr}, P), \perp)$
    $checkConfig()$
    return $C_{curr}$

On $\langle error, D\rangle$ do
    $C_{curr} \leftarrow D$
    restart operation

procedure $checkConfig()$
    $\langle D, S\rangle \leftarrow Check(C_{curr})$
    while $D! = C_{curr}$ do
        $transferState(\langle D, S\rangle, \perp)$
        $\langle D, S\rangle \leftarrow Check(C_{curr})$

procedure $transferState(\langle D, S\rangle, value)$
    for each $C \in S$ do
        $tmp \leftarrow C.RRRead()$
        if $tmp.ts > version.ts$ then
            $version \leftarrow tmp$
    if $value \neq \perp \ \vee \ pickTS = true$ then
        $version \leftarrow \langle value, \langle version.ts.num + 1, i\rangle\rangle$
        $pickTS \leftarrow false$
    $D.RRWrite(version)$
    $C_{curr} \leftarrow D$
```

The pseudocode appears in Algorithm 1. The *transferState* method reads the register's version from the entire speculation set S and writes the latest version to the new configuration D. The *checkConfig* method repeatedly calls *transferState* until the configuration returned by *Check* stops changing. During the loop execution, an operation on an expired configuration may incur an exception, with a notification of the form $\langle error, D\rangle$ (see line 18). In this case, the loop is aborted and the operation starts over at configuration D. In case write is restarted after it has chosen a new timestamp, it skips the timestamp selection step.

We say that a configuration C becomes *stable* when some version is written to C in step (iii). We refer to the first version written to C as the *opening* version of C. Consider a completed operation (Read, Write, or ChangeConfig) *op* and let C be the last configuration in which *op* writes some version v, we say that *op* commits v in C when it completes. The correctness of the register emulation is based on the following key invariant:

▶ Invariant 1. For every stable configuration C, the opening version of C is higher than or equal to the highest version committed in any configuration C' s.t. $|C'| < |C|$.

In other words, a larger stable configuration always holds a newer (or equal) version of the register's value than that committed in a smaller activated one.

Algorithm 2 Dynamic atomic max-register using Reconfiguration.

```
   Client local variables:
1:     configuration $C_{curr}$, initially $C_{init}$
2:     $value \in \mathbb{V}$, initially $v_0$

   Code for client $c_i \in \Pi$:

3: MRead()
4:     $transferState(Check(C_{curr}), \perp)$
5:     $checkConfig()$
6:     return $value$

7: MWrite(v)
8:     $transferState(Check(C_{curr}), v)$
9:     $checkConfig()$
10:    return ok

11: ChangeConfig(P)
12:    $transferState(Propose(C_{curr}, P), \perp)$
13:    $checkConfig()$
14:    return $C_{curr}$

15: On $\langle error, D\rangle$ do
16:    $C_{curr} \leftarrow D$
17:    restart operation

18: procedure $checkConfig()$
19:    $\langle D, S\rangle \leftarrow Check(C_{curr})$
20:    while $D! = C_{curr}$ do
21:        $transferState(\langle D, S\rangle, \perp)$
22:        $\langle D, S\rangle \leftarrow Check(C_{curr})$

23: procedure $transferState(\langle D, S\rangle, v)$
24:    if $v \neq \perp$ then
25:        $value \leftarrow v$
26:    for each $C \in S$ do
27:        $tmp \leftarrow C.MRead()$
28:        if $tmp > value$ then
29:            $value \leftarrow tmp$
30:    $D.MWrite(value)$
31:    $C_{curr} \leftarrow D$
```

Complexity. We measure complexity in terms of the number of accesses to low level objects, namely static atomic registers. Note that Read/Write/collect operations on static registers are emulated in a constant number of rounds using ABD. The complexity of the dynamic register's operations is determined by (1) the complexity of the operations inside the Checks invoked during the loop (plus possibly one Propose); and (2) the sum of the sizes of all speculation sets returned by Propose/Check operations in this loop (where the register's implementation performs Reads).

In a run with n ChangeConfig proposals, clearly, the best complexity we can hope for is $O(n)$. In the next section we present our algorithm for Reconfiguration, which achieves the asymptotically optimal $O(n)$ complexity.

5.2 Dynamic atomic max-register

The emulation of a max-register on top of Reconfiguration is similar to the read/write register emulation. It differs in how we keep and transfer the state, i.e., the register's value. First, instead of a (static) ranked register in each configuration, we use a (static) max-register. Second, instead of timestamps, we use the actual written values, that is, a writer writes its value in step (iii) only if it is higher than all the values read in step (ii) (Otherwise, it transfers the highest value it read). The pseudocode appears in Algorithm 2.

5.3 Read/write register correctness proof

We start with notations:

Notation. We say that a version v is *higher* than a version v' if $v.ts > v'.ts$. An operation (Read, Write, or ChangeConfig) *stores* a version v in configuration C when it performs *C.RRWrite(v)* (line 34). A configuration C becomes *stable* when some operation stores a version in C. We say that the *opening* version of a stable configuration C is the first version that is stored in C. A configuration C *holds* a version v at time t if every *C.RRRead()* performed after time t returns a version that is higher than or equal to v. Consider a completed operation (Read, Write, or ChangeConfig) *op* and let C be the last configuration in which *op* stores some version v, we say that *op commits* v in C when it completes.

The following observation follows immediately from the specification of *ranked register*:

▶ Observation 1. A stable configuration holds at time t the highest version that was stored in it before time t.

The following observation follows immediately from the definitions of activated, stable and nominated configurations, and the code:

▶ Observation 2. Every activated configuration is Stable and every stable configuration is nominated.

The following observations follow immediately from the code:

▶ Observation 3. Every completed operation commits a version.

▶ Observation 4. A Check and Propose are always called with a stable configuration.

We now ready to prove the correctness of our register, which rely on the following invariant:

Invariant 1 (restated). *For every stable configuration C, the opening version of C is higher than or equal to the highest version committed in any configuration C' s.t. $|C'| < |C|$.*

Proof. Assume in a way of contradiction that there is a time t at which the invariant does not hold. Let C_{st} be the smallest stable configuration that violate the invariant at time t, let vst be the opening version of C_{st}, and let op_{st} be the operation that stores v_{st} in C_{st}. Now let t_1^{st} be the time when a *Check* or a *Propose* performed by op_{st} returns $\langle C_{st}, S\rangle$ for some S for the first time. Denote this *Check* or *Propose* by CP_{st}, and let t_2^{st} be the time when op_{st} invokes procedure *transferState*$(\langle C_{st}, S\rangle, \perp)$ (line 26). Note that $t_1^{st} < t_2^{st} < t$.

By the contradiction assumption there is at least one version, committed in a smaller configuration than C until time t, that is higher than v_{st}. Let op_c be an operation that commits version $v_c < v_{st}$ in configuration C_c until time t s.t. $|C_c| < C_{st}$. Now let t_1^c be the time when op_c stores v in C_c and let $t_2^c > t_1^c$ be the time when op_c invokes $Check(C_c)$ for the last time. Note that this $Check(C_c)$ returns $\langle C_c, S\rangle$ for some S. Now consider two case:

- First, $t_1^{st} < t_2^c$. Meaning that op_c invokes a $Check(C)$ that nominates C after op_{st} nominates C_{st}. A contradiction to property D_2 (Real time order) of the Reconfiguration abstraction.
- Second, $t_1^{st} > t_2^c$. Therefore, $t_2^{st} > t_1^c$, meaning that op_{st} performs *transferState*($\langle C_{st}, S\rangle, \perp$) after op_c stores v in C_c. Let C_{in} be the input configuration to CP_{st}. Now consider two case:
  - Fist, $C_c \in S$. Since op_{st} performs $C_c.RRRead()$ (in procedure *transferState*) after t_1^c, and since *transferState* stores the highest version it reads, we get that v_{st} is higher than or equal to v_c. A contradiction.
  - Second, $C_c \notin S$. By property S_1 (Speculation) of the Reconfiguration abstraction, this case is possible only if (1) $|C_c| < |C_{in}| \leq |C_{st}|$ and $C_{in} \in S$ or (2) S includes an activated configuration C_a s.t. $|C_c| < |C_a| \leq |C_{st}|$. By Observations 2 and 4, C_a and C_{in} are stable. Therefore, in both cases, S includes a stable configuration C' s.t. $|C_c| < |C'| \leq |C_{st}|$. Now consider two case:
    1. First, $|C'| = |C_{st}|$, meaning that C_{st} is stable at time t_1^{st}. Therefore, some operation stores a version in C_{st} before op_{st}. A contradiction to v_{st} being the opening version of C_{st}.
    2. Second, $|C'| < |C_{st}|$. Since C_{st} is the smallest stable configuration that violates the invariant at time t, C_a's opening version is higher than or equal to v_c. By Observation 1, and since *transferState* stores the highest version it reads from configurations in S, we get that v_{st} is higher than or equal to v_c. A contradiction.

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From Invariant 1 and Observation 1 we get the following corollary:

▶ **Corollary 2.** *At every time t a stable configuration C holds a version that is higher than or equal to the highest version committed in any configuration C' s.t. $|C'| \leq |C|$ before time t.*

The following lemma follows from Corollary 2 and the specification of the reconfiguration abstraction:

▶ **Lemma 3.** *Consider an operation op_1 that commits a version v_1 in configuration C_1, and an operation op_2 that begins after op_1 returns. Let $Check_2$ be a Check performed by op_2, and let $\langle C_2, S_2\rangle$ be the valued it returns. Then, S_2 includes a stable configuration that holds a version higher than or equal to v_1.*

Proof. Note that op_1 nominates C_1, and since op_1 precedes op_2, op_1 performs a Check that nominates C_1 before op_2 invokes $Check_2$. Thus, by property D_2 (real time order) of the reconfiguration abstraction, $|C_1| \leq |C_2|$. Moreover, by Observation 4 and property S_1 (speculation) of the reconfiguration abstraction, S_2 includes a stable configuration C_s s.t. $|C_s| \geq |C_1|$. By Corollary 2, C_s holds a version with a timestamp $ts_s \geq ts_1$.

◀

Notice that every completed Write operation picks exactly one timestamp (line 32), every picked timestamp is unique (ties are broken by clients ids), and there are no two different versions with the same timestamp. Thus, we say that Write operations are *associated* with the timestamp they pick. Note also that a Read operation returns the value in the version is commits. Therefore, we say that Read operations are associated with the timestamps of the versions they commits. The following Observation follows immediately from the code:

▶ Observation 5. A completed Write operation that is associated with timestamp ts commits a version with timestamp $ts' \geq ts$.

▶ **Lemma 4.** *Consider two completed Write operations w_1, w_2 that are associated with timestamps ts_1, ts_2, respectively. If w_1 precedes w_2, then $ts_1 < ts_2$.*

Proof. Let ts_1^c be the timestamp of the version committed by w_1. By Observation 5, $ts_1^c \geq ts_1$. Let $Cheak_2$ be the last check w_2 performs before picking a timestamp, and let $\langle C_2, S_2\rangle$ be its return value. By Lemma 3, S_2 includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1^c \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 > ts_s \geq ts_1^c \geq ts_1$. ◀

▶ **Lemma 5.** *Consider two completed Read operations rd_1, rd_2 that are associated with timestamps ts_1, ts_2, respectively. If rd_1 precedes rd_2, then $ts_1 \leq ts_2$.*

Proof. Let $Cheak_2$ be the Check operation that rd_2 performs before storing a version with timestamp ts_2, and let $\langle C_2, S_2\rangle$ be the return value of $check_2$. By Lemma 3, S_2 includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 \geq ts_s \geq ts_1$. ◀

▶ **Lemma 6.** *Consider a Write operation w_1 associated with timestamp ts_1, and a Read operation rd_2 associated with timestamp ts_2. If w_1 precedes rd_2, then $ts_1 \leq ts_2$.*

Proof. Let ts_1^c be the timestamp of the version committed by w_1. By Observation 5, $ts_1^c \geq ts_1$. Let $Cheak_2$ be the Check operation that rd_2 performs before storing a version with timestamp ts_2, and let $\langle C_2, S_2\rangle$ be the return value of $check_2$. By Lemma 3, S_2 includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1^c \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 \geq ts_s \geq ts_1^c \geq ts_1$. ◀

▶ **Lemma 7.** *Consider a Read operation rd_1 associated with timestamp ts_1, and a Write operation w_2 associated with timestamp ts_2. If rd_1 precedes w_2, then $ts_1 < ts_2$.*

Proof. Let $Cheak_2$ be the last check w_2 performs before picking a timestamp, and let $\langle C_2, S_2\rangle$ be its return value. By Lemma 3, S_2 includes a stable configuration that holds a version with a timestamp $ts_s \geq ts_1$. Therefore, by the code of *transferState*, $ts_2 > ts_s \geq ts_1$. ◀

▶ **Definition 8** (lineraization)**.** For every run r we define the sequential run σ_r as follows: All the Write operations in r are ordered in σ_r by the timestamp they are associated with, and every Read operation associated with a timestamp ts in r is ordered in σ_r immediately after the Write that is associated with ts. Read operations that are associated with the same timestamps are ordered (among themselves) by the time they return.

▶ **Theorem 9.** *The algorithm emulates an atomic register.*

Proof. We need to show that for every run r, σ_r is a linearization of r. The sequential specification is satisfied by construction, and the real time order follows from Lemmas 4, 5, 6, and 7.

◀

6 The Reconfiguration Abstraction Implementation

In this section we present an optimal and modular Reconfiguration implementation. In Section 6.1 we introduce the *Common Set (CoS)* building block, which is used by the Reconfiguration abstraction in every configuration. In Section 6.2 we show how CoS is used for non-optimal Reconfiguration and in Section 6.3 we optimize the algorithm. Formal proofs for correctness and complexity are given in Sections 6.4 and 6.5, respectively.

6.1 CoS building block

The *Common Set (CoS)* building clock is a static shared object, emulated in every configuration C over a set of (static) registers. Its API consists of a single operation, denoted C.CoS(P), where P is a set of arbitrary values. C.CoS returns an output set of sets satisfying the following:

▶ **Definition 10** (*Common Set* in configuration C).
(CoS_1) Each set in the output is the union of some of the inputs and strictly contains C;
(CoS_2) if a client's input strictly contains C, then its output is not empty;
(CoS_3) there is a common non-empty set in all non-empty outputs; and
(CoS_4) every C.CoS invocation that strictly follows a C.CoS call that returns a non-empty output returns a non-empty output.

For example, consider three concurrent clients that input to $C.CoS$ the sets P1, P2, and P3, all of which contain C. A possible outcome is for their outputs to be $\{P1\}, \{P1, P1 \cup P2\}$, and $\{P1, P2, P3\}$, respectively. The intuitive explanation behind using CoS is that it builds a *common sequence* of configurations inductively: The first configuration in the sequence is C_{init}, the next is the common configuration returned by $C_{init}.CoS$ (property CoS_3), and so on. Although this sequence is not known to the clients themselves, every client observes this sequence starting with some activated configuration. Every configuration in this sequence contains the previous one.

CoS can be implemented directly using consensus or atomic snapshot, as illustrated in [24]. In Algorithm 3, (without the PreCompute function, which is an optimization and will be discussed later), we give an implementation based on DynaStore's weak snapshot [3]. In the pseudocode, we denote by $\bigcup S$ the union of all sets in a set of sets S. If the proposal P strictly contains C, p_i has something new to propose and it writes P into its cell in the "weak" snapshot array *Warr* (lines 9-10). (Note that *Warr* is a static array emulated in the configuration where CoS is implemented). Either way, it collects *Warr* (line 11). In case the collect is not empty, p_i collects *Warr* again and returns the set of collected proposals (lines 12-15). The second collect ensures that the intersection of non-empty outputs includes the first written input, implying CoS_3; the remaining properties are satisfied by a single collect.

6.2 Simple Reconfiguration

Given CoS, we can solve Reconfiguration in a generic way as shown in Algorithm 4 (ignore the shaded areas for now). Both Check and Propose use the auxiliary procedure *reconfig*.

Algorithm 3 Efficient CoS; algorithm of client p_i in configuration C; optimization code shaded.

```
1: Local variables:                                          ▷ flags accessible outside CoS
2:     firstTime set by reconfig and read by CoS
3:     drop set by CoS and read by reconfig

4: Shared variables (emulated in configuration $C$):
5:     Boolean startingPoint, initially false               ▷ Is $C$ a starting point for some client
6:     Mapping from client to registers Warr and Sarr, initially {}.

7: procedure CoS($P$)
8:     $P \leftarrow PreCompute(P)$                          ▷ optimization
9:     if $P \supset C$ then
                ▷ Something new to propose
10:        $C.Warr[i] \leftarrow P$
11:    $ret \leftarrow C.collect(Warr)$
12:    if $ret = \{\}$ then
13:        return $ret$
14:    else
15:        return $C.collect(Warr)$

16: procedure PreCompute($P$)
17:    if firstTime then
18:        $C.startingPoint \leftarrow true$
19:    $C.Sarr[i] \leftarrow P$
20:    $drop \leftarrow false$
21:    if $\neg C.startingPoint$ then
22:        return $P$
       ▷ repeat collect until $P$ stops changing.
23:    $drop \leftarrow true$
24:    $tmp \leftarrow \bigcup C.collect(Sarr)$
25:    while $tmp \neq P$ do
26:        $P \leftarrow tmp$
27:        $tmp \leftarrow \bigcup C.collect(Sarr)$
28:    return $P$
```

Propose(C, P) first sets a local variable *proposal* to the union of C and P, whereas Check(C) initiates *proposal* to be C. Both then execute the loop in line 40. Each iteration selects the smallest configuration in *ToTrack*; we say that the iteration *tracks* this configuration. The loop tracks all configurations returned by CoS, smallest to largest, starting with C. In each tracked configuration C', the client introduces C', invokes $C'.CoS(proposal)$ and adds to *proposal* the union of the configurations returned from $C'.CoS$. This repeats for every configuration C' returned from CoS until there are no more configurations to track. Recall that by the liveness condition, if some configuration C' is expired and no longer supports $C'.CoS$, then the client gets in return to $C'.CoS$ an exception with some newer activated configuration C_a. In this case, *reconfig* starts over from C_a. At the end, Propose and Check return *proposal* and the set of all tracked configurations.

The common sequence starts with C_{init}, and is inductively defined as follows: If $C_k.CoS$ has a non-empty output, then C_{k+1} is the smallest common configuration returned by all non-empty $C_k.CoS$s. By CoS_3, all non-empty return values have at least one configuration in common, and if there is more than one such configuration, then we pick the smallest, breaking ties using lexicographic order. By CoS_1, each configuration in the common sequence strictly contains the previous one.

Algorithm 4 Generic Reconfiguration algorithm; optimization code shaded.

```
29: Propose(C, P)
30:     return $reconfig(C, P)$

31: Check(C)
32:     $ret \leftarrow reconfig(C, \{\})$
33:     if $ret = \langle C, * \rangle$ then $activate(C)$
34:     return $ret$

35: procedure $reconfig(C, P)$
36:     $proposal \leftarrow P \cup C$
37:     $ToTrack \leftarrow \{C\}$
38:     $speculation \leftarrow \{\}$
39:     $firstTime \leftarrow true$
40:     while $ToTrack \neq \{\}$ do
41:         $C' \leftarrow \operatorname{argmin}_{C'' \in ToTrack} |C''|$          ▷ smallest configuration
42:         $introduce(C')$
43:         $speculation \leftarrow speculation \cup \{C'\}$
44:         $ret \leftarrow C'.CoS(proposal)$
45:         if $ret = \langle$"error"$, C_a \rangle$ then          ▷ $C'$ is expired - restart from $C_a$
46:             return $reconfig(C_a, proposal)$
47:         $ToTrack \leftarrow (ToTrack \cup ret) \setminus \{C'\}$
48:         $firstTime \leftarrow false$
49:         if $drop = true$ then          ▷ drop old configurations in $ToTrack$
50:             $ToTrack \leftarrow ret$
51:         $proposal \leftarrow proposal \cup \bigcup ToTrack$
52:     $C_{curr} \leftarrow proposal$
53:     return $\langle proposal, speculation \rangle$
```

Correctness. The validity property (D_1) immediately follows from CoS property CoS_1 and the observation that *proposal* is set to include P at beginning of *reconfig* and never decreases.

To provide intuition for the remaining properties, we discuss the case in which all operations start in C_{init} and no exceptions occur; the proof for the general case appears in Section 6.4 Observe that since *proposal* always contains $\bigcup$ *ToTrack* and configurations are traversed from smallest to largest, we get from property CoS_2 that $C.CoS$ returns an empty set only if C includes *ToTrack*, i.e., C is the last traversed configuration. The key correctness argument is that all nominated configurations belong to the common sequence, and are thus related by containment:

▶ **Lemma 11.** *For every reconfig that returns $\langle D, S \rangle$, D belongs to the common sequence.*

Proof - sketch for the special case (starting in C_{init}, no exceptions). Assume by way of contradiction that D_j is returned by reconfig operation rec_j but does not belong to the common sequence. Note that C_{init} is in the common sequence and is tracked by rec_j. Let $\tilde{C_j}$ be the last configuration tracked by rec_j that belongs to the common sequence. By assumption, $\tilde{C_j} \neq D_j$, and thus, rec_j gets a non-empty output from $\tilde{C_j}.CoS$ (it gets an output since we assume that there are no exceptions). But, this output includes some configuration in the common sequence, so rec_j tracks a configuration in the common sequence after $\tilde{C_j}$. A contradiction.

Liveness follows since (i) every call to CoS returns, either successfully or with an exception; and (ii) tracked configurations are monotonically increasing, and, provided that the number of reconfigurations is finite, they are bounded.

6.3 Optimal Reconfiguration

The key to the efficiency of our new algorithm is in its thrifty CoS implementation, and the signals it conveys to the reconfiguration algorithm, which minimize the number of tracked configurations. To this end, the efficient solution for CoS shares (local) state variables *firstTime* and *drop* with the Reconfiguration implementation.

To explain the intuition behind our algorithm, let us first consider a scenario in which all clients invoke register operations (Read, Write, or ChangeConfig) in the same starting configuration C_0 (e.g., C_0 may be C_{init}), and no exceptions occur. If n of the clients invoke Propose, then there are n sets $P_1, \ldots, P_n$ proposed by $\mathit{reconfig}(C, P_i)$ operations. The unoptimized (weak snapshot-based) CoS may return up to 2^n different subsets in CoS responses (assuming many clients invoke Read/Write operations), inducing high complexity.

Our algorithm reduces this complexity by running a pre-computation phase in *PreCompute*, which imposes a containment order on all configurations passed to, and hence returned from, CoS. This is done by running a variant of (strong) atomic snapshot [1] on all client proposals in configuration C_0. Specifically, each process writes its own proposal P (line 19) to the “strong” array *Sarr*, and then (lines 24-27) repeatedly collects the union of all *Sarr* cells into P, until P stops changing. Like an atomic snapshot, this ensures that all results of PreCompute are related by containment. Note, however, that unlike an atomic snapshot, the complexity of this pre-computation is linear in the number of *different* proposals written, rather than in the number of participating processes; if collect encounters a newly written value that does not change the union of written values, PreCompute returns. In case all operations start in C_0, there are no new proposals in other configurations, and so the containment order is preserved throughout the computation. This ensures that the number of different configurations tracked by all clients is at most n.

Next, we account for the case that clients invoke (or restart due to exceptions) their operations in different starting configurations. We have to identify configurations where some client starts, and run PreCompute in them too. To this end, we have clients signal (by raising the startingPoint flag) if a configuration is their starting point. Every client that later runs $C.CoS$ sees this flag true, and executes the pre-computation. If a client p_i sees the flag false in $C.CoS$, p_i does not run the pre-computation. Nevertheless, since p_i checks the flag after writing its value to *Sarr*, p_i's proposal is already in the array before new clients that start in this configuration perform their collects, and so p_i's proposal is contained in theirs. Thus, at this new starting point, all clients obtain proposals that are related by containment among themselves.

The tricky part is that old proposals that were included in *ToTrack* before the new starting point are not necessarily ordered relative to ensuing proposals, as in the following scenario:

- Clients p_1 and p_2 start in C_0 and propose $C_0 \cup \{+a\}$ and $C_0 \cup \{+b\}$, respectively; p_1 gets $\{C_1\}$, where $C_1 = C_0 \cup \{+a\}$, from $C_0.CoS$ and p_2 gets $\{C_1, C_2\}$, where $C_2 = C_0 \cup \{+a, +b\}$.
- Client p_1 tracks C_1, gets an empty set from $C_1.CoS$, and activates it. Client p_3 starts in C_1, (which is now activated), proposes $C_3 = C_1 \cup \{+c\}$ in $C_1.CoS$, and gets $\{C_3\}$.
- Later, p_2 tracks C_1, and gets C_3 in $C_1.CoS$'s output. At this point p_2's *ToTrack* contains C_2 and C_3, neither of which contains the other.

To achieve linear complexity, we have clients *drop* all configurations previously returned from CoS at all the starting points they encounter. One subtle point is ensuring safety in the presence of such drops, and our proof of the general case of Lemma 11 addresses this issue.

Intuitively, since the purpose of tracking all configurations is to ensure that clients traverse the common sequence, once we know C is in the common sequence, there is no need to continue to track any configuration older than C. So, the drop is safe.

A second subtle point is preserving linear complexity despite executing PreCompute in multiple starting points. But since (i) the worst-case complexity of a single pre-computation is linear in the number of different proposals written to it, (ii) each CoS begins with a proposal that reflects all those seen in previous CoSs, and (iii) there are n new proposals overall, the combined complexity of *all* pre-computations is $O(n)$.

Finally, we provide intuition for the complexity of the high-level dynamic atomic register given in Section 5. The full proof, which wraps this intuition into a technical induction, appears in the next sections. Recall that the register emulation performs a loop in which it repeatedly calls Check(C), where C is the configuration returned from the previous Check/Propose, until some Check(C') returns $\langle C', S\rangle$ for some C' and S. The loop performs a constant number of operations in every configuration returned in a speculated set S from Check. Therefore, we want the Checks in this loop to return the optimal number of configurations, and have optimal complexity themselves.

Since all the configurations introduced (and returned in speculation sets) by our algorithm are related by containment, we immediately conclude that the number of configurations returned in speculated sets S of all Checks together is bounded by n. Now we show that the complexity of all Checks combined is $O(n)$. First observe that all Checks combined invoke at most n CoSs. Second, each CoS writes at most three times to shared registers (lines 10, 18, and 19), reads once (in line 21), and performs each of the collects in lines 11, 15, and 24 at most once. Now observe that CoS performs the collect in line 27 only if the previous collect (in line 24 or 27) contained a proposal $P_1 \not\subseteq P$, which means that none of the CoSs collected P_1 before. Since there are at most n proposals, all CoSs together perform the collect in line 27 at most n times. All in all, we get that the complexity of all Checks is $O(n)$.

6.4 Reconfiguration Correctness Proof

The proof makes use of the following simple observation:

▶ Observation 6. The output of *PreCompute* contains its input.

▶ **Lemma 12.** *Algorithm 3 implements CoS.*

Proof. We show the four properties of Definition 10:

***CoS*$_1$.** Each CoS's output is a set of proposals all of which were returned from PreCompute and, by line 9, strictly contain C. In PreCompute, P is always a subset of the union of inputs to CoS.

***CoS*$_2$.** Consider a configuration C and a client p_i that invokes $C.CoS(P)$ s.t. $P \supset C$. By Observation 6, $P \supset C$ also in line 9. Therefore, p_i writes P into $Warr[i]$ (line 10), and since no other client writes into the same cell, p_i collects a non-empty set in its collects (lines 11 and 15). Thus, p_i returns a non-empty set.

***CoS*$_3$.** If there are no non-empty outputs, then we are done. Otherwise, there is at least one client that writes its P to $Warr$. Let p_i be the first such client (because $Warr$ consists of atomic registers and atomicity is composable, the first is well-defined), and denote its P as P_i. We next show that every returned non-empty set contains P_i.

Consider a client p_j that returns a non-empty set. Then p_j collects $Warr$ twice, and the first collect is not empty. Therefore, p_j completes its first collect after p_i writes P_i into $Warr[i]$, and thus, it is guaranteed that p_j reads P_i from $Warr[i]$ during its second collect, and returns a set that contains P_i.

CoS_4. Consider two complete C.CoS calls op_i and op_j s.t. op_i strictly follows op_j, and op_j returns a non-empty set. There is a non-empty cell in *Warr* before op_j completes, and since nothing is erased from *Warr*, op_i's collects are not empty.

◀

We continue with the following notations:

Notation. We start with some notation that we use throughput the proof. The *common sequence* of configurations $C_{init}, C_1, C_2, ...$, which only an outside observer can view, is inductively defined as follows: It starts with C_{init}, and if $C_k.CoS$ has a non-empty output, then C_{k+1} is the smallest common configuration returned in all non-empty $C_k.CoS$ outputs (by CoS property CoS_3, the intersection is not empty), with ties broken in lexicographic order.

We say that a set is *monotonic* if all its elements are related by containment. A sequence is monotonic if every element $C^k \neq C^0$ contains C^{k-1}. By CoS property CoS_1, the common sequence is monotonic. For a *reconfig*(C, P) operation rec_j, we say a configuration is *tracked* by rec_j if it is selected as C' in line 41. We define $tracked(j) = C_j^0, C_j^1, \ldots, C_j^m$ to be the sequence of configurations tracked by rec_j in the last recursive call of *reconfig* in the order they are tracked. Note that $tracked(j)$ does not include configurations tracked before an exception is received. We further denote rec_j's return value by $\langle D_j, S_j \rangle$.

The following observations follow immediately from the code:

▶ Observation 7. Consider a reconfig operation rec_j, then $D_j = C_j^m \supseteq C_j^0 \cup \cdots \cup \ C_j^m$.

▶ Observation 8. If some reconfig reads drop=true in $C.CoS$ at time t, then there is a reconfig rec_j s.t. $C_j^0 = C$ that starts before time t.

The next claim stipulates that *reconfig* returns once CoS returns it an empty set.

▶ Claim 9. Consider a reconfig operation rec_j and a configuration $C' \in tracked(j)$. If $C'.CoS$ called during rec_j returns an empty set, then $C' = D_j$.

Proof. By CoS property CoS_2, since $C'.CoS$ returns an empty set, when $C'.CoS$ is called (line 44), $proposal \not\supset C'$. Moreover, whenever CoS is called during a *reconfig* operation, $proposal \supseteq \bigcup ToTrack$. Together, we get that C' is not strictly contained in $\bigcup ToTrack$. Now notice that C' is selected in line 41 as $\underset{C \in ToTrack}{\text{argmin}} \ |C|$. Therefore, we get that $ToTrack = \{C'\}$ when $C'.CoS$ is called (line 44). By the assumption, $C'.CoS$ returns $\{\}$. Hence, in line line:ToTrackUpdate1 50/47, *ToTrack* becomes $\{\}$, and thus the *reconfig* exits the while loop, and $C' = D_j$.

◀

The following observation follows from the usage of *reconfig* and the oracle behavior:

▶ Observation 10. For every reconfig operation rec_j, C_j^0 is a nominated configuration that is returned by some reconfig before rec_j is invoked.

The next claim shows that our algorithm drops old configurations only upon tracking a configuration in the common sequence.

▶ Claim 11. Assume that for every reconfig operation rec_j that returns before some time t, D_j belongs to the common sequence. Now consider a configuration C that belongs to the common sequence, and a reconfig operation rec_i that tracks C and returns at time t. If rec_i gets a non-empty output from $C.CoS$, then rec_i tracks another configuration belonging to the common sequence after C.

Proof. First observe that rec_i gets a configuration C' that belongs to the common sequence from $C.CoS$, and rec_i's *ToTrack* contains C' at the end of the corresponding while loop. If rec_i tracks C', we are done. Otherwise, rec_i drops C' after calling some $C''.CoS$ (while tracking C''). By Observation 8, there is a *reconfig* rec_j s.t. $C_j^0 = C''$ that starts before time t. By our assumption and by Observation 10, C'' belongs to the common sequence, and we are done. ◀

We now show that every nominated configuration belongs to the common sequence.

Lemma 11 (restated). *For every reconfig that returns $\langle D, S\rangle$, D belongs to the common sequence.*

Proof. We prove by induction on time $t \geq 0$ that for every *reconfig* operation rec_j that returns at time t, D_j belongs to the common sequence.

Base: Since no operation returns at time 0 or earlier, the lemma holds for $t = 0$.

Step: We now assume that the lemma holds for some $t \geq 0$, and prove for $t + 1$. Let rec_j be a *reconfig* operation that returns at time t. By Observation 10, C_j^0 was returned by some *reconfig* before time t. Therefore, by the induction assumption C_j^0 belongs to the common sequence. Now assume in a way of contradiction that D_j does not belong to the common sequence. Let C be the last configuration tracked by rec_j that belongs to the common sequence (there is such a configuration since C_j^0 belongs to the common sequence). By the contradiction assumption, $C \neq D_j$, and thus by Claim 9, rec_j gets a non-empty output from $C.CoS$. Therefore, by Claim 11, rec_j tracks a configuration belonging to the common sequence after C. A get a contradiction. ◀

The following is an immediate conclusion from Lemma 11 and the monotonicity of the common sequence.

▶ **Corollary 13.** *The set of nominated configurations is monotonic.*

▶ Claim 12. Consider a reconfig(C, P) operation rec_j that returns D_j. Then rec_j tracks every configuration in the common sequence between C and D_j.

Proof. Assume by way of contradiction that there is a configuration C' in the common sequence between C and D_j that rec_j does not track. Let C'' be the last configuration before C' in the common sequence that is tracked by rec_j. (There must be such a configuration because $C = C_j^0$ is in the common sequence.) By Claim 9, rec_j gets a non-empty output from $C''.CoS$, and by the common sequence definition, this output contains the next configuration C^{next} in the common sequence. Now recall that rec_j tracks configurations from smallest to largest, and it does not track C^{next}. Therefore, rec_j drops C^{next} after calling CoS in some configuration $\tilde{C}$ not in the common sequence. By Lemma 11, $\tilde{C}$ is not nominated, and thus not activated. Therefore, by the oracle definition no *reconfig* starts in $\tilde{C}$, and thus the *drop* flag in $\tilde{C}.CoS$ is always false. Hence, rec_j does not drop configurations after calling $\tilde{C}.CoS$. A contradiction. ◀

▶ **Theorem 14.** *Algorithms 4 and 3 implement the Reconfiguration abstraction.*

Proof. We now show that all Reconfiguration properties are satisfied:

D_1 (i) Consider a Propose(C, P) operation that calls *reconfig*(C, P) and returns $\langle D, S\rangle$. We have to show that $D \supseteq P$. Now observe that *proposal* is set to contain P at the beginning of the *reconfig*, and never decreases, including recursive calls to *reconfig*. The property follows from line 53. (ii) Consider a Check or a Propose operation *op* that introduces, activates, or return configuration C', and consider $e \in C' \setminus C \cup P$. Observe that *op* gets e by some CoS output. The property follows by inductively using CoS_1.

D_2 Consider an operation (Propose or Check) op_j that strictly precedes another operation op_i. Let rec_j be the *reconfig* operation that is called during op_j, and let rec_i be the *reconfig* operation that is called during op_i. Notice that rec_j strictly precedes rec_i. By Lemma 11, D_j and D_i are in the common sequence. From CoS_1, either $D_i \supseteq D_j$ or $D_j \supseteq D_i$. Assume by contradiction that $|D_i| < |D_j|$. Thus, $D_j \supset D_i$, and D_i precedes D_j in the common sequence. This means that $D_i.CoS$ returned a non-empty output to some *reconfig* operation before D_j was added to the common sequence, and so before rec_j returned D_j. Since rec_j strictly precedes rec_i, we get that $D_i.CoS$ returned a non-empty output before rec_i invoked $D_i.CoS$, and so by CoS_4, $D_i.CoS$ returns a non-empty output also to rec_i, a contradiction to the assumption that rec_i returns D_i.

S_1 Consider a Check(C) or Propose(C, P) operation *op* that returns $\langle D, S\rangle$. Let $rec_i =$ *reconfig(*C', P'*)* be the last *reconfig* operation that is called during *op*. By the oracle definition, by Lemma 11, and since every activate configuration is also nominated, *reconfig*(C, P) recursively calls *reconfig*$(C_a, *)$ only if C_a is activated and nominated, and $|C_a| > |C|$. Thus, if $C' \neq C$, then C' is activated and belongs to the common sequence. By Lemma 11, all the nominated configurations are in the common sequence. Therefore, by Claim 12 and by the observation that the *speculation* set of rec_i is *tracked*(i), S includes all nominated configurations C'' s.t. $|C'| \leq |C''| \leq |D|$.

◀

6.5 Reconfiguration Complexity Proof

We use the following additional notations:

Notation. For every introduced configuration C:

1. We define *OldProps(C)* to be the proposals suggested in $C.CoS$ by *reconfig* operations starting their traversals before C. That is,

   $OldProps(C) \triangleq \{P \mid \exists$*reconfig* that calls C.CoS(P) while its flag *firstTime* $=$ *false*$\}$

2. Since by Corollary 13, the set of nominated configurations is monotonic, there is at most one nominated configuration of a given size. Thus, we can define *Pred(C)* to be the biggest nominated configuration C' s.t. $|C'| < |C|$ for $C \neq C_{init}$. For completeness, we define $Pred(C_{init}) \triangleq C_{init}$.

For every nominated configuration C':

1. All introduced configurations that have C' as their Pred are its successors:

   $Successors(C') \triangleq \{C'\} \cup \{C \text{ is introduced} \mid C' = Pred(C)\},$ and

   $successors^i(C) \triangleq \{C' \in successors(C) \mid |C'| \leq i\}.$

2. All configurations that are in *ToTrack* of *reconfig* operations after tracking C' are potential successors of C':

$$PotentialSuccessors(C') \triangleq \{C \mid \text{some } reconfig \text{ calls } C'.CoS \text{ in line 44 and}$$

$$C \in ToTrack \text{ in line 51 in the same iteration}\}$$

An illustration of *Successors* and *PotentialSuccessors* appears in Figure 2.

We denote $introducedSet^i \triangleq \{C \text{ is introduced} \mid |C| \leq i\}$, and $nominatedSet^i \triangleq \{C \text{ is nominated} \mid |C| \leq i\}$

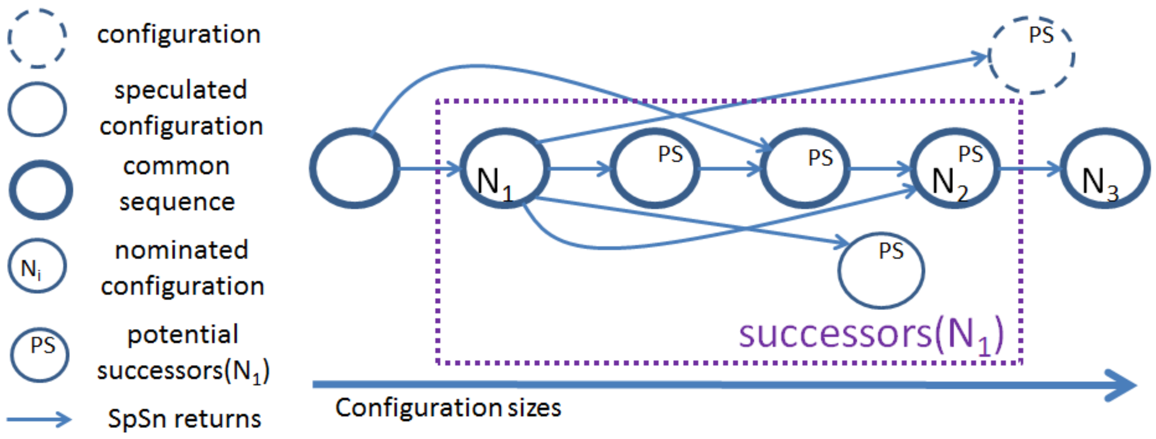

Figure 2 Example run of client p_i of our algorithm. The dashed configuration is in p_i's *ToTrack* after calling $N_1.CoS$, and thus it is in *potentialSuccessors*(N_1). But it is dropped after p_i calls $N_2.CoS$, and thus it is never introduced, and so it is not in *successors*(N_2).

We now state some observations that follow immediately from the code.

▶ Observation 13. Whenever CoS is called during a reconfig operation, $proposal \supseteq \bigcup ToTrack$.

▶ Observation 14. Consider a reconfig operation rec that calls $C.CoS(P)$ s.t. $P \supseteq C$ and returns ret. Then $\exists C' \in ret$ s.t. $P \subseteq C'$.

▶ Observation 15. The return value of every PreCompute that reads startPoint=false in $C.CoS$ is in OldProps(C).

▶ Observation 16. If a configuration C' is returned by $C.CoS$, then there is a $C.CoS$ invocation in which PreCompute returns C'.

▶ **Corollary 15.** *If all reconfig operations that call $C.CoS$ read startPoint=false, then all $C.CoS$'s return values in OldProps(C).*

▶ Observation 17. Let P_1 and P_2 be two proposals returned by PreCompute executions during $C.CoS$ pc_1 and pc_2, respectively. If pc_1 and pc_2 execute lines 24 to 27 (repeat collecting $Sarr$ until P stops changing), then P_1 and P_2 are related by containment.

▶ Claim 18. Consider configuration C. If $OldProps(C)$ is monotonic, then all configurations returned from $C.CoS$ invocations are related by containment.

Proof. By Observation 16, we need to show that all proposals returned from PreCompute invoked during $C.CoS$ are related by containment. Let P_1 and P_2 be two proposals returned by PreCompute executions during $C.CoS$ pc_1 and pc_2, respectively. We show that P_1 and P_2 are related by containment. Consider three cases:

1. First, pc_1 and pc_2 read *startPoint=false* in line 21 executions during $C.CoS$. By Observation 15, $P_1, P_2 \in OldProps(C)$. Since by the assumption $OldProps(C)$ is monotonic, we are done.
2. Second, pc_1 reads *startPoint=false*, and pc_2 reads *startPoint=true*. Note that since pc_1 reads *startPoint=false*, it is called with P_1. Now since *startPoint* never changes from *true* to *false*, pc_1 reads *startPoint* before pc_2. Thus, pc_1 writes P_1 into $Sarr$ in line 19 before pc_2 reads *startPoint=true*. Therefore, pc_2 sees P_1 in all the collects in lines 24 to 27, and thus $P_2 \supseteq P_1$.
3. Third, pc_1 and pc_2 read *startPoint=true*. Therefore, both execute lines 24 to 27, and thus by Observation 17, P_1 and P_2 are related by containment.

◀

▶ Claim 19. If a configuration C is nominated, then C is introduced.

Proof. By Lemma 11, C belongs to the common sequence, and so by the common sequence definition C is introduced.

◀

▶ Claim 20. Consider a reconfig operation rec_j that introduces configuration C' with firstTime=true and later introduces C. Then, $C' \subset C$.

Proof. Note that when C' is introduced, $ToTrack = \{C'\}$. Recall that if an error is returned, then rec_j is aborted, and no later configurations are introduced by rec_j. Thus, no error is returned when rec_j introduces C', and by by CoS property CoS_1, at the end of the iteration all the configurations in $ToTrack$ are strictly contain C'. Therefore, the claim follows by inductively repeating this argument.

◀

▶ **Corollary 16.** *Consider an introduced configuration C. Then, $C \supseteq C_{init}$.*

Proof. Let rec_j be a *reconfig*(C', P') that introduces C. Observe that rec_j introduces C' with *firstTime=true*. Thus, by Claim 20, $C \supseteq C'$. By Lemma 11, C' belongs to the common sequence, and by monotonicity of the common sequence, $C' \supseteq C_{init}$. Therefore, $C \supseteq C_{init}$.

◀

▶ Claim 21. Consider a *reconfig* operation rec_j that introduces configuration C in iteration *iter*. If firstTime=false at the beginning of *iter*, then there is at least one nominated configuration that is smaller than C, and rec_j tracks Pred(C) before *iter*.

Proof. Let C' be the configuration that rec_j is called with. Since C' and $Pred(C)$ are nominated, by Corollary 13, C' and $Pred(C)$ belong to the common sequence. Now observe that rec_j introduces C' with *firstTime=true* before it introduces C. Thus, by Claim 20, $|C'| < |C|$. If $C' = Pred(C)$, we are done, Otherwise, $|C'| < |Pred(C)| < |C|$. The Claim follows from Claim 12, and the observation that rec_j tracks configurations from the smallest to the biggest.

◀

▶ **Corollary 17.** *Consider an introduced configuration $C \neq C_{init}$. Then there is at least one nominated configuration that is smaller than $|C|$ and a reconfig operation that tracks $Pred(C)$ and introduces C.*

Proof. Let rec_j be a *reconfig* operation that introduces C in an iteration *iter*, and consider two cases:

1. First, flag *firstTime=false* at the beginning of *iter* during rec_j. The Corollary follows by Claim 21.
2. Otherwise, C is the parameter rec_j is called with, and thus C nominated. Let rec_i be the first *reconfig* that tracks C, and let t be that time. Therefore, no *reconfig* returns C before time t. hence, rec_i is not called with C, and thus *firstTime=false* when rec_i tracks C. The Corollary follows by Claim 21.

◀

▶ Claim 22. Consider a nominated configuration C, and an introduced configuration $C^{i+1} \in successors(C)$ of size $i+1 > |C|$. Assume that for every $C' \in successors^i(C)$, $PotentialSuccessors(C') \subseteq PotentialSuccessors(C)$, and $PotentialSuccessors(C)$ is monotonic. Then:

(a) $C^{i+1} \in PotentialSuccessors(C)$
(b) $OldProps(C^{i+1}) \subseteq PotentialSuccessors(C)$

Proof. (a) Since $C^{i+1} \neq C_{init}$, by Corollary 17, there is a *reconfig* operation rec_j that tracks $C = Pred(C^{i+1})$ and introduces C^{i+1}. Now let C'^{i+1} be the biggest configuration in $successors^i(C)$ that is tracked by rec_j. By the assumptions $PotentialSuccessors(C'^{i+1}) \subseteq PotentialSuccessors(C)$, and thus monotonic. Therefore, there is at most one configuration in $PotentialSuccessors(C'^{i+1})$ whose size is $i+1$. And since a *reconfig* operation tracks configurations by the order of their sizes, there is at least one configuration in $PotentialSuccessors(C'^{i+1})$ whose size is $i+1$. Therefore, there is exactly one configuration C''^{i+1} in $PotentialSuccessors(C'^{i+1})$ of size is $i+1$, and rec_j tracks C''^{i+1} immediately after C'^{i+2}. By CoS property CoS_1, $C''^{i+2} = C^{i+2}$, and we are done.

(b) Consider some $P \in OldProps_{C^{i+1}}$, we need to show that $P \in PotentialSuccessors(C)$. Let rec_k be a *reconfig* operation that calls $C^{i+1}.CoS$ while its flag *firstTime= false*. By Claim 21, rec_k tracks C. Let C'''^{i+1} be the biggest configuration in $successors^i(C)$ that is tracked by rec_k. Since rec_k tracks configurations according to their sizes and since by (a) C^{i+1} is the only configuration in $successors(C)$ of size is $i+1$, rec_k tracks C^{i+1} immediately after it tracks C'''^{i+1}. By Observation 14, rec_k's proposal in $C'''^{i+1}.CoS$ is included by some configuration in rec_k's *ToTrack* before rec_k introduces C^{i+1}, and by definition, rec_k's *ToTrack* is included in $PotentialSuccessors(C'''^{i+1})$ before rec_k introduces C^{i+1}. Now By the assumptions, $PotentialSuccessors(C'''^{i+1}) \subseteq PotentialSuccessors(C)$, and thus monotonic. Therefore, $P \in PotentialSuccessors(C'''^{i+1}) \subseteq PotentialSuccessors(C)$, and we are done. ◀

▶ Claim 23. Consider a nominated configuration C. Assume that no more configurations of size $|C|$ are introduced and $\{C\} \cup$PotentialSuccessors(C) is monotonic. Denote $x \triangleq max(\{j \mid \exists C' \in successors(C) : |C'| = j\})$. Then for every $|C| \leq i < x$:

- For every $C' \in successors^i(C)$, $PotentialSuccessors(C') \subseteq PotentialSuccessors(C)$

Proof. We prove by induction on i.

Base: $i = |C|$. If $i = x$, we are done. Otherwise, by the assumption, C is the only introduced configuration of size $|C|$, and the lemma follows.

Step: Now assume that the lemma holds for some $|C| \leq i < x$, we show that it holds for $i+1$. If $i+1 \geq x$, we are done. If there is no configuration in $successors(C)$ whose size is $i+1$, then

(1) follows by induction. Otherwise, by the Claim 22 (a) and since $PotentialSuccessors(C)$ is monotonic, there is exactly one configuration $C^{i+1} \in successors^{i+1}(C)$ of size is $i+1$. Since $i+1 < x$, C^{i+1} is not nominated. Therefore, whenever $C^{i+1}.CoS$ is called by a *reconfig* operation, its *firstTime = false*. Thus, all *reconfig* operations that call $C^{i+1}.CoS$ read *startingPoint=false*. Now let $C'^{i+1} \in PotentialSuccessors(C^{i+1})$, we show that $C'^{i+1} \in PotentialSuccessors(C)$. If C'^{i+1} is returned by $C^{i+1}.CoS$, then by Corollary 15, $C'^{i+1} \in OldProps(C^{i+1})$. By Claim 22 (b), $OldProps(C^{i+1}) \subseteq PotentialSuccessors(C)$. Therefore, $C'^{i+1} \in PotentialSuccessors(C)$, and we are done. Otherwise, there is a *reconfig* operation *rec* that has C'^{i+1} in its *ToTrack* before it invokes $C^{i+1}.CoS$. Thus *rec*'s *firstTime=false* when it introduces $C^{i+1}.CoS$, and thus by Claim 21, *rec* track C. Now let C''^{i+1} be the last configuration *rec* tracks before C^{i+1}, and note that $C''^{i+1} \in successors^{i}(C)$ and $C'^{i+1} \in PotentialSuccessors(C''^{i+1})$. By the induction assumption, $PotentialSuccessors(C''^{i+1}) \subseteq PotentialSuccessors(C)$. Therefore, $C'^{i+1} \in PotentialSuccessors(C)$, and we are done. ◀

The following corollary immediately follows from Claims 22 and 23:

▶ **Corollary 18.** *Consider two nominated configurations C, C' s.t. $C = Pred(C')$, and assume that no more configurations with size $|C|$ are introduced. If $\{C\} \cup$ PotentialSuccessors(C) is monotonic, then (1) successors(C) is monotonic, (2) OldProps$(C') \subseteq PotentialSuccessors(C)$, and (3) for every $C'' \in successors^{|C'|-1}(C)$, $PotentialSuccessors(C'') \subseteq PotentialSuccessors(C)$.*

▶ Claim 24. Consider two nominated configurations C, C' s.t. $C = Pred(C')$, and let S be a monotonic set. If $OldProps(C') \subseteq S$ and $\forall C'' \in successors^{|C'|-1}(C)$, $PotentialSuccessors(C'') \subseteq S$, then $\{C'\} \cup PotentialSuccessors(C')$ is monotonic.

Proof. Consider a configuration $C_1 \in PotentialSuccessors(C')$, we start by showing that $C_1 \supset C'$. By definition, there is a *reconfig* rec_1 that calls $C'.CoS$ in line 44 and $C_1 \in ToTrack$ in line 51 in the same iteration. Now consider two cases:

1. $C'.CoS$ in rec_1 returns C_1. Therefore, by CoS property CoS_1, $C_1 \supset C'$.
2. C_1 is in rec_1's *ToTrack* before it invokes $C'.CoS$. Thus, rec_1 calls $C'.CoS$ while its *firstTime = false*, and so by Claim 21, rec_1 tracks C. Now let C'' be the last configuration rec_1 tracks before C', and note that $C', C_1 \in PotentialSuccessors(C'')$. By the assumption, $PotentialSuccessors(C'')$ is monotonic, and by the observation that rec_j tracks configurations from smallest to biggest, $C_1 \supset C'$.

Consider another configuration $C_2 \in PotentialSuccessors(C')$. By definition, there is a *reconfig* rec_2 that calls $C'.CoS$ in line 44 and $C_2 \in ToTrack$ in line 51 in the same iteration. We now show that C_1 and C_2 are related by containment. there are three cases:

1. Both C_1 and C_2 are returned by $C'.CoS$. Therefore, by Claim 18, C_1 and C_2 are related by containment.
2. Neither C_1 nor C_2 is returned by $C'.CoS$. Thus C_1 (C_2) is in rec_1's (respectively, rec_2's) *ToTrack* before it invokes $C'.CoS$. Thus, rec_1 and rec_2 call $C'.CoS$ while their *firstTime = false*, and so by Claim 21, rec_1 and rec_2 track C. Now let C'_1 (C'_2) be the last configuration rec_1 (respectively, rec_2) tracks before C', and note that $C_1 \in PotentialSuccessors(C'_1)$ (and $C_2 \in PotentialSuccessors(C'_2)$). By the assumption, $PotentialSuccessors(C'_1)$,
$PotentialSuccessors(C'_2) \subseteq S$, and thus $C_1, C_2 \in S$. Now since S is monotonic, we are done.

3. One of the configurations, w.l.o.g. C_1 is returned by $C'.CoS$, and C_2 is not. In this case, C_2 is in rec_2's *ToTrack* before it invokes $C'.CoS(P_2)$ and thus, by Observation 13, $P_2 \supseteq C_2$, and as in above $C_2 \in S$. In addition, since rec_2 does not drop C_2 after $C'.CoS$ returns, it reads *startingPoint=false* during $C'.CoS$. By Observation 16, there is a *reconfig* operation rec'_1 that gets C_1 from PreCompute during $C'.CoS$. Now consider two cases:
    1. rec'_1 reads *startingPoint=false*. Therefore, rec'_1 calls $C'.CoS$ while its *firstTime=false*, and PreCompute returns its input which is therefore C_1. Thus, by definition, $C_1 \in OldProps(C')$. By assumption, $C_1 \in S$, and we are done.
    2. rec'_1 reads *startingPoint=true*. Since rec_2 reads *startingPoint=false* during $C'.CoS(P_2)$ and writes its proposal to *Sarr* (line 19) before reading *startingPoint* 21, and since *startingPoint* never changes from *true false* and rec'_1 finds it true, rec'_1 collects P_2 in *Sarr* in line 24. Therefore, $C_1 \supseteq P_2 \supseteq C_2$.

◀

▶ Claim 25. The set $PotentialSuccessors(C_{init}) \cup \{C_{init}\}$ is monotonic.

Proof. By Corollary 16 Claim 21, all *reconfig* operations that calls $C_{init}.CoS$ do so with *firstTime=true*. Therefore, $OldProps(C_{init}) = \{\}$ and all configurations in $PotentialSuccessors(C_{init})$ are returned from $C_{init}.CoS$. Thus, by CoS property CoS_1, all configurations in $PotentialSuccessors(C_{init})$ contain C_{init}, and by Claim 18, they are related by containment. The claim follows.

◀

▶ **Lemma 19.** *The set of introduced configurations is monotonic.*

Proof. Let $x = |C_{init}|$. We will show by induction on $i \geq x$ that the following are satisfied:

(a) The set $introducedSet^i$ is monotonic.
(b) For every configuration $C \in nominatedSet^i$, $\{C\} \cup PotentialSuccessors(C)$ is monotonic.

The lemma will follow from (a). **Base:** we prove for $i = x$. By Corollary 16, there is no introduced configuration other than C_{init} whose size is smaller than or equal to x. Hence, (a) is satisfied; (b) follows from Claim 25.

Step: assume by induction that (a) and (b) hold for $i \geq x$, we prove for $i + 1$. By Claim 19, every nominated configuration is also introduced, so if there is no introduced configuration whose size is $i + 1$, then we are done. Otherwise, let C be an introduced configuration s.t. $|C| = i + 1$, and let $C_p = Pred(C)$. Note that by definition, $C \in successors(C_p)$. Since $|C| = i + 1 > x = |C_{init}|$, $|C_p| < |C| = i + 1$. Therefore, by the induction assumption (b), $\{C_p\} \cup PotentialSuccessors(C_p)$ is monotonic, and by the induction assumption (a), C_p is the only introduced configuration of size $|C_p|$.

(a): By the first induction assumption $introducedSet^{|C_p|}$ is monotonic, thus it is enough to show that C contains or equals every configuration $C' \in introducedSet$ s.t. $|C_p| \leq |C'| \leq i+1$. Note that, by definition, $C' \in successors(C_p)$. By Corollary 18 (1), $successors(C_p)$ is monotonic. Therefore, C and C' are related by containment, and since $|C| = i + 1 \geq |C'|$, C contains or equals C', as requested.

(b): By (a), C is the only introduced configuration whose size is $i + 1$. If C is not nominated, then we are done. Otherwise, let $S = PotentialSuccessors(C_p)$ by Corollary 18 (2), $OldProps(C) \subseteq S$ and by 18 (3), for every $C'' \in successors^{|C|-1}(C_p)$, $PotentialSuccessors(C'') \subseteq S$. Therefore, by Claim 24, $\{C\} \cup PotentialSuccessors(C)$ is monotonic, as needed.

◀

We are now ready to conclude the complexity of the dynamic objects that uses our algorithm, which is captured by the following lemma:

▶ **Lemma 20.** *Consider an execution of a dynamic objects that uses our algorithm, and consider a loop in which Check(C) is repeatedly called, s.t. C is the configuration returned from the previous Check, until some Check(C') returns $\langle C', *\rangle$. Let n be the number of Propose(P) operations in the execution. Then:*

1. *All the Checks in the loop (together) return $O(n)$ configurations in the speculations sets.*
2. *The complexity of all Checks in the loop combined is $O(n)$.*

Proof. Since every Check in the loop starts where the previous returns, the Checks in the loop introduce different configurations. Thus by Lemma 19, we immediately conclude that the number of configurations returned in speculated sets of all Checks in the loop together is bounded by n. Moreover, by CoS_1, no configuration is returned more than once in the speculation sets. It is left to show that the complexity of all Checks combined is $O(n)$. First observe (again by Lemma 19) that all Checks combined invokes at most n CoSs. Second, each CoS writes at most three times to shared registers (lines 10, 18, and 19), reads once (in line 21), and performs each of the collects in lines 11, 15, and 24 at most once.

Now observe that CoS performs the collect in line 27 only if the previous collect (in line 24 or 27) contained a proposal $P_1 \not\subseteq P$, which means that none of the CoSs collected P_1 before. Since there are at most n proposals, all CoSs together perform the collect in line 27 at most n times. All in all, we get that the complexity of all Checks in the loop is $O(n)$.

◀

7 Conclusions

We defined a dynamic model with a clean failure condition that allows an administrator to reconfigure an object and switch a removed server off once the reconfiguration operation completes. In this model, we have captured a succinct abstraction for consensus-less reconfiguration, which dynamic objects like atomic read/write register and max-register may use. We demonstrated the power of our abstraction by providing an optimal implementation of a dynamic register, which has better complexity than previous solutions in the same model.

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