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Sampling Without Input Constraints: Consistent Reconstruction in Arbitrary Spaces

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ABSTRACT
In this chapter we develop a general framework for sampling and reconstruction procedures. The procedures we develop allow for almost arbitrary sampling and reconstruction spaces, as well as arbitrary input signals. The rudimentary constraint we impose on the reconstruction is that if the input lies in the reconstruction space, then the reconstruction will be equal to the original signal, so that our framework includes the more restrictive perfect reconstruction theories as special cases.

In our development we consider both nonredundant sampling and redundant sampling in which the samples constitute an overcomplete representation of the signal. In this case reconstruction is obtained using the oblique dual frame vectors, that lead to frame expansions in which the analysis and synthesis frame vectors are not constrained to lie in the same space as in conventional frame expansions. As we show, the algorithms we develop are consistent, so that the reconstructed signal has the property that although it is not necessarily equal to the original signal, it nonetheless yields the same measurements. Building upon this property of our algorithms, we develop a general procedure for constructing signals with prescribed properties.

1 Introduction

Sampling is the process of representing a signal f by a sequence of numbers, which can be interpreted as measurements of f. The classical approach is to choose the measurements as samples of f. A more recent approach [29, 20, 3, 27, 28, 11, 8, 33, 32] is to consider measurements that can be expressed as inner products of f with a set of sampling vectors that span a subspace $\mathcal{S}$, which is referred to as the sampling space. Examples include multiresolution decompositions [20, 7], and spline decompositions [27]. The problem then is to reconstruct f from these measurements, using a set of reconstruction vectors that span a subspace $\mathcal{W}$, which is referred to as the reconstruction space.

Standard sampling problems that have been studied extensively in the sampling literature are the problems of perfectly reconstructing a signal f from a given set of measurements, and the dual problem of sampling f so that it can be perfectly reconstructed using a given set of reconstruction vectors. If f does not lie in the reconstruction space $\mathcal{W}$, then it cannot be perfectly reconstructed using only reconstruction vectors that span $\mathcal{W}$. The traditional approach is therefore to assume that f lies in $\mathcal{W}$, or to choose a sampling method such that the reconstructed signal is the minimal-error approximation to f, *i.e.,* closest to f in a least-squares (l_2) sense. However, this requires the sampling space $\mathcal{S}$ to be equal to the reconstruction space $\mathcal{W}$.

An interesting problem first studied by Unser and Aldroubi [29] in the context of shift invariant spaces and later in [30, 31, 28, 9, 8], is the problem in which both the sampling vectors and the reconstruction vectors are specified, so that we have no freedom in choosing $\mathcal{S}$ and $\mathcal{W}$, and the input signal f can lie in an arbitrary space $\mathcal{H}$ containing $\mathcal{W}$, that may be larger than $\mathcal{W}$. If f does not lie in $\mathcal{W}$ and the sampling scheme is such that $\mathcal{S} \neq \mathcal{W}$, then the minimal-error approximation cannot be obtained. Therefore, in this general setting, our problem is to construct a good approximation of f given both a sampling method and a reconstruction method.

In this chapter we develop a broad sampling framework for nonredundant and redundant sampling and reconstruction. The procedures we develop allow for almost arbitrary sampling and reconstruction spaces, as well as arbitrary input signals. The rudimentary constraint we impose on the reconstruction is that if the input f lies in $\mathcal{W}$, then the reconstruction will be equal to f so that our framework includes the more restrictive perfect reconstruction theories as special cases. We will show that this requirement uniquely determines the reconstructed signal. Furthermore this reconstructed signal is a *consistent reconstruction* [29] of f, namely it has the property that although if f does not lie in $\mathcal{W}$ then it is not equal to f, it nonetheless yields the same measurements.

In our development, we consider both the case of nonredundant sampling and the case of redundant sampling in which the measurements constitute an overcomplete representation of the signal. To obtain a consistent reconstruction of f in this case, we introduce a generalization of the well known dual frame vectors [6], referred to as the *oblique dual frame vectors* [8]. These frame vectors have properties that are very similar to those of the conventional dual frame vectors. However, in contrast with the dual frame vectors, they are not constrained to lie in the same space as the original frame vectors.

By allowing for arbitrary sampling and reconstruction spaces, the sampling algorithms can be greatly simplified in many cases with only a minor increase in approximation error [29, 27, 28, 30, 4, 5]. Using oblique dual frame vectors we can further simplify the sampling and reconstruction processes while still retaining the flexibility of choosing the spaces almost arbitrarily, due to the extra degrees of freedom offered by the use of frames that allow us to construct frames with prescribed properties [14, 1]. Furthermore, using the redundant procedure we can reduce the quantization error when the measurements are quantized prior to reconstruction [8].

Building upon a geometric interpretation of the consistent sampling procedures, we also develop a general framework for constructing signals with prescribed properties. For example, using this framework we can construct a signal with specified lowpass coefficients and specified values over a time interval.

For simplicity of exposition the results in this chapter are derived for the finite-dimensional case; however, most of the results can be extended to the infinite-dimensional case, under certain mild constraints. For example, we must assume that the sampling and reconstruction spaces are closed, and that the sampling and reconstruction vectors satisfy certain norm constraints. This more general case is the subject of ongoing work.

In Section 2 we consider the sampling framework in detail, and develop a geometric interpretation of the sampling and reconstruction scheme that provides further insight into the problem. Section 3 considers explicit reconstruction methods and develops an interpretation of the sampling scheme in terms of the oblique pseudoinverse. The stability and reconstruction error resulting from our general scheme are analyzed in Section 4. In Section 5 we consider nonredundant sampling schemes, and illustrate the reconstruction in the context of concrete examples. In Section 6 we use the concept of oblique dual frame vectors to develop redundant sampling procedures, and to derive some general properties of our algorithms. Based on our consistent reconstruction algorithms, in Section 7 we develop a general framework for constructing signals with prescribed properties.

2 Consistent Reconstruction

2.1 Notation and Definitions

We denote vectors in an arbitrary Hilbert space $\mathcal{H}$ by lowercase letters, and the elements of a vector $c \in \mathbb{C}^N$ by $c[k]$. $P_{\mathcal{S}}$ denotes the orthogonal projection operator onto the space $\mathcal{S}$, and $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ denote the null space and range space of the corresponding operator, respectively. The *Moore-Penrose pseudoinverse* [12] of a transformation T is denoted by $T^\dagger$, and T^* denotes the adjoint of T. The inner product between vectors $x, y \in \mathcal{H}$ is denoted by $\langle x, y\rangle = x^* y$.

A *set transformation* $X\colon\ \mathbb{C}^N \rightarrow \mathcal{H}$ corresponding to $\{x_k \in \mathcal{H}, 1 \leq k \leq N\}$ is defined by $Xa = \sum_{k=1}^N a[k]x_k$ for any $a \in \mathbb{C}^N$. From the definition of the adjoint $X^*\colon\ \mathcal{H} \rightarrow \mathbb{C}^N$ it follows that if $a = X^* y$, then $a[k] = \langle x_k, y\rangle$.

A set of vectors $\{x_k \in \mathcal{H}, 1 \leq k \leq N\}$ forms a frame for an M-dimensional space $\mathcal{H}$ if there exists constants $A > 0$ and $B < \infty$ such that

$$A\|y\|^2 \leq \sum_{k=1}^{N} |\langle y, x_k\rangle|^2 \leq B\|y\|^2, \tag{2.1}$$

for all $y \in \mathcal{H}$ [6]. Although in principle N maybe infinite, we assume throughout that N is finite. With X denoting the set transformation corresponding to the vectors x_k, (2.1) is equivalent to

$$A\|Pa\|^2 \leq \langle a, X^*Xa\rangle \leq B\|Pa\|^2, \tag{2.2}$$

for any $a \in \mathbb{C}^N$, where $P = P_{\mathcal{N}(X)^\perp}$ is the orthogonal projection onto $\mathcal{N}(X)^\perp$. The lower bound in (2.1) ensures that the vectors x_k span $\mathcal{H}$; thus we must have $N \geq M$. If $N < \infty$, then the right hand inequality of (2.1) is always satisfied with $B = \sum_{k=1}^N \langle x_k, x_k\rangle$. Thus, any finite set of vectors that spans $\mathcal{H}$ is a frame for $\mathcal{H}$. If the bounds $A = B$ in (2.1), then the frame is called a tight frame. The redundancy of the frame is defined as $r = N/M$.

A set of vectors $\{x_k \in \mathcal{H}\}$ is a Riesz basis for $\mathcal{H}$ if it is complete, *i.e.,* the closure of the span of $\{x_k\}$ equals $\mathcal{H}$, and there exists constants $\alpha > 0$ and $\beta < \infty$ such that

$$\alpha \sum_k |a[k]|^2 \leq \left\| \sum_k a[k]x_k \right\|^2 \leq \beta \sum_k |a[k]|^2,$$

for all $a \in l_2$. We note that if $\mathcal{H}$ is a finite-dimensional space, then any basis for $\mathcal{H}$ is a Riesz basis.

2.2 Consistency Condition

Suppose we are given measurements $c[k]$ of a signal f that lies in an arbitrary Hilbert space $\mathcal{H}$. The measurements $c[k] = \langle s_k, f\rangle$ are obtained by taking the inner products of f with a set of N sampling vectors $\{s_k, 1 \leq k \leq N\}$ that span an M-dimensional subspace $\mathcal{S} \subseteq \mathcal{H}$, which is referred to as the sampling space. We construct an approximation $\hat{f}$ of f using a given set of N reconstruction vectors $\{w_k, 1 \leq k \leq N\}$ that span an M-dimensional subspace $\mathcal{W} \subseteq \mathcal{H}$, which is referred to as the reconstruction space. In the case of nonredundant sampling $N = M$ so that the sampling and reconstruction vectors form a basis for $\mathcal{S}$ and $\mathcal{W}$, respectively; in the case of redundant sampling $N > M$ and the sampling and reconstruction vectors form a frame for $\mathcal{S}$ and $\mathcal{W}$, respectively. We do not require the sampling space $\mathcal{S}$ and the reconstruction space $\mathcal{W}$ to be equal.

The reconstruction $\hat{f}$ has the form $\hat{f} = \sum_{k=1}^N d[k]w_k$ for some coefficients $d[k]$ that are a linear transformation of the measurements $c[k]$, so that $d = Hc$ for some H. With W and S denoting the set transformations corresponding to the vectors w_k and s_k respectively,

$$\hat{f} = \sum_{k=1}^N d[k]w_k = Wd = WHc = WHS^*f. \tag{2.3}$$

The sampling and reconstruction scheme is illustrated in Fig. 1.

Since $\hat{f}$ given by (2.3) always lies in $\mathcal{W}$, if $f \notin \mathcal{W}$, then $\hat{f} \neq f$. Because we are allowing the space of signals $\mathcal{H}$ to be larger than $\mathcal{W}$, we must replace the requirement for perfect reconstruction of $f \notin \mathcal{W}$ with a less stringent requirement.

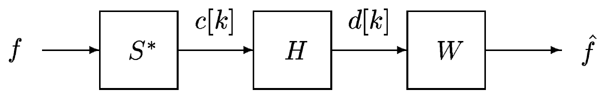


FIGURE 1. General sampling and reconstruction scheme.

Therefore, our problem is to choose H in Fig. 1 so that $\hat{f}$ is a good approximation of f. In particular, we require that if $f \in \mathcal{W}$, then $\hat{f} = f$. To this end we must have that $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$. For suppose that x is a nonzero signal in $\mathcal{W} \cap \mathcal{S}^{\perp}$. Then $c[k] = \langle s_k, x \rangle = 0$ for all k, and clearly x cannot be reconstructed from the measurements $c[k]$. Consequently, throughout this chapter we explicitly assume that $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$, and that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$. Note that if $\mathcal{W}$ and $\mathcal{S}$ both have finite and equal dimension, then $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$ implies that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$. However, this is not true in general for infinite-dimensional spaces.

The condition $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$ can be equivalently stated in terms of the cosine of the "angle" between the subspaces $\mathcal{S}$ and $\mathcal{W}$ [29, 2], which is defined by

$$\cos(\theta(\mathcal{W}, \mathcal{S})) = \inf_{w \in \mathcal{W}, \|w\|=1} \|P_{\mathcal{S}} w\|. \tag{2.4}$$

Specifically, if $\mathcal{W} + \mathcal{S}^{\perp}$ is closed in $\mathcal{H}$, then $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$ if and only if $\cos(\theta(\mathcal{W}, \mathcal{S})) > 0$ [26]. Furthermore, $\cos(\theta(\mathcal{W}, \mathcal{S})) > 0$ and $\cos(\theta(\mathcal{S}, \mathcal{W})) > 0$ if and only if $\mathcal{W} \oplus \mathcal{S}^{\perp} = \mathcal{H}$. Note that in general $\cos(\theta(\mathcal{S}, \mathcal{W})) \neq \cos(\theta(\mathcal{W}, \mathcal{S}))$ but we always have $\cos(\theta(\mathcal{S}, \mathcal{W})) = \cos(\theta(\mathcal{W}^{\perp}, \mathcal{S}^{\perp}))$. If $\mathcal{W} \oplus \mathcal{S}^{\perp} = \mathcal{H}$, then $\cos(\theta(\mathcal{S}, \mathcal{W})) = \cos(\theta(\mathcal{W}, \mathcal{S}))$ [26]. We also have equality when $\mathcal{W}$ and $\mathcal{S}$ are both shift invariant spaces [29].

Since we are requiring that $\hat{f} = WHS^*f = f$ for all $f \in \mathcal{W}$ it follows immediately that with $G = WHS^*$, $GGf = Gf$ for any $f \in \mathcal{W}$. Furthermore, since any $x \in \mathcal{H}$ can be expressed as $x = w + v$ with $w \in \mathcal{W}$ and $v \in \mathcal{S}^{\perp}$ and $S^*v = 0$, $Gx = Gw = w$, so that for any $f \in \mathcal{H}$, $GGf = Gf$. We conclude that G must be a projection operator. To specify G, we need to determine its null space $\mathcal{N}(G)$ and its range space $\mathcal{R}(G)$. Since $G = WHS^*$, $\mathcal{N}(G) \supseteq \mathcal{N}(S^*) = \mathcal{S}^{\perp}$ and $\mathcal{R}(G) \subseteq \mathcal{R}(W) = \mathcal{W}$. But since $Gf = f$ for all $f \in \mathcal{W}$ we have that $\mathcal{R}(G) = \mathcal{W}$ which immediately implies that $\mathcal{N}(G) = \mathcal{S}^{\perp}$. Therefore $G = WHS^*$ is an oblique[1] projection [15, 2, 18] with $\mathcal{R}(G) = \mathcal{W}$ and $\mathcal{N}(G) = \mathcal{S}^{\perp}$, denoted by $E_{\mathcal{W}\mathcal{S}^{\perp}}$. The oblique projection $E_{\mathcal{W}\mathcal{S}^{\perp}}$ is the unique operator satisfying

$$\begin{aligned} E_{\mathcal{W}\mathcal{S}^{\perp}} w &= w \text{ for any } w \in \mathcal{W}; \\ E_{\mathcal{W}\mathcal{S}^{\perp}} v &= 0 \text{ for any } v \in \mathcal{S}^{\perp}. \end{aligned} \tag{2.5}$$

[1] An oblique projection is a projection operator E satisfying $E^2 = E$ that is not necessarily Hermitian. The notation $E_{\mathcal{W}\mathcal{S}^{\perp}}$ denotes an oblique projection with range space $\mathcal{W}$ and null space $\mathcal{S}^{\perp}$. If $\mathcal{W} = \mathcal{S}$, then $E_{\mathcal{W}\mathcal{S}^{\perp}}$ is an orthogonal projection onto $\mathcal{W}$ which we denote by $P_{\mathcal{W}}$.

We therefore have the following theorem:

Theorem 2.1. *Let $\{c[k] = \langle s_k, f\rangle\}$ denote measurements of $f \in \mathcal{H}$ with sampling vectors $\{s_k\}$ that span a subspace $\mathcal{S} \subseteq \mathcal{H}$, and let the reconstruction vectors $\{w_k\}$ span a subspace $\mathcal{W} \subseteq \mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$. Then $\hat{f}$ is a reconstruction of f that reduces to a perfect reconstruction for all $f \in \mathcal{W}$ if and only if*

$$\hat{f} = E_{\mathcal{W}\mathcal{S}^\perp} f. \tag{2.6}$$

The reconstruction (2.6) has the additional property that it satisfies the consistency requirement as formulated by Unser and Aldroubi in [29]. A *consistent reconstruction* $\hat{f}$ of f has the property that if we measure it using the measurement vectors s_k, then the measurements will be equal to the measurements $c[k]$ of f. Since $\hat{f} = E_{\mathcal{W}\mathcal{S}^\perp} f$ it follows immediately that $S^*\hat{f} = S^* E_{\mathcal{W}\mathcal{S}^\perp} f = S^* f$, so that $\hat{f}$ is a consistent reconstruction of f. Furthermore, any consistent reconstruction $\hat{f}$ of f reduces to a perfect reconstruction for $f \in \mathcal{W}$. Indeed, if $f \in \mathcal{W}$ and $\hat{f}$ is a consistent reconstruction of f, then $\langle s_k, \hat{f}\rangle = \langle s_k, f\rangle$ for all k, so that $\langle s_k, f - \hat{f}\rangle = 0$, which implies that $f - \hat{f} \in \mathcal{S}^\perp$. But $f - \hat{f}$ also lies in $\mathcal{W}$, and since $\mathcal{W} \cap \mathcal{S}^\perp = \{0\}$ we conclude that $f = \hat{f}$. We therefore have the following corollary to Theorem 2.1:

Corollary 2.1. *Let $\{c[k] = \langle s_k, f\rangle\}$ denote measurements of $f \in \mathcal{H}$ with sampling vectors $\{s_k\}$ that span a subspace $\mathcal{S} \subseteq \mathcal{H}$, and let the reconstruction vectors $\{w_k\}$ span a subspace $\mathcal{W} \subseteq \mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$. Then $\hat{f}$ is a consistent reconstruction of f if and only if $\hat{f} = E_{\mathcal{W}\mathcal{S}^\perp} f$.*

Theorem 2.1 describes the form of the unique consistent reconstruction if it exists, however it does not establish its existence. In Section 3 we show that a consistent reconstruction can always be obtained, and we derive explicit reconstruction procedures. This then implies that if $f \in \mathcal{W}$, then f can be perfectly reconstructed from the measurements $c[k]$. Therefore, our results can also be used to generate new sampling theorems that yield perfect reconstruction. We will illustrate these ideas in the context of a concrete example in Section 5.2. Before proceeding to the detailed methods, in the next section we present a geometric interpretation of the sampling and reconstruction that provide further insight into the problem.

2.3 Geometric Interpretation of Sampling and Reconstruction

Let us first consider the case of perfect reconstruction for signals in $\mathcal{W}$. Thus, we would like to determine conditions under which any $f \in \mathcal{W}$ can be reconstructed from the measurements $c[k] = \langle f, s_k\rangle$. We first note that sampling f with measurement vectors in $\mathcal{S}$, is equivalent to sampling the orthogonal projection of f onto $\mathcal{S}$, denoted by $f_{\mathcal{S}} = P_{\mathcal{S}} f$. This follows from the relation

$$\langle s_k, f\rangle = \langle P_{\mathcal{S}} s_k, f\rangle = \langle s_k, P_{\mathcal{S}} f\rangle. \tag{2.7}$$

We may therefore decompose the sampling process into two stages, as illustrated in Fig. 2. In the first stage the signal f is (orthogonally) projected onto the sampling space $\mathcal{S}$, and in the second stage the projected signal $f_{\mathcal{S}}$ is measured. Since $f_{\mathcal{S}} \in \mathcal{S}$ and the vectors s_k span $\mathcal{S}$, $f_{\mathcal{S}}$ is uniquely determined by the measurements $c[k]$. Therefore, knowing $c[k]$ is equivalent to knowing $f_{\mathcal{S}}$.

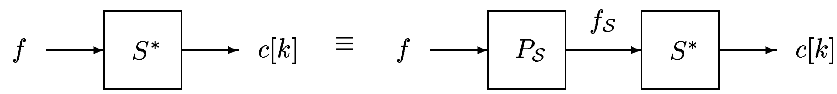


FIGURE 2. Decomposition of the sampling process into two stages.

In view of the interpretation of Fig. 2, our problem can be rephrased as follows. Can we reconstruct a signal in $\mathcal{W}$, given the orthogonal projection of the signal onto $\mathcal{S}$, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$? Fig. 3(a) depicts the orthogonal projection of an unknown signal $f \in \mathcal{W}$ onto $\mathcal{S}$, denoted $f_{\mathcal{S}}$. The problem then is to determine f from this projection. Since the "direction" of $\mathcal{W}$ is known, there is only one vector in $\mathcal{W}$ whose orthogonal projection onto $\mathcal{S}$ is $f_{\mathcal{S}}$; this vector is illustrated in Fig. 3(b). Thus, from this geometrical interpretation we conclude that for $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$, perfect reconstruction of any $f \in \mathcal{W}$ from the measurements $c[k]$ is always possible.

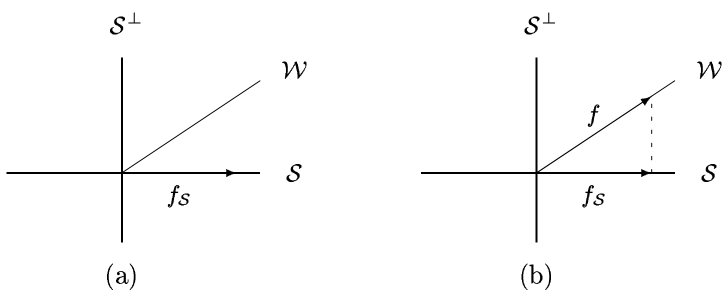


FIGURE 3. Illustration of perfect reconstruction of $f \in \mathcal{W}$ from $f_{\mathcal{S}} = P_{\mathcal{S}} f$, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$ (a) orthogonal projection of unknown signal in $\mathcal{W}$ onto $\mathcal{S}$ (b) unique signal in $\mathcal{W}$ with the given projection.

We now discuss consistent reconstruction for signals $f \in \mathcal{H}$. If $\hat{f}$ is a consistent reconstruction of f, then f and $\hat{f}$ have the same measurements: $c[k] = \langle s_k, f\rangle = \langle s_k, \hat{f}\rangle$. From our previous discussion it then follows that $f_{\mathcal{S}} = \hat{f}_{\mathcal{S}}$ where $\hat{f}_{\mathcal{S}} = P_{\mathcal{S}}\hat{f}$. Thus, geometrically a consistent reconstruction $\hat{f}$ of f is a signal in $\mathcal{W}$ whose orthogonal projection onto $\mathcal{S}$ is equal to the orthogonal projection of f onto $\mathcal{S}$, as illustrated in Fig. 4. Evidently, the consistent reconstruction is unique and always exists. We have seen in Theorem 2.1 that this reconstruction has a nice geometrical interpretation: It is the oblique projection of f onto $\mathcal{W}$ along $\mathcal{S}^\perp$. This interpretation is illustrated in Fig. 5, from which it is apparent that $E_{\mathcal{W}\mathcal{S}^\perp} f$ and f have the same orthogonal projection onto $\mathcal{S}$ and consequently yield the same measurements.

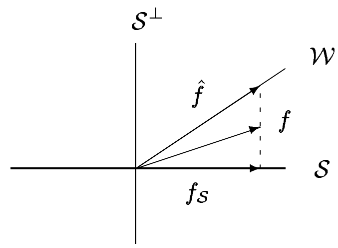


FIGURE 4. Illustration of consistent reconstruction of an arbitrary f from $f_\mathcal{S}$, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$.

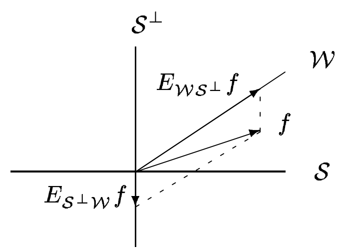


FIGURE 5. Decomposition of f into its components in $\mathcal{W}$ and $\mathcal{S}^\perp$ given by $E_{\mathcal{W}\mathcal{S}^\perp} f$ and $E_{\mathcal{S}^\perp\mathcal{W}} f$, respectively.

In summary, by considering a geometric interpretation of the sampling process and the consistency requirement we have demonstrated that perfect reconstruction of signals in $\mathcal{W}$ is always possible as long as $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$, and we illustrated the reconstruction geometrically. We also showed that under the same condition consistent reconstruction is always possible, and illustrated the reconstruction. It is important to note that the geometric interpretation (and Theorem 2.1) hold irrespective of whether the sampling process is nonredundant or redundant. In the next section we derive an explicit reconstruction scheme. We then specialize the results to the nonredundant case in Section 5, and to the redundant case in Section 6.

3 Reconstruction Scheme

3.1 Reconstruction Algorithm

From Theorem 2.1 and the geometric interpretation of Section 2.3 it follows that to obtain a consistent reconstruction $\hat{f}$ of f we need to determine H in Fig. 1 such that $G = WHS^* = E_{\mathcal{W}\mathcal{S}^\perp}$, *i.e.*, such that G satisfies (2.5). The following proposition establishes that with $H = (S^*W)^\dagger$,

$$\hat{f} = \sum_{k=1}^{N} d[k]w_k = Wd = W(S^*W)^\dagger c = W(S^*W)^\dagger S^* f, \tag{3.8}$$

is a consistent reconstruction of f for all $f \in \mathcal{H}$.

Proposition 3.1 ([8, 9]). *Let the vectors $\{s_k, 1 \leq k \leq N\}$ corresponding to S span an M-dimensional subspace $\mathcal{S} \subseteq \mathcal{H}$, and let the vectors $\{w_k, 1 \leq k \leq N\}$ corresponding to W span an M-dimensional subspace $\mathcal{W} \subseteq \mathcal{H}$, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$. Then $E_{\mathcal{W}\mathcal{S}^\perp}$ can be expressed as $E_{\mathcal{W}\mathcal{S}^\perp} = W(S^*W)^\dagger S^*$.*

If $f \in \mathcal{W}$ then $\hat{f} = E_{\mathcal{W}\mathcal{S}^\perp} f = f$, and f can be perfectly reconstructed from the measurements $c[k]$ using (3.8). By choosing different spaces $\mathcal{H}$, $\mathcal{W}$ and $\mathcal{S}$ and using (3.8), we can arrive at a variety of new and interesting perfect reconstruction sampling theorems.

From (3.8), $\hat{f}$ is obtained by first transforming the measurements $c[k]$ into "corrected" measurements $d[k]$ corresponding to $d = (S^*W)^\dagger c = Tf$, where $T = (S^*W)^\dagger S^*$. As we now show, T has an interesting interpretation: It is the *oblique pseudoinverse* of W on $\mathcal{V} = \mathcal{N}(W)^\perp$ along $\mathcal{S}^\perp$.

Alternatively, we can obtain $\hat{f}$ by transforming the reconstruction vectors w_k into "corrected" reconstruction vectors q_k corresponding to $Q = W(S^*W)^\dagger$, so that $\hat{f} = \sum_k c[k]q_k$. In analogy to T, in the next section we show that Q is the oblique pseudoinverse of S^* on $\mathcal{W}$ along $\mathcal{Q} = \mathcal{N}(S)$.

3.2 Oblique Pseudoinverse Interpretation of Reconstruction

Let $T\colon\ \mathcal{K} \to \mathcal{U}$ be a linear transformation and let $\mathcal{K} = \mathcal{G} \oplus \mathcal{N}(T)$ and $\mathcal{U} = \mathcal{R}(T) \oplus \mathcal{Z}$. The *oblique pseudoinverse* of T on $\mathcal{G}$ along $\mathcal{Z}$, denoted $T^\#_{\mathcal{G}\mathcal{Z}}$, is the unique transformation satisfying [21, 9]

$$TT^\#_{\mathcal{G}\mathcal{Z}} = E_{\mathcal{R}(T)\mathcal{Z}}; \tag{3.9}$$

$$T^\#_{\mathcal{G}\mathcal{Z}}T = E_{\mathcal{G}\mathcal{N}(T)}; \tag{3.10}$$

$$\mathcal{R}(T^\#_{\mathcal{G}\mathcal{Z}}) = \mathcal{G}. \tag{3.11}$$

As shown in [9], (3.9)–(3.11) imply that $T^\#_{\mathcal{G}\mathcal{Z}}$ inverts T between $\mathcal{G}$ and $\mathcal{R}(T)$, while nulling out any vector in $\mathcal{Z}$. This interpretation is illustrated in Fig. 6, from which it follows that the pseudoinverse $T^\dagger$ is a special case of the oblique pseudoinverse $T^\#_{\mathcal{G}\mathcal{Z}}$ for which $\mathcal{G} = \mathcal{N}(T)^\perp$ and $\mathcal{Z} = \mathcal{R}(T)^\perp$.

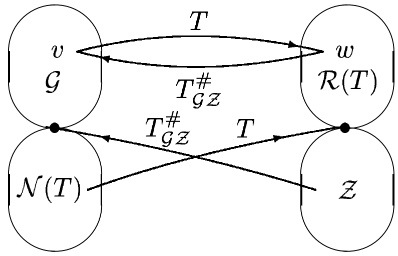


FIGURE 6. The action of T and $T^{\#}_{\mathcal{G}\mathcal{Z}}$ on the subspaces $\mathcal{G}, \mathcal{N}(T), \mathcal{R}(T)$ and $\mathcal{Z}$.

Proposition 3.2. *Let the vectors* $\{s_k, 1 \leq k \leq N\}$ *corresponding to* S *span an* M*-dimensional subspace* $\mathcal{S} \subseteq \mathcal{H}$*, and let the vectors* $\{w_k, 1 \leq k \leq N\}$ *corresponding to* W *span an* M*-dimensional subspace* $\mathcal{W} \subseteq \mathcal{H}$*, with* $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$*. Then*

1. *the oblique pseudoinverse of* W *on* $\mathcal{V} = \mathcal{N}(W)^{\perp}$ *along* $\mathcal{S}^{\perp}$ *can be expressed as* $W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}} = (S^*W)^{\dagger}S^*$*;*
2. *the oblique pseudoinverse of* S^* *on* $\mathcal{W}$ *along* $\mathcal{Q} = \mathcal{N}(S)$ *can be expressed as* $(S^*)^{\#}_{\mathcal{W}\mathcal{Q}} = W(S^*W)^{\dagger}$.

Proof. The proof of (1) is given in [8]; the proof of (2) is analogous to the proof of (1) and is therefore omitted. □

The actions of $W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}}$ and $(S^*)^{\#}_{\mathcal{W}\mathcal{Q}}$ are illustrated in Fig. 7.

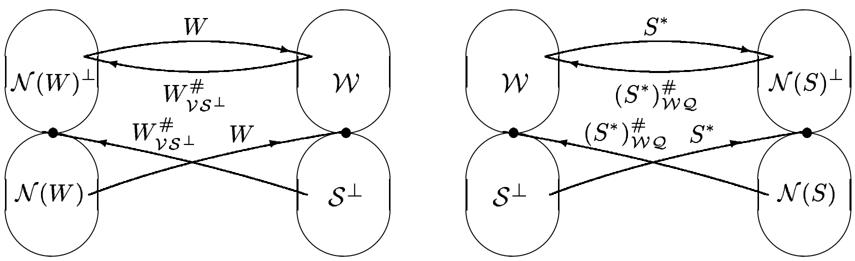


FIGURE 7. (a) The action of $W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}}$ with $\mathcal{V} = \mathcal{N}(W)^{\perp}$ (b) the action of $(S^*)^{\#}_{\mathcal{W}\mathcal{Q}}$ with $\mathcal{Q} = \mathcal{N}(S)$.

Comparing (3.8) with Proposition 3.2 we see that $\hat{f} = Wd$ where $d = W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}}f$. Thus $d[k] = \langle v_k, f\rangle$ where v_k are the vectors corresponding to $(W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}})^*$. Since $\mathcal{R}((W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}})^*) = \mathcal{N}(W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}})^{\perp} = \mathcal{S}$, the vectors $v_k \in \mathcal{S}$ span $\mathcal{S}$. Therefore, in the case of nonredundant sampling, *i.e.*, $N = M$, the vectors v_k form a basis for

$\mathcal{S}$, and in the case of redundant sampling, *i.e.,* $N > M$, the vectors v_k form a frame for $\mathcal{S}$. These basis and frame vectors have special properties which we discuss in Sections 5 and 6, respectively. Specifically, in Section 5 we show that in the case of nonredundant sampling, the vectors v_k form a basis for $\mathcal{S}$ that is biorthogonal to the basis vectors w_k. In Section 6 we show that in the case of redundant sampling, the vectors v_k form the *oblique dual frame* [8] of w_k on $\mathcal{S}$, which has properties analogous to the dual frame vectors.

Alternatively, $\hat{f} = Qc = \sum_{k=1}^{N} c[k]q_k$ where $Q = (S^*)^{\#}_{\mathcal{W}\mathcal{Q}}$ and the vectors q_k correspond to Q. Since $\mathcal{R}(Q) = \mathcal{W}$, the vectors $q_k \in \mathcal{W}$ span $\mathcal{W}$. As we show in Sections 5 and 6, when $N = M$ the vectors q_k form a basis for $\mathcal{W}$ that is biorthogonal to the basis vectors s_k, and when $N > M$ the vectors q_k form a frame for $\mathcal{W}$ that is the oblique dual frame of s_k on $\mathcal{W}$.

In Section 6 we will see that both of these interpretations are useful in developing properties of our general reconstruction scheme.

Before developing the properties of the vectors v_k and q_k, in the next section we consider the performance of our reconstruction algorithm. Specifically, we derive a figure of merit characterizing the stability of the reconstruction, and provide bounds on the norm of the reconstruction error.

4 Stability and Performance Analysis

4.1 Stability

We first consider the stability of our algorithm, namely the affect of a small perturbation of the measurements on the reconstructed signal.

Since $\hat{f} = Qc$ with $Q = W(S^*W)^{\dagger}$, and the vectors $\{q_k, 1 \le k \le N\}$ form a frame for $\mathcal{W}$,

$$A_q \|Pc\|^2 \le \|\hat{f}\|^2 \le B_q \|Pc\|^2, \tag{4.12}$$

for some constants $0 < A_q \le B_q$ and any $c \in \mathbb{C}^N$, where P is the orthogonal projection onto $\mathcal{N}(Q)^{\perp} = \mathcal{N}(S)^{\perp}$. Therefore, if the measurements $c[k]$ are perturbed by a sequence $e[k]$, then the perturbation in the output f_e satisfies

$$\sqrt{\frac{A_q}{B_q}} \frac{\|Pe\|^2}{\|Pc\|^2} \le \frac{\|f_e\|^2}{\|f\|^2} \le \sqrt{\frac{B_q}{A_q}} \frac{\|Pe\|^2}{\|Pc\|^2}. \tag{4.13}$$

Based on (4.13), Unser and Zerubia [30] propose the condition number

$$\kappa = \sqrt{\frac{B_q}{A_q}}, \tag{4.14}$$

as an indicator of the stability of the reconstruction algorithm. Here it is assumed that B_q and A_q are the tightest possible bounds in (4.12).

To derive κ for our reconstruction algorithm, let $G = (W^*S)^\dagger (W^*W)^{1/2}$ so that $GG^* = Q^*Q$. Then from (4.12),

$$A_q \|a\|^2 \leq \langle a, G^*Ga \rangle \leq B_q \|a\|^2, \tag{4.15}$$

for any $a \in \mathcal{N}(W)^\perp$, so that the tightest possible frame bounds are $A_q = \sigma_M^2(G)$ and $B_q = \sigma_1^2(G)$, where $\sigma_k(X)$ denote the nonzero singular values of X, $\sigma_1(X) = \max_k \sigma_k(X)$ and $\sigma_M(X) = \min_k \sigma_k(X)$. Thus,

$$\kappa = \frac{\sigma_1(G)}{\sigma_M(G)}. \tag{4.16}$$

Although (4.16) provides an explicit formula for κ, it is not very informative. In particular, it is difficult to see from this expression how the properties of the frame vectors s_k, and the subspaces $\mathcal{S}$ and $\mathcal{W}$ affect the stability of the reconstruction. In what follows, we develop an upper bound on κ that depends only on the angles between the different subspaces, and on the frame bounds of the vectors s_k, but not on the specific choice of frame vectors. Furthermore, we show that in many cases this bound is tight.

Since $((S^*S)^{1/2})^\dagger (S^*S)^{1/2} = P_{\mathcal{N}(S)^\perp}$ and $\mathcal{R}((W^*S)^\dagger) = \mathcal{N}(S)^\perp$, we can express G as $G = ZT$, with

$$Z = ((S^*S)^{1/2})^\dagger, \tag{4.17}$$

and

$$T = (S^*S)^{1/2}(W^*S)^\dagger (W^*W)^{1/2}. \tag{4.18}$$

It then follows that [16]

$$\sigma_1^2(G) \leq \sigma_1^2(Z)\sigma_1^2(T), \tag{4.19}$$

and

$$\sigma_M^2(G) \geq \sigma_M^2(Z)\sigma_M^2(T). \tag{4.20}$$

We have equality in (4.19) (resp. (4.20)) if and only if the right singular vector corresponding to $\sigma_1(Z)$ (resp. $\sigma_M(Z)$) is equal to the left singular vector corresponding to $\sigma_1(T)$ (resp. $\sigma_M(T)$). In particular, we have equality in both (4.19) and (4.20) if S^*S and TT^* commute.

Since the vectors s_k form a frame for $\mathcal{S}$, for any $a \in \mathcal{N}(S)^\perp$,

$$A_s \|a\|^2 \leq \langle a, S^*Sa \rangle \leq B_s \|a\|^2, \tag{4.21}$$

where the tightest bounds are $A_s = \sigma_M^2(S)$ and $B_s = \sigma_1^2(S)$. Therefore $\sigma_1^2(Z) = 1/A_s$ and $\sigma_M^2(Z) = 1/B_s$. Finally, $\sigma_1^2(T) = 1/\sigma_M^2(T^\dagger)$ and $\sigma_M^2(T) = 1/\sigma_1^2(T^\dagger)$ where using the fact that for two matrices D and H, $(DH)^\dagger = H^\dagger D^\dagger$ if $\mathcal{R}(H) = \mathcal{N}(D)^\perp$, $\mathcal{N}(DH) = \mathcal{N}(H)$ and $\mathcal{R}(DH) = \mathcal{R}(D)$, we have

$$T^\dagger = ((W^*W)^{1/2})^\dagger W^*S((S^*S)^{1/2})^\dagger. \tag{4.22}$$

Since $\mathcal{N}(T^\dagger) = \mathcal{N}(S)$ and $P_{\mathcal{W}} = W(W^*W)^\dagger W^*$,

$$\sigma_M^2(T^\dagger) = \min_{z \in \mathcal{N}(S)^\perp, \|z\|=1} \|T^\dagger z\|^2 = \min_{s \in \mathcal{S}, \|s\|=1} \|P_{\mathcal{W}} s\|^2 = \cos^2(\theta(\mathcal{S}, \mathcal{W})), \quad (4.23)$$

where $\cos(\theta(\mathcal{S}, \mathcal{W}))$ is defined by (2.4), and

$$\sigma_1^2(T^\dagger) = \max_{s \in \mathcal{S}, \|s\|=1} \|P_{\mathcal{W}} s\|^2 = 1 - \min_{s \in \mathcal{S}, \|s\|=1} \|(I - P_{\mathcal{W}})s\|^2 = 1 - \cos^2(\theta(\mathcal{S}, \mathcal{W}^\perp)). \quad (4.24)$$

We conclude that

$$1 \le \kappa \le \sqrt{\frac{B_s}{A_s}} \frac{\sqrt{1 - \cos^2(\theta(\mathcal{S}, \mathcal{W}^\perp))}}{\cos(\theta(\mathcal{S}, \mathcal{W}))}. \quad (4.25)$$

If S^*S and TT^* commute, then

$$\kappa = \sqrt{\frac{B_s}{A_s}} \frac{\sqrt{1 - \cos^2(\theta(\mathcal{S}, \mathcal{W}^\perp))}}{\cos(\theta(\mathcal{S}, \mathcal{W}))}, \quad (4.26)$$

so that in this case the stability of our algorithm does not depend on the specific choice of frame vectors s_k or on the specific choice of subspaces $\mathcal{S}$ and $\mathcal{W}$, but only on the frame bounds and the appropriate angles between the subspaces.

If the vectors s_k form a tight frame for $\mathcal{S}$, then $SS^* = AP_{\mathcal{N}(S)^\perp}$ for some $A > 0$, and S^*S is an orthogonal projection. In this case $S^*S = (S^*S)^{1/2}$, and S^*S and TT^* commute. In addition $A_s = B_s = A$, so that $\kappa = \sqrt{1 - \cos^2(\theta(\mathcal{S}, \mathcal{W}^\perp))} / \cos(\theta(\mathcal{S}, \mathcal{W}))$.

As we show in Section 5.1, in the special case in which $f = f(t)$ lies in L_2, and $\mathcal{S}$ and $\mathcal{W}$ are shift invariant spaces generated by integer translates of appropriately chosen functions, each of the infinite matrices S^*S, W^*S and W^*W are Toeplitz matrices which are diagonalized by a Fourier transform matrix. Therefore in this case TT^* and S^*S commute and κ is given by (4.26). An explicit expression for $\cos(\theta(\mathcal{S}, \mathcal{W}))$ in this case is given in [29].

4.2 Performance Analysis

Since our reconstruction algorithm does not yield perfect reconstruction for all $f \in \mathcal{H}$, there is an error associated with the reconstruction. If $\mathcal{S} = \mathcal{W}$, then $E_{\mathcal{W}\mathcal{S}^\perp} = P_{\mathcal{W}}$ and our algorithm will result in a reconstruction that minimizes the norm of the reconstruction error. If $\mathcal{S} \ne \mathcal{W}$, then the minimal error approximation cannot be obtained. Nonetheless, the norm of the reconstruction error $f - E_{\mathcal{W}\mathcal{S}^\perp} f$ can be bounded based on results derived in [29],

$$\|f - P_{\mathcal{W}} f\| \le \|f - E_{\mathcal{W}\mathcal{S}^\perp} f\| \le \frac{1}{\cos(\theta_{\mathcal{W}\mathcal{S}})} \|f - P_{\mathcal{W}} f\|, \quad (4.27)$$

where $\|f - P_{\mathcal{W}} f\|$ is the minimal norm of the reconstruction error corresponding to the case in which $\mathcal{W} = \mathcal{S}$.

From (4.27) we see that there is a penalty for the flexibility offered by choosing $\mathcal{S}$ (almost) arbitrarily: The norm of the reconstruction error for $f \notin \mathcal{W}$ is increased. However, in many practical applications this increase in error is very small [28, 30, 4, 5].

5 Reconstruction From Nonredundant Measurements

Suppose that the sampling vectors $\{s_k, 1 \leq k \leq M\}$ form a basis for $\mathcal{S}$ and the reconstruction vectors $\{w_k, 1 \leq k \leq M\}$ form a basis for $\mathcal{W}$. Then, from Proposition 5.1 below S^*W is invertible so that the general reconstruction formula (3.8) reduces to

$$\hat{f} = \sum_{k=1}^{M} d[k]w_k = Wd = W(S^*W)^{-1}S^*f. \tag{5.28}$$

Proposition 5.1 **([8]).** *Let the vectors $\{s_k, 1 \leq k \leq M\}$ corresponding to S denote a basis for an M-dimensional subspace $\mathcal{S}$ of $\mathcal{H}$, and let the vectors $\{w_k, 1 \leq k \leq M\}$ corresponding to W denote a basis for an M-dimensional subspace $\mathcal{W}$ of $\mathcal{H}$. Then S^*W is invertible if and only if $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$.*

A similar proposition was proved in [9] for the infinite-dimensional case in which the vectors s_k and w_k form Riesz bases for $\mathcal{S}$ and $\mathcal{W}$ respectively. Furthermore, since S^*W is bounded, the open map theorem implies that $(S^*W)^{-1}$ is also bounded.

The resulting measurement and reconstruction scheme is depicted in Fig. 8. Note, that since $\hat{f}$ is unique and the vectors w_k are linearly independent, the coefficients $d[k]$ are unique. The reconstruction scheme of Fig. 8 can also be used in the case in which $\mathcal{S}$ and $\mathcal{W}$ are infinite-dimensional spaces generated by the Riesz bases $\{s_k\}$ and $\{w_k\}$ respectively, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$.

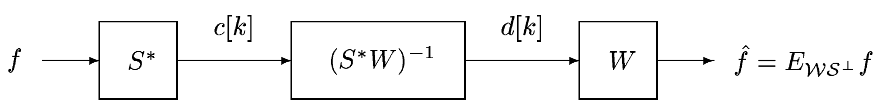


FIGURE 8. Consistent reconstruction of f using nonredundant sampling vectors s_k and nonredundant reconstruction vectors w_k, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$.

We may interpret the reconstruction scheme of Fig. 8 in terms of a basis expansion for signals in $\mathcal{W}$. Since for $f \in \mathcal{W}$, $\hat{f} = f$, any $f \in \mathcal{W}$ can be represented as $f = \sum_k d[k]w_k$ where $d[k] = \langle v_k, f \rangle$ and the vectors $v_k \in \mathcal{S}$ correspond to $V = (W^{\#}_{\mathcal{V}\mathcal{S}^{\perp}})^* = S(W^*S)^{-1}$. We have already seen in Section 3.2 that the vectors v_k form a basis for $\mathcal{S}$, and since $(W^*S)^{-1}$ is bounded, these vectors form a Riesz basis. Since $V^*W = (S^*W)^{-1}S^*W = I$, these basis vectors have the property that they are biorthogonal to w_k: $\langle v_k, w_m\rangle = \delta_{km}$. Therefore Fig. 8 provides an explicit method for constructing (Riesz) basis vectors for an arbitrary space $\mathcal{S}$ with $\mathcal{W} \oplus \mathcal{S}^{\perp} = \mathcal{H}$, that are biorthogonal to the (Riesz) basis vectors w_k.

Alternatively, any $f \in \mathcal{W}$ can be represented as $f = \sum_k c[k]q_k$ where the vectors $q_k \in \mathcal{W}$ correspond to $Q = (S^*)^{\#}_{\mathcal{W}Q} = W(S^*W)^{-1}$. Since Q is bounded and $S^*Q = I$, these vectors form a (Riesz) basis for $\mathcal{W}$ that is biorthogonal to the (Riesz) basis vectors s_k.

To illustrate the details of the sampling and reconstruction scheme of Fig. 8 we now consider two examples. Specifically, reconstruction in shift invariant spaces, and bandlimited sampling of time-limited sequences.

5.1 Reconstruction In Shift Invariant Spaces

We first consider the sampling procedure of Fig. 8 for the special case in which $f = f(t)$ lies in L_2, and $\mathcal{S}$ and $\mathcal{W}$ are shift invariant spaces generated by integer translates of appropriately chosen functions. Thus $\mathcal{W} = \{f(t) = \sum_{k\in\mathbb{Z}} x[k]w(t-k)\}$ and $\mathcal{S} = \{f(t) = \sum_{k\in\mathbb{Z}} x[k]s(t-k)\}$. To ensure that the vectors $\{s_k(t) = s(t-k)\}$ and $\{w_k(t) = w(t-k)\}$ form Riesz bases for $\mathcal{S}$ and $\mathcal{W}$ respectively, we must have that [3] $\alpha \le \sum_k |W(\omega - 2\pi k)|^2 \le \beta$ where $0 < \alpha < \beta$ and $\gamma \le \sum_k |S(\omega - 2\pi k)|^2 \le \delta$ where $0 < \gamma < \delta$. Here $W(\omega)$ and $S(\omega)$ denote the continuous-time Fourier transforms of $w(t)$ and $s(t)$, respectively. The sampling procedure in this case was first considered by Unser and Aldroubi in [29].

The measurements $c[k] = \langle s_k(t), f(t)\rangle = \int s(t-k)f(t)dt$ correspond to samples at times $t = k$ of the output of a filter with impulse response $s(-t)$ with $f(t)$ as its input. The reconstructed signal corresponds to the output of a filter with impulse response $w(t)$, with an impulse train whose values are the corrected measurements $d[k]$ as its input, where $d = (S^*W)^{-1}c$ and S and W are the set transformations corresponding to the vectors $s_k(t)$ and $w_k(t)$ respectively. Since $\langle s_i(t), w_k(t)\rangle = g(k-i)$, where $g(t) = \int s(\tau)w(\tau - t)d\tau = s(t) * w(-t)$, S^*W is an infinite Toeplitz matrix, and is therefore equivalent to a filtering operation with a discrete-time filter whose impulse response is given by $p[k] = g(k) = \int s(t)w(t-k)dt$. The frequency response of the filter is

$$P(\omega) = \sum_{k=-\infty}^{\infty} g(k)e^{-j\omega k} = 2\pi \sum_{k=-\infty}^{\infty} S(\omega + 2\pi k)W^*(\omega + 2\pi k), \tag{5.29}$$

where we used the Poisson sum formula [23]. Since $d = (S^*W)^{-1}c$, d is obtained by filtering the sequence c with a discrete-time filter with frequency response

$$H(\omega) = \frac{1}{P(\omega)} = \frac{1}{2\pi \sum_{k=-\infty}^{\infty} S(\omega + 2\pi k)W^*(\omega + 2\pi k)}. \tag{5.30}$$

Therefore, the sampling scheme of Fig. 8 reduces to the sampling scheme depicted in Fig. 9, which is equivalent to that proposed in [29].

Note that from Proposition 5.1 it follows that the filter $P(\omega)$ is invertible if and only if $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$, or alternatively, if and only if $\cos(\theta(\mathcal{W}, \mathcal{S})) > 0$.

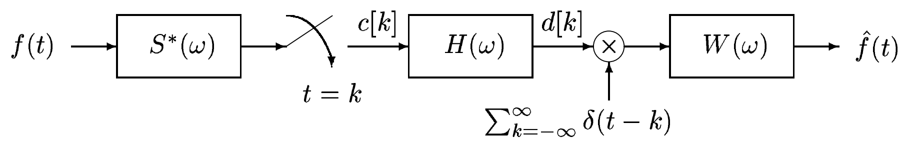


FIGURE 9. Consistent reconstruction of $f(t)$ in shift invariant spaces.

5.2 Bandlimited Sampling of Time-Limited Sequences

We now consider an example that was also considered in [8, 11], in which $\mathcal{H}$ is the space of sequences $x[n]$ such that $x[n] = 0$ for $n < 0, n \geq N$, $\mathcal{W}$ is the space of sequences $x[n]$ such that $x[n] = 0$ for $n < 0, n \geq M$ where $M = 2M' + 1 < N$, and $\mathcal{S}$ is the space of "bandlimited" sequences $x[n]$ such that $X[k] = 0$ for $M' < k < N - M'$, where $X[k], 0 \leq k \leq N-1$ denotes the N point DFT of $x[n]$. The bases for $\mathcal{S}$ and $\mathcal{W}$ are chosen as the sequences $s_k[n], 0 \leq k \leq M-1$ and $w_k[n], 0 \leq k \leq M-1$ respectively, given by $s_k[n] = e^{j2\pi(k-M')n/N}$ for $0 \leq n \leq N-1$ and 0 otherwise, and $w_k[n] = \delta[k-n]$.

Consider an arbitrary sequence $f[n]$ in $\mathcal{H}$. The measurements $c[k], 0 \leq k \leq M-1$ of $f[n]$ are

$$c[k] = \sum_{n=0}^{N-1} s_k^*[n] f[n] = \sum_{n=0}^{N-1} f[n] e^{-j2\pi(k-M')n/N} = F[((k-M'))_N], \qquad (5.31)$$

where $F[k], 0 \leq k \leq N-1$ is the N point DFT of $f[n]$, and $((p))_N = p \bmod N$. Thus, the measurements $c[k]$ are the M lowpass DFT coefficients of the N point DFT of $f[n]$. To obtain a consistent reconstruction of $f[n]$ we need to determine $(S^*W)^{-1}$. The kmth element of S^*W is

$$\langle s_k, w_m \rangle = \sum_{n=0}^{N-1} s_k^*[n] w_m[n] = s_k^*[m] = Z^{km} B^m, \qquad (5.32)$$

where $Z = e^{-j2\pi/N}$ and $B = e^{j2\pi M'/N}$. We can therefore express S^*W as

$$S^*W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & Z & Z^2 & \cdots & Z^{M-1} \\ & & \vdots & & \\ 1 & Z^{M-1} & Z^{2(M-1)} & \cdots & Z^{(M-1)^2} \end{bmatrix} D. \qquad (5.33)$$

Eq. (5.33) is the product of a Vandermonde matrix and a diagonal matrix D with nonzero diagonal elements $B^m, 0 \leq m \leq M-1$. Therefore, S^*W is always invertible which implies by Proposition 5.1 that $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$. We can compute the inverse of S^*W using any of the formulas for the inverse of a Vandermonde matrix (see *e.g.,* [19, 24]). The corrected measurements $d[k]$ are then given by the elements of $d = (S^*W)^{-1}c$ where c is the vector with elements $c[k]$ given by

(5.31), and $\hat{f}[n] = \sum_{k=0}^{N-1} w_k[n]d[k] = d[n]$ for $0 \leq n \leq M-1$ and 0 otherwise. The consistency requirement implies that $\hat{F}[((k-M'))_N] = F[((k-M'))_N]$ for $0 \leq k \leq M-1$, where $\hat{F}[k]$ is the N point DFT of $\hat{f}[n]$. Thus $\hat{f}[n]$ is a time-limited sequence that has the same lowpass DFT coefficients as $f[n]$.

In Section 7 we develop a systematic method for constructing signals in a subspace $\mathcal{W}$ with specified properties in a subspace $\mathcal{S}$. We also consider the more general problem of constructing a signal in $\mathcal{H}$ with specified properties in both $\mathcal{W}$ and $\mathcal{S}$. Using these methods we can generalize our construction here to produce a signal with specified lowpass coefficients *and* specified values on a given time interval.

Now, suppose that $f[n]$ is a length M sequence in $\mathcal{W}$, and we are given M lowpass DFT coefficients $F[((k-M'))_N]$, $0 \leq k \leq M-1$. We can then perfectly reconstruct $f[n]$ from these coefficients using the method described above. This implies the intuitive result that a time-limited discrete-time sequence can be reconstructed from a lowpass segment of its DFT transform. This result is the analogue for the finite length discrete-time case of Papoulis' theorem [22], which implies that a time-limited function can be recovered from a lowpass segment of its Fourier transform. The reconstruction based on Papoulis' theorem is typically obtained using iterative algorithms such as those discussed in [22, 25]. By choosing appropriate sampling and reconstruction vectors in the general scheme of Fig. 8, we obtained a finite length discrete-time version of this theorem together with a simple non-iterative reconstruction method. This example illustrates the type of procedure that might be followed in using our framework to generate new sampling theorems.

6 Reconstruction From Redundant Measurements

Suppose now that we are given a set of redundant measurements $\tilde{c}[k] = \langle x_k, f \rangle$ of a signal $f \in \mathcal{H}$, where the vectors $\{x_k, 1 \leq k \leq N\}$ form a frame for $\mathcal{S}$ and reconstruction is obtained using the reconstruction vectors $\{y_k, 1 \leq k \leq N\}$ which form a frame for $\mathcal{W}$. From the general reconstruction formula (3.8), $\hat{f}$ is obtained using the frame vectors y_k by transforming the measurements $\tilde{c}[k]$ into corrected measurements $\tilde{d} = (X^*Y)^\dagger \tilde{c}$, as depicted in Fig. 10.

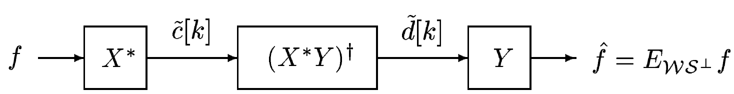


FIGURE 10. Consistent reconstruction of f using redundant sampling vectors x_k and redundant reconstruction vectors y_k, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$.

One of the reasons for using redundant measurements is to reduce the average power of the quantization error, when quantizing the corrected measurements

$\tilde{d}[k]$ prior to reconstruction. If $\mathcal{S} = \mathcal{W}$, then it is well known that using a redundant procedure the quantization error can be reduced by the redundancy of the frame [6, 13]. In [8] this result is extended to the case in which $\mathcal{S} \neq \mathcal{W}$, so that we can choose a frame y_k for $\mathcal{W}$ such that when using the redundant sampling procedure of Fig. 10 we can reduce the average power of the reconstruction error by the redundancy, in comparison with the nonredundant scheme of Fig. 8.

In the next section we show that the redundant sampling scheme of Fig. 10 can be interpreted as a frame expansion of $f \in \mathcal{W}$ in terms of the *oblique dual frame vectors* of y_k on $\mathcal{S}$. Alternatively, we can interpret the redundant sampling scheme as a frame expansion of $f \in \mathcal{W}$ in terms of the oblique dual frame vectors of x_k on $\mathcal{W}$. Based on the properties of the oblique dual frame vectors, in Section 6.2 we show that our reconstruction algorithm has some desirable properties. Specifically, the coefficients $\tilde{d}[k]$ in Fig. 10 have minimal l_2-norm from all possible coefficients leading to consistent reconstruction. Furthermore, if the measurements $\tilde{c}[k]$ are perturbed, then the sampling scheme of Fig. 10 results in a reconstruction $\hat{f}$ whose measurements using the measurement vectors x_k are as close as possible to the measurements $\tilde{c}[k]$ of f, in an l_2-norm sense.

6.1 Oblique Dual Frame Vectors

Definition 6.1. [8] Let the vectors $\{y_k \in \mathcal{W}, 1 \leq k \leq N\}$ corresponding to Y denote a frame for an M-dimensional subspace $\mathcal{W}$ of $\mathcal{H}$, and let $\mathcal{S}$ be an M-dimensional subspace of $\mathcal{H}$ with $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$. Then the oblique dual frame vectors of y_k on $\mathcal{S}$ are the frame vectors $\{\tilde{y}_k^{\mathcal{S}} \in \mathcal{S}, 1 \leq k \leq N\}$ corresponding to the oblique dual frame operator $(Y^{\#}_{\mathcal{V}\mathcal{S}^\perp})^*$ where $\mathcal{V} = \mathcal{N}(Y)^\perp$.

Note that from the discussion following Proposition 3.2, the vectors $\tilde{y}_k^{\mathcal{S}}$ form a frame for $\mathcal{S}$. As we show in the next section, these frame vectors have properties which are analogous to the properties of the conventional dual frame vectors [17, 6], and therefore justify our choice of terminology.

From (3.8) and Proposition 3.2, the corrected measurements $\tilde{d}[k]$ in Fig. 10 are the inner products of f with the oblique dual frame vectors of y_k on $\mathcal{S}$: $\tilde{d}[k] = \langle \tilde{y}_k^{\mathcal{S}}, f \rangle$. Since $YY^{\#}_{\mathcal{V}\mathcal{S}^\perp} = E_{\mathcal{W}\mathcal{S}^\perp}$, any $f \in \mathcal{W}$ can be expressed as

$$f = E_{\mathcal{W}\mathcal{S}^\perp} f = \sum_{k=1}^{N} \langle \tilde{y}_k^{\mathcal{S}}, f \rangle y_k. \tag{6.34}$$

Eq. (6.34) is just a frame expansion of a signal $f \in \mathcal{W}$. However, in contrast with conventional frame expansions, here the synthesis frame vectors y_k lie in $\mathcal{W}$, while the analysis frame vectors $\tilde{y}_k^{\mathcal{S}}$ lie in an arbitrary space $\mathcal{S}$, such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$. In the special case in which $\mathcal{S} = \mathcal{W}$, $Y^{\#}_{\mathcal{V}\mathcal{S}^\perp} = Y^\dagger$ and the oblique dual frame operator reduces to the conventional dual frame operator [6]. Then any $f \in \mathcal{W}$ can be expressed as $f = \sum_{k=1}^{N} \langle \tilde{y}_k, f \rangle y_k$, where $\tilde{y}_k \in \mathcal{W}$ are the dual frame vectors [6] of y_k in $\mathcal{W}$, corresponding to $(Y^\dagger)^*$.

Alternatively, we can express $\hat{f}$ in Fig. 10 as $\hat{f} = \sum_{k=1}^{N} \tilde{c}[k]\tilde{x}_k^{\mathcal{W}}$, where from Proposition 3.2 the vectors $\tilde{x}_k^{\mathcal{W}}$ are the frame vectors corresponding to $(X^*)^{\#}_{\mathcal{W}\mathcal{Q}} = W(S^*W)^{\dagger}$, with $\mathcal{Q} = \mathcal{N}(X)$. From the properties of the oblique pseudoinverse we have that $(X^*)^{\#}_{\mathcal{W}\mathcal{Q}} = (X^{\#}_{\mathcal{Q}^{\perp}\mathcal{W}^{\perp}})^*$, so that from Definition 6.1, the vectors $\tilde{x}_k^{\mathcal{W}}$ are the oblique dual frame vectors of x_k on $\mathcal{W}$. Since $(X^*)^{\#}_{\mathcal{W}\mathcal{Q}}X^* = E_{\mathcal{W}\mathcal{S}^{\perp}}$, any $f \in \mathcal{W}$ can also be expressed as a frame expansion,

$$f = E_{\mathcal{W}\mathcal{S}^{\perp}} f = \sum_{k=1}^{N} \langle x_k, f\rangle \tilde{x}_k^{\mathcal{W}}. \tag{6.35}$$

Eq. (6.35) and (6.34) generalize the concept of a frame expansion to the case in which the analysis and synthesis vectors are not constrained to lie in the same space. Specifically, given a frame $\{y_k\}$ for $\mathcal{W}$, any $f \in \mathcal{W}$ can be expressed as $f = \sum_k \langle \tilde{y}_k^{\mathcal{S}}, f\rangle y_k$ where $\tilde{y}_k^{\mathcal{S}}$ are the oblique dual frame vectors of y_k on $\mathcal{S}$, and $\mathcal{S}$ is an arbitrary subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$. Similarly, given a frame $\{x_k\}$ for $\mathcal{S}$, any $f \in \mathcal{W}$ can be expressed as $f = \sum_k \langle x_k, f\rangle \tilde{x}_k^{\mathcal{W}}$ where $\tilde{x}_k^{\mathcal{W}}$ are the oblique dual frame vectors of x_k on $\mathcal{W}$, and $\mathcal{W}$ is an arbitrary subspace such that $\mathcal{H} = \mathcal{S} \oplus \mathcal{W}^{\perp}$, or equivalently, $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$.

6.2 Properties of the Oblique Dual Frame Vectors

Given a frame y_k for $\mathcal{W}$, there are many choices of coefficients $\tilde{d}[k]$ that correspond to measurements of f using a frame for $\mathcal{S}$, and such that $E_{\mathcal{W}\mathcal{S}^{\perp}} f = \sum_k \tilde{d}[k] y_k$. The particular choice $\tilde{d}[k] = \langle \tilde{y}_k^{\mathcal{S}}, f\rangle$ has the minimal l_2-norm from all possible coefficients, as incorporated in the following proposition.

Proposition 6.1 ([8]). *Let $\{y_k, 1 \le k \le N\}$ denote a frame for an M-dimensional subspace $\mathcal{W} \subseteq \mathcal{H}$, and let $\mathcal{S} \subseteq \mathcal{H}$ denote an M-dimensional subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$. Then from all possible coefficients $d[k]$ that satisfy $E_{\mathcal{W}\mathcal{S}^{\perp}} f = \sum_{k=1}^{N} d[k] y_k$ for all $f \in \mathcal{H}$, the coefficients $\tilde{d}[k]$ corresponding to $\tilde{d} = Y^{\#}_{\mathcal{V}\mathcal{S}^{\perp}} f$ with $\mathcal{V} = \mathcal{N}(Y)^{\perp}$ have minimal l_2-norm.*

Since in Fig. 10, $\tilde{d}[k] = \langle \tilde{y}_k^{\mathcal{S}}, f\rangle$, from Proposition 6.1 these coefficients have minimal l_2-norm from all possible coefficients leading to consistent reconstruction.

We can consider the property stated in Proposition 6.1 from a slightly different point of view. Since the vectors y_k form a frame for $\mathcal{W}$, any $f \in \mathcal{W}$ can be expressed as $f = Yd$ for some d. However, since the vectors y_k are linearly dependent, d is not unique. The minimal norm coefficients are the unique coefficients that lie in $\mathcal{N}(Y)^{\perp} = \mathcal{V}$. We may express these coefficients as $d = Y^{\dagger} f$; indeed $Yd = YY^{\dagger} f = P_{\mathcal{W}} f = f$. Alternatively, $d = Y^{\#}_{\mathcal{V}\mathcal{S}^{\perp}} f$ where $\mathcal{S}^{\perp}$ is an arbitrary subspace of $\mathcal{H}$ such that $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$; indeed $Yd = YY^{\#}_{\mathcal{V}\mathcal{S}^{\perp}} f = E_{\mathcal{W}\mathcal{S}^{\perp}} f = f$. Thus, although the minimal norm coefficients $d[k]$ are unique, the resulting sampling vectors t_k such that $d[k] = \langle t_k, f\rangle$ are not unique. If in addition we impose the constraint that $t_k \in \mathcal{S}$, then the unique sampling vectors that result in coefficients with minimal norm correspond to $(Y^{\#}_{\mathcal{V}\mathcal{S}^{\perp}})^*$. This interpretation is useful in applications in which a signal $f \in \mathcal{W}$ is corrupted by noise that is known to lie in some subspace $\mathcal{S}^{\perp}$. By using appropriate sampling vectors in $\mathcal{S}$, we can totally eliminate this noise and at the same time recover the minimal norm coefficients.

Given measurements $\tilde{c}[k] = \langle x_k, f\rangle$ using a set of frame vectors x_k for $\mathcal{S}$, there are many choices of frame vectors q_k for $\mathcal{W}$ such that $E_{\mathcal{W}\mathcal{S}^{\perp}} f = \sum_k \tilde{c}[k] q_k$. The particular choice $q_k = \tilde{x}_k^{\mathcal{W}}$ has the property that if the measurements $\tilde{c}[k]$ are perturbed, then the measurements of the reconstruction $\hat{f}$ will be as close as possible to the measurements $\tilde{c}[k]$ of f, in an l_2-norm sense.

Proposition 6.2 ([8]). *Let $\hat{f} = \sum_{k=1}^{N} b[k] w_k$ for some vectors $\{w_k, 1 \le k \le N\}$ that form a frame for $\mathcal{W}$, and are to be determined. Let $\{t_k, 1 \le k \le N\}$ denote a given set of sampling vectors corresponding to T. Then the vectors w_k corresponding to $(T^{\#}_{\mathcal{N}(T)^{\perp}\mathcal{W}^{\perp}})^* = (T^*)^{\#}_{\mathcal{W}\mathcal{N}(T)}$ result in $\hat{f}$ with measurements $\langle t_k, \hat{f}\rangle$ that are as close as possible to $b[k]$ in an l_2-norm sense.*

It follows from Proposition 6.2 that if the coefficients $\tilde{c}[k]$ in Fig. 10 are perturbed, then our reconstruction algorithm will lead to a reconstruction $\hat{f}$ whose measurements using the given sampling vectors x_k are as close as possible to the measurements $\tilde{c}[k]$ of f in an l_2-norm sense.

It is interesting to note that the oblique dual frame vectors of $\tilde{y}_k^{\mathcal{S}}$ on $\mathcal{W}$ are the vectors y_k [8]. Thus not only do we have $f = \sum_{k=1}^{N} \langle \tilde{y}_k^{\mathcal{S}}, f\rangle y_k$ for any $f \in \mathcal{W}$ but also $f = \sum_{k=1}^{N} \langle y_k, f\rangle \tilde{y}_k^{\mathcal{S}}$ for any $f \in \mathcal{S}$. Similarly, the oblique dual frame vectors of $\tilde{x}_k^{\mathcal{W}}$ on $\mathcal{S}$ are the vectors x_k, so that $f = \sum_{k=1}^{N} \langle \tilde{x}_k^{\mathcal{W}}, f\rangle x_k$ for any $f \in \mathcal{S}$.

7 Constructing Signals With Prescribed Properties

A potential class of interesting applications of the consistent sampling procedures we developed in the previous sections is to the problem of constructing signals with prescribed properties that can be described in terms of inner products of the signal with a set of vectors. For example, we may consider constructing an odd signal with specified local averages, or constructing a signal with specified odd part *and* specified local averages. Exploiting the results we derived in the context of consistent reconstruction, in this section we develop a general framework for constructing signals of this form.

We first consider the simpler case in which we wish to construct a signal f to lie in a subspace $\mathcal{W}$, and to have some additional properties in a subspace $\mathcal{S}$ that can be described in terms of a set of mathematical constraints of the form $\langle s_k, f\rangle$ for a set of vectors s_k that span $\mathcal{S}$. We then consider the problem of constructing a signal f with properties in two subspaces $\mathcal{W}$ and $\mathcal{S}$ that can be described in terms of mathematical constraints of the form $\langle s_k, f\rangle$ for a set of vectors s_k that span $\mathcal{S}$, and $\langle w_k, f\rangle$ for a set of vectors w_k that span $\mathcal{W}$.

We assume for simplicity that the constraints are nonredundant, and that the vectors s_k and w_k form Riesz bases for $\mathcal{S}$ and $\mathcal{W}$, respectively. Using the oblique dual frame vectors, the results extend in a straightforward way to the redundant case.

Our first problem can be solved immediately by noting that it is equivalent to a consistent reconstruction problem. Specifically, let $c[k] = \langle s_k, f \rangle$ denote the constraints on the signal f. Then the problem is to construct a signal $f \in \mathcal{W}$ so that its measurements taken with respect to the sampling vectors s_k are equal to $c[k]$. If $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^{\perp}$, then the unique signal f follows immediately from (5.28),

$$f = W(S^*W)^{-1}c, \tag{7.36}$$

where W is a set transformation corresponding to a Riesz basis for $\mathcal{W}$, and S is the set transformation corresponding to the vectors s_k.

Next, suppose that we want to construct a signal f with specific properties in two spaces $\mathcal{W}$ and $\mathcal{S}$ with $\mathcal{W} \cap \mathcal{S} = \{0\}$, *i.e.*, we want to construct f such that $\langle s_k, f \rangle = c[k]$ and $\langle w_k, f \rangle = d[k]$. In view of the geometric interpretation of Fig. 2 it follows that constructing f such that $\langle s_k, f \rangle = c[k]$ and $\langle w_k, f \rangle = d[k]$ is equivalent to constructing f to have a specified orthogonal projection $f_{\mathcal{S}}$ onto $\mathcal{S}$ and a specified orthogonal projection $f_{\mathcal{W}}$ onto $\mathcal{W}$. Fig. 3(a) depicts the orthogonal projections of an unknown signal f onto $\mathcal{S}$ and $\mathcal{W}$. The problem then is to construct a signal f with these orthogonal projections. With $\mathcal{U} = \mathcal{W} \oplus \mathcal{S}$, it is obvious that f can be arbitrary on $\mathcal{U}^{\perp}$. However, there is a unique vector $f \in \mathcal{U}$ compatible with the given projections; this vector is illustrated in Fig. 3(b). From this geometrical interpretation we conclude that for $\mathcal{W} \cap \mathcal{S} = \{0\}$, we can always construct a signal with the desired properties. Furthermore, the orthogonal projection of this signal onto $\mathcal{U}$ is unique.

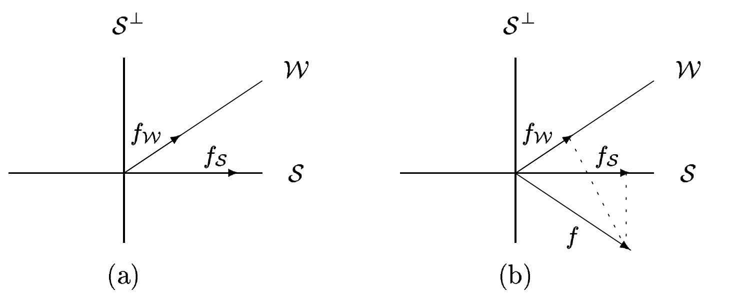


FIGURE 11. Illustration of a construction of a signal f with specified orthogonal projections $f_{\mathcal{S}} = P_{\mathcal{S}} f$ and $f_{\mathcal{W}} = P_{\mathcal{W}} f$ with $\mathcal{W} \cap \mathcal{S} = \{0\}$ (a) orthogonal projection of unknown signal onto $\mathcal{S}$ and $\mathcal{W}$ (b) unique signal in $\mathcal{U} = \mathcal{W} \oplus \mathcal{S}$ with the given projections.

We now explicitly construct the unique vector $f \in \mathcal{U}$ satisfying the required constraints. First we note that any signal $f \in \mathcal{U}$ can be written as $f = s + v$

where $s \in \mathcal{S}$ and $v \in \widetilde{\mathcal{S}}$ with $\widetilde{\mathcal{S}} = \mathcal{S}^\perp \cap \mathcal{U}$. Then, since $\langle s_k, f\rangle = \langle s_k, s\rangle$ for all k, constructing a signal f such that $\langle s_k, f\rangle = c[k]$ is equivalent to constructing a signal $s \in \mathcal{S}$ such that $\langle s_k, s\rangle = c[k]$. Since the vectors s_k form a Riesz basis for $\mathcal{S}$, S^*S is invertible and the unique vector $s \in \mathcal{S}$ such that $S^*s = c$ is given by $s = S(S^*S)^{-1}c$. Once we determined s, the problem reduces to finding $v \in \widetilde{\mathcal{S}}$ such that $\langle w_k, v\rangle = d[k] - \langle w_k, s\rangle \stackrel{\triangle}{=} d'[k]$, which is again equivalent to a consistent reconstruction problem: We need to construct a signal $v \in \widetilde{\mathcal{S}}$ so that its measurements using the sampling vectors w_k are equal to $d'[k]$. Since the orthogonal complement $\widetilde{\mathcal{S}}^\perp$ of $\widetilde{\mathcal{S}}$ in $\mathcal{U}$ is equal to $\mathcal{S}$, $\mathcal{U} = \widetilde{\mathcal{S}}^\perp \oplus \mathcal{W}$, and we can apply (7.36) to obtain $v = V(W^*V)^{-1}d' = V(W^*V)^{-1}(d - W^*s)$, where V is a set transformation corresponding to a basis for $\widetilde{\mathcal{S}}$. Finally, the unique $f \in \mathcal{U}$ satisfying the desired constraints is

$$f = S(S^*S)^{-1}c + V(W^*V)^{-1}(d - W^*S(S^*S)^{-1}c). \tag{7.37}$$

We can immediately verify that indeed $S^*f = c$ and $W^*f = d$.

Note that there are many alternative methods of constructing f. Specifically, instead of utilizing the decomposition $f = s + v$ we can decompose f as $f = x + v$ where $v \in \widetilde{\mathcal{S}}$ and x is a subspace $\mathcal{X}$ such that $\mathcal{X} \oplus \widetilde{\mathcal{S}} = \mathcal{U}$. We then construct f by first finding the unique vector $x \in \mathcal{X}$ such that $\langle s_k, x\rangle = c[k]$, and then finding the unique $v \in \widetilde{\mathcal{S}}$ such that $\langle w_k, v\rangle = d[k] - \langle w_k, x\rangle$. We may also change the roles of $\mathcal{S}$ and $\mathcal{W}$ and utilize a decomposition of the form $f = w + y$ where now $w \in \mathcal{W}$ and $y \in \mathcal{W}^\perp \cap \mathcal{U}$.

As a final comment, we can also construct f by defining the combined basis $\{t_i\}$ for $\mathcal{W} \oplus \mathcal{S}$, consisting of the vectors $\{s_i\}$ and $\{w_i\}$. Then with T denoting the set transformation corresponding to the vectors t_i,

$$f = T(T^*T)^{-1}a, \tag{7.38}$$

where a is the concatenation of c and d. Although our construction scheme is mathematically equivalent to (7.38), it provides further insight into the construction, so that in many cases f can be constructed simply by inspection, without having to formally employ (7.38).

In Section 5.2 we considered an application of consistent sampling to the construction of a time-limited signal with specified lowpass coefficients. Using (7.37) we can now extend this construction to produce a signal with specified lowpass coefficients and specified values on a time interval. By choosing different spaces $\mathcal{W}$ and $\mathcal{S}$ and using (7.37), we can construct signals with a variety of different properties. We consider some specific examples in the next section.

7.1 Examples of Signal Construction

To illustrate the details of the framework for constructing signals with prescribed properties, in this section we consider the problem of constructing a signal with prescribed local averages and prescribed odd part, the problem of constructing a signal with prescribed recurrent nonuniform samples, and the problem of constructing a signal with prescribed samples using a given reconstruction filter.

Constructing a signal with prescribed local averages and prescribed odd part

As an illustration of the framework, we consider an example in which we wish to construct a sequence $f \in l_2$ with local averages $f[2k] + f[2k+1] = c[k]$, where $c[-1] = c[0] = c[1] = 1$ and $c[k] = 0$ otherwise, and with odd part $f[k] - f[-k] = d[k]$, where $d[1] = 1, d[2] = 2, d[3] = 1$ and $d[k] = 0, k \geq 4$.

To this end we first determine a set of vectors s_k and a set of vectors w_k such that the desired properties can be expressed in the form $\langle s_k, f\rangle = \tilde{c}[k], k \geq 1$ and $\langle w_k, f\rangle = d[k], k \geq 1$ where $\tilde{c}[k]$ is a reordering[2] of $c[k]$:

$$\tilde{c}[k] = \begin{cases} c[(k-1)/2], & k \geq 1, k \text{ odd;} \\ c[-k/2], & k \geq 2, k \text{ even.} \end{cases} \tag{7.39}$$

Let

$$s_k[n] = \begin{cases} \delta[n-k+1] + \delta[n-k], & k \geq 1, k \text{ odd;} \\ \delta[n+k-1] + \delta[n+k], & k \geq 2, k \text{ even,} \end{cases} \tag{7.40}$$

and $w_k[n] = \delta[n-k] - \delta[n+k]$. Then, $\tilde{c}[k] = \langle s_k, f\rangle$ and $d[k] = \langle w_k, f\rangle$ for $k \geq 1$.

In this example, $\mathcal{S}$ is the subspace of signals x that satisfy $x[2n] = x[2n+1]$ for all n, and $\mathcal{W}$ is the subspace of odd signals. It is immediate that $\mathcal{S} \cap \mathcal{W} = \{0\}$. To apply (7.37) we need to select a basis v_k for $\mathcal{S}^{\perp}$, which is the subspace of signals x that satisfy $x[2n] = -x[2n+1]$ for all n. A possible basis is

$$v_k[n] = \begin{cases} \delta[n-k+1] - \delta[n-k], & k \geq 1, k \text{ odd;} \\ \delta[n+k-1] - \delta[n+k], & k \geq 2, k \text{ even.} \end{cases} \tag{7.41}$$

To determine f we need to calculate the semi-infinite matrices $(S^*S)^{-1}$, $(W^*V)^{-1}$, and W^*S, where S, W and V are the set transformations corresponding to the vectors s_k, w_k and v_k, respectively. Since $\mathcal{W} \oplus \mathcal{S}$ is not closed, the inverse $(W^*V)^{-1}$ exists but is not bounded. We may still apply our framework to construct a signal of the desired form as long as the sequences c and d are absolutely summable, as they are in our example.

Since $S^*S = 2I$, $(S^*S)^{-1} = (1/2)I$. Now,

$$\langle w_k, v_j\rangle = (-1)^{j+1}(\delta_{k,j-1} - \delta_{k,j}), \quad k, j \geq 1, \tag{7.42}$$

[2] The purpose of the reordering is to ensure that the index set of the vectors s_k and the vectors w_k is the same.

so that

$$W^*V = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ & & \vdots & & & \end{bmatrix}, \tag{7.43}$$

and

$$(W^*V)^{-1} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & \cdots \\ 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & -1 & -1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ & & \vdots & & & \end{bmatrix}. \tag{7.44}$$

Finally,

$$\langle w_k, s_j \rangle = (-1)^{j+1}(\delta_{k,j-1} + \delta_{k,j}), \quad k, j \geq 1, \tag{7.45}$$

so that

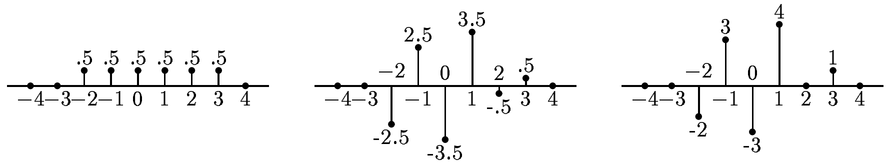
$$W^*S = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 1 & \cdots \\ & & \vdots & & & \end{bmatrix}. \tag{7.46}$$

Applying (7.37) results in

$$f = \frac{1}{2} S\tilde{c} + Vg, \tag{7.47}$$

where $g = (W^*V)^{-1}h$ with $h = d - (1/2)e$ and $e = W^*S\tilde{c}$. Thus, $f[n] = f_1[n] + f_2[n]$ where $f_1[n] = (1/2)\sum_{k=1}^{\infty} \tilde{c}[n]s_k[n]$ and $f_2[n] = \sum_{k=1}^{\infty} g[n]v_k[n]$. The sequence f_1 lies in $\mathcal{S}$ and has the desired local averages: $f_1[2n] + f_1[2n+1] = c[n]$ for all n. The sequence f_2 lies in $\mathcal{S}^{\perp}$, and completes the odd part of f_1 to the desired odd part.

In our example, $\tilde{c}[1] = \tilde{c}[2] = \tilde{c}[3] = 1$ and $\tilde{c}[k] = 0$ for $k \geq 4$. Then $e[1] = e[2] = 0$, $e[3] = 1$, and $e[k] = 0$ for $k \geq 4$. Finally, $g[1] = -\sum_{k=1}^{3}(d[k] - 1/2e[k]) = -3.5$, $g[2] = \sum_{k=2}^{3}(d[k] - 1/2e[k]) = 2.5$, $g[3] = -d[3] + 1/2e[3] = -0.5$. Thus, f is the sum of the two sequences depicted in Fig. 12. Fig. 12(a) depicts the unique signal $f_1 \in \mathcal{S}$ with the desired local averages, so that $f_1[2k] + f_1[2k+1] = c[k]$. Fig. 12(b) depicts the unique signal $f_2 \in \mathcal{S}^{\perp}$ with odd part satisfying $f_2[k] - f_2[-k] = d[k] - x[k]$, where $x = W^*f_1$ is the odd part of f_1. Note that, as we expect, the local averages of f_2 are all equal 0. Fig. 12(c) depicts $f = f_1 + f_2$ which is the unique sequence with the desired local averages and the desired odd part.



(a) (b) (c)

FIGURE 12. Constructing a sequence f with specified local averages and specified odd part (a) unique signal $f_1 \in \mathcal{S}$ with required local averages (b) unique signal $f_2 \in \mathcal{S}^\perp$ with odd part equal to the difference between the required odd part and the odd part of f_1 (c) unique signal $f = f_1 + f_2$ with both the required local averages and the required odd part.

Constructing a signal with prescribed recurrent nonuniform samples

As a second illustration of the framework, suppose we want to construct a continuous-time signal $f(t)$ bandlimited to $\omega_0 = \pi/T_Q$ with specified samples, where the sampling points are divided into groups of N points each, and the group has a recurrent period $T = NT_Q$. Each period consists of N nonuniform sampling points. Denoting the points in one period by $t_i, i = 1, 2, \ldots N$, the complete set of sampling points is

$$t_i + lT, \quad i = 1, 2, \ldots N, \; l \in \mathbb{Z}. \tag{7.48}$$

Thus our problem is to find an $f \in \mathcal{W}$ where $\mathcal{W}$ is the space of all signals bandlimited to $\omega_0 = \pi/T_Q$, such that $\langle s_k, f\rangle = c[k]$ where $s_k(t) = \delta(t - lT - t_i)$ with $k = lN + i, 0 \leq i \leq N-1$ and $\langle s_k, f\rangle = \int s_k(t) f(t) dt$. The unique f with these samples is the signal given by $f = (S^*W)^{-1}c$ where S is the set transformation corresponding to the signals $s_k(t)$, and W is a set transformation corresponding to a basis $w_k(t)$ for $\mathcal{W}$. A possible choice is $w_k(t) = \sin(\omega_0(t - kT_Q))/(\omega_0(t - kT_Q))$. With this choice, if $y = S^*Wc$, then y can be obtained as the output of the filter bank depicted in Fig. 13, where the filters $H_i(\omega)$ have impulse response $h_i[n] = (-1)^n \sin(\omega_o t_i)/(\omega_o t_i - n\pi)$.

To determine $(S^*W)^{-1}$ we need to invert the filter bank of Fig. 13. The inverse filter bank has the form depicted in Fig. 14, where the filters $G_i(\omega)$ have been determined in [10] and are equal to the filters in [10, Fig. 9] given by $G_i(\omega) = (1/T_Q)R_i(\omega/T_Q)e^{-jt_i\omega/T_Q}$, for $|\omega| \leq \pi$, where $R_i(\omega)$ is the frequency response of the filter with impulse response

$$r_i(t) = a_i T \frac{\sin(\pi t/T)}{\pi t} \prod_{\substack{q=0\\ q\neq i}}^{N-1} \sin(\pi(t + t_i - t_q)/T), \tag{7.49}$$

and

$$a_i = \frac{1}{\prod_{q=0, q\neq i}^{N-1} \sin(\pi(t_i - t_q)/T)}. \tag{7.50}$$

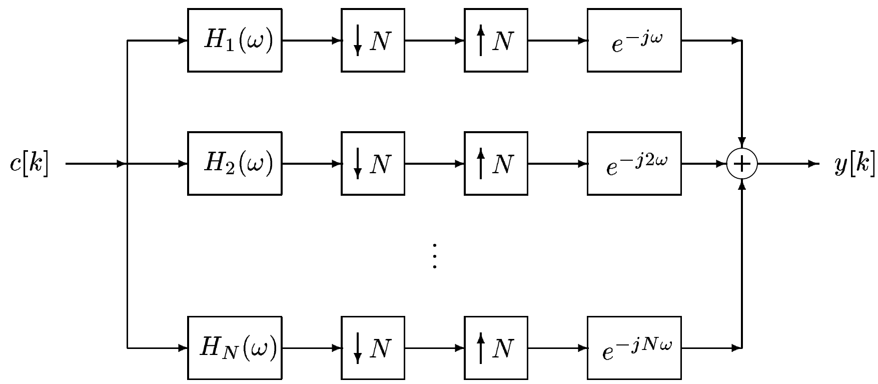


FIGURE 13. Filter bank implementation of $y = S^*Wc$.

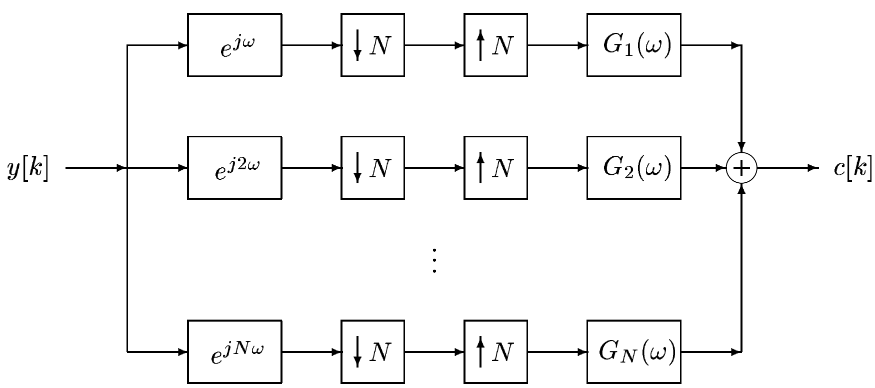


FIGURE 14. Filter bank implementation of $c = (S^*W)^{-1}y$.

Therefore to construct $f(t)$, we first obtain y using the filter bank of Fig. 14. Then $f(t) = \sum_k y[k]w_k(t)$ which can be implemented by modulating the samples $y[k]$ onto a uniformly spaced impulse train with period T_Q, and then filtering the modulated impulse train with a continuous-time lowpass filter with cutoff frequency π/ω_0.

Constructing signals with prescribed samples

As a third illustration of our framework, suppose we wish to construct a continuous-time signal $f(t)$ to have prescribed samples so that $f(k) = c[k], k \in \mathbb{Z}$, where $f(k)$ denotes the value of $f(t)$ at $t = k$. The signal $f(t)$ is constrained to lie in the subspace $\mathcal{W}$ generated by the integer translates $\{w(t-k), \in \mathbb{Z}\}$ of a given function $w(t)$, so that $f(t) = \sum_k x[k]w(t-k)$ for some coefficients $x[k]$. We assume that $\alpha \leq \sum_k |W(\omega - 2\pi k)|^2 \leq \beta$ where $0 < \alpha < \beta$ and $W(\omega)$ is the continuous-time Fourier transform of $w(t)$, which ensures that $\{w(t-k)\}$ forms a Riesz basis for $\mathcal{W}$ [3]. The signal $f(t)$ can be obtained as the output of a filter with impulse response $w(t)$, with an impulse train whose values are the coefficients $x[k]$ as its input. The problem then is to find the coefficients $x[k]$ so that $f(k) = c[k]$.

We can express $f(k)$ as $f(k) = \langle s_k(t), f(t)\rangle$ where $s_k(t) = \delta(t-k)$ and $\langle y(t), r(t)\rangle = \int y(t)r(t)dt$. Note that $s(t) = \delta(t)$ does not generate a Riesz basis. However, a sufficient condition for the coefficients $f[k]$ to be in l_2 is that $f(t)$ is continuous and decays sufficiently fast [29]. From (7.36) it then follows that $x = (S^*W)^{-1}c$ where S and W are the set transformations corresponding to the vectors $s_k(t)$ and $w_k(t)$ respectively. Since $\langle s_i(t), w_k(t)\rangle = w(i-k)$, S^*W is an infinite Toeplitz matrix, and is therefore equivalent to a filtering operation with a filter whose impulse response is given by $\langle s_0(t), w_k(t)\rangle = w(k)$. Using the Poisson sum formula [23], the frequency response of the filter is

$$\sum_{k=-\infty}^{\infty} w(k)e^{-j\omega k} = 2\pi \sum_{k=-\infty}^{\infty} W(\omega + 2\pi k). \tag{7.51}$$

It follows that if $x = (S^*W)^{-1}c$, then x is obtained by filtering the sequence c with a discrete-time filter with frequency response

$$G(\omega) = \frac{1}{2\pi \sum_k W(\omega + 2\pi k)}, \tag{7.52}$$

as depicted in Fig. 15.

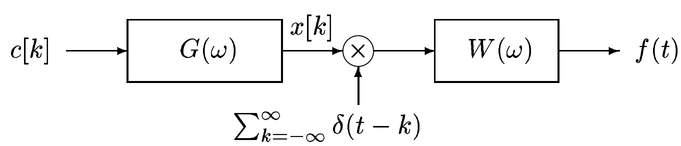


FIGURE 15. Constructing a signal $f(t)$ with samples $f(k) = c[k]$ using a given filter with frequency response $W(\omega)$, where $G(\omega)$ is given by (7.52).

In the special case in which $W(\omega)$ is the frequency response of an ideal lowpass filter with cutoff frequency $\omega_0 = \pi$, $G(\omega) = 1$ so that $x[k] = c[k]$.

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