Block Sparsity and Sampling over a Union of Subspaces

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DSP

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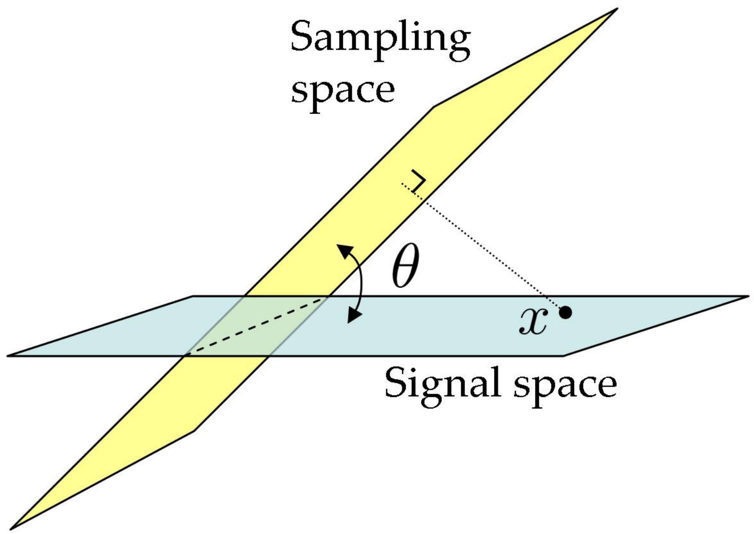
Subspace Sampling

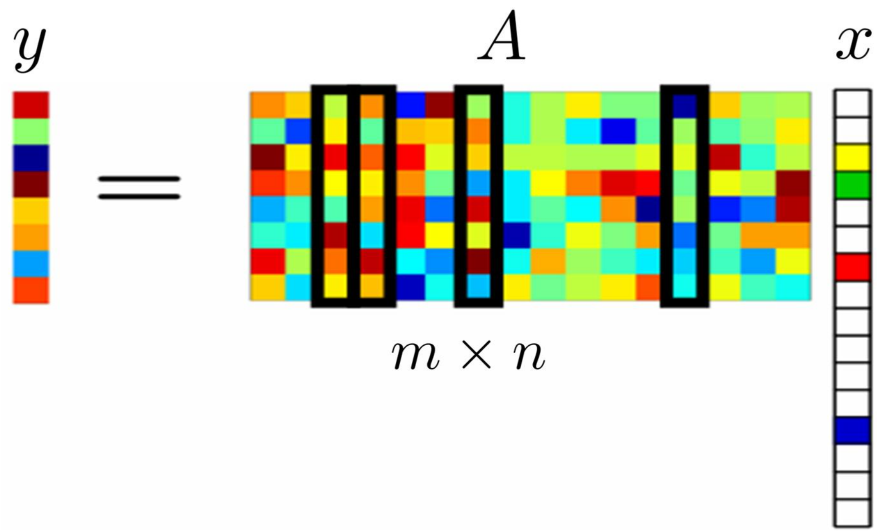


Sampling	Recovery Algorithms	Guarantees
$\theta < 90^\circ$: unique x $\theta \ll 90^\circ$: stable inversion	Oblique projection	$\theta < 90^\circ$ Perfect Reconstruction

Compressed Sensing

- Sample using few measurements

length m



length n k-sparse

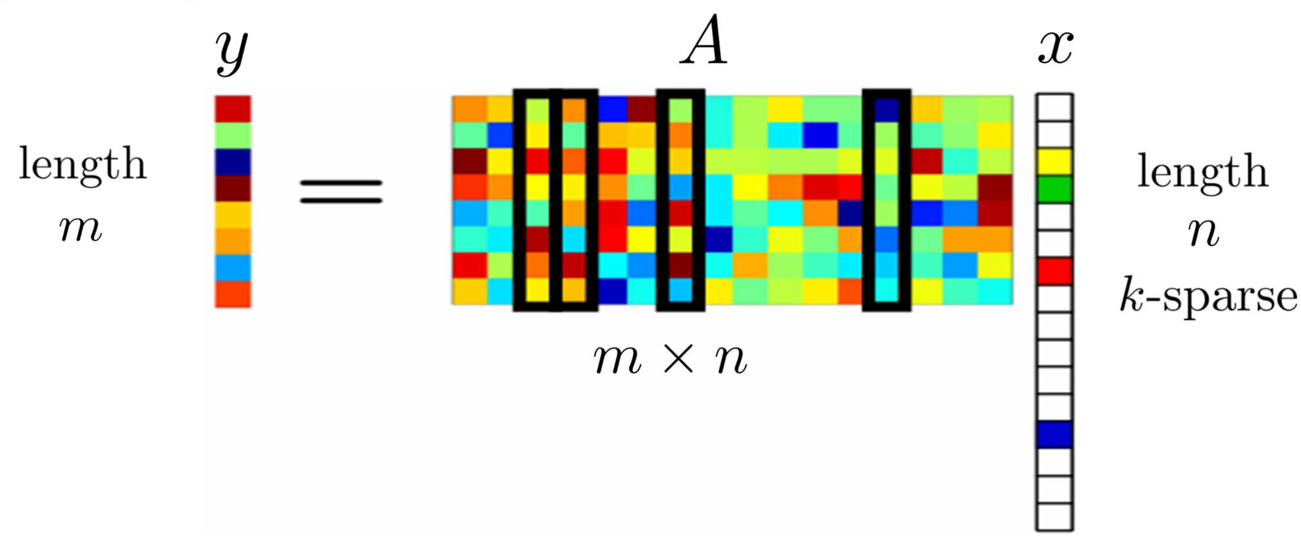
Sampling | **Recovery Algorithms** | **Guarantees**

$y \longleftrightarrow x$

Unique/Stable mapping?

Compressed Sensing

- Sample using few measurements



Sampling

$y \longleftrightarrow x$

Unique/Stable mapping?

Recovery Algorithms

- $\min \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$
- $\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$
- Orthogonal matching pursuit

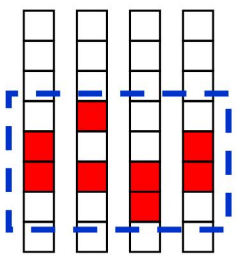
Guarantees

- Small RIP constant
- Small coherence

$$m = O\left(k(\log(n/k) + 1)\right)$$

Sparsity Priors

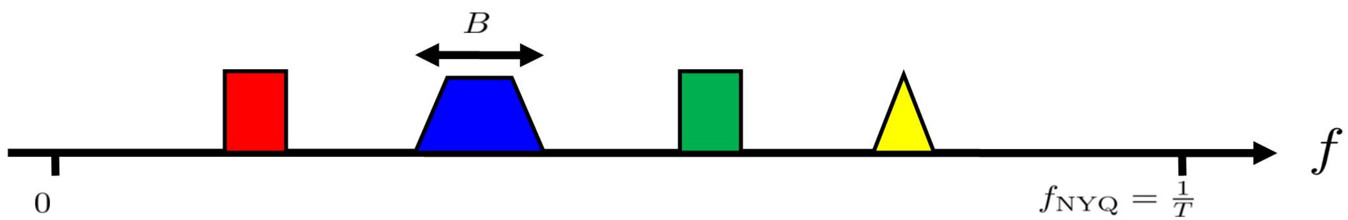
- $\mathbf{x} =$  Well understood ✓
- d columns

  $\mathbf{X} =$  Joint sparsity Many papers ✓

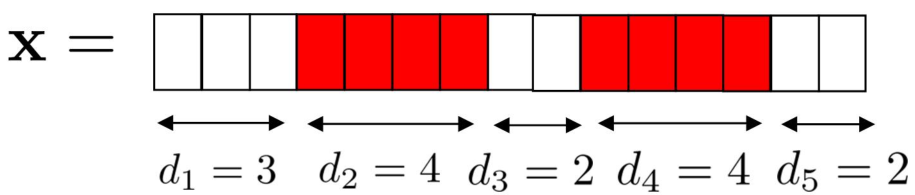
Sparsity = Nonzeros

Sparsity Priors

- Multiband signal:



- Block-sparse signal:

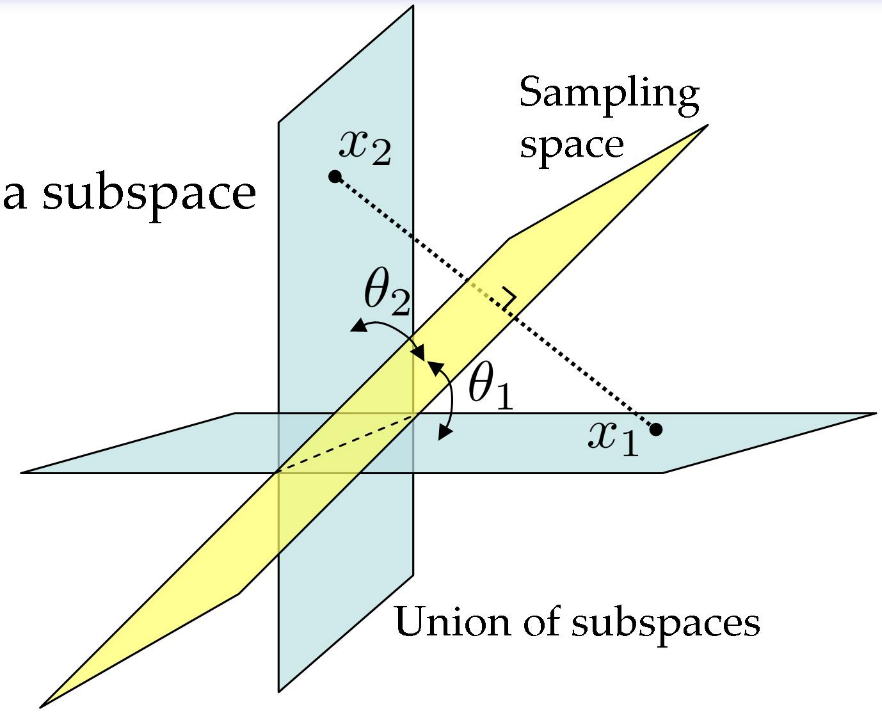


More general notion of sparsity needed!

Union Sampling

$x \in \bigcup_i \mathcal{U}_i$

where each $\mathcal{U}_i$ is a subspace



Sampling → **Recovery Algorithms** → **Guarantees**

Conditions for Unique/Stable mapping:

- *Lu and Do, 08*
- *Blumensath and Davies, 08*

?

?

Goal: Develop stable and efficient recovery algorithms over a union

Outline

- Key observation: Need structured union
- Finite settings: Develop recovery algorithms
  Prove equivalence guarantees
- Infinite union: Intro+Application

Sampling → **Recovery Algorithms** → **Guarantees**

Conditions for
Unique/Stable mapping:

- *Lu and Do, 08*
- *Blumensath and Davies, 08*

- Convex relaxation
- Subspace OMP

- Block RIP
- Block coherence

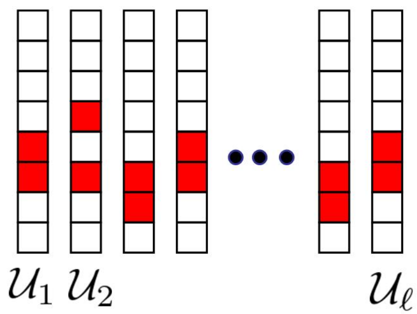
(Eldar and Mishali) DSP'09 paper
(Eldar and Bolcskei) ICASSP'09 paper

Examples: Unions of Subspaces

$x \in \bigcup_i \mathcal{U}_i$ where each $\mathcal{U}_i$ is a subspace

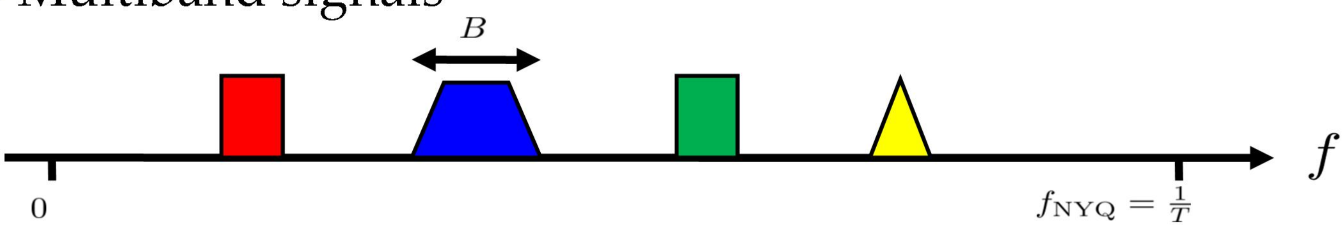
- Sparsity

2 - sparse



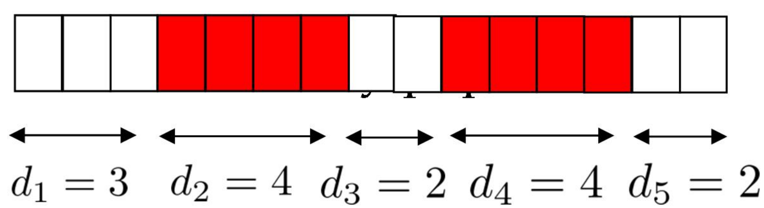
$\ell = \binom{8}{2}$

- Multiband signals



- Block-Sparsity

$\mathbf{x} =$

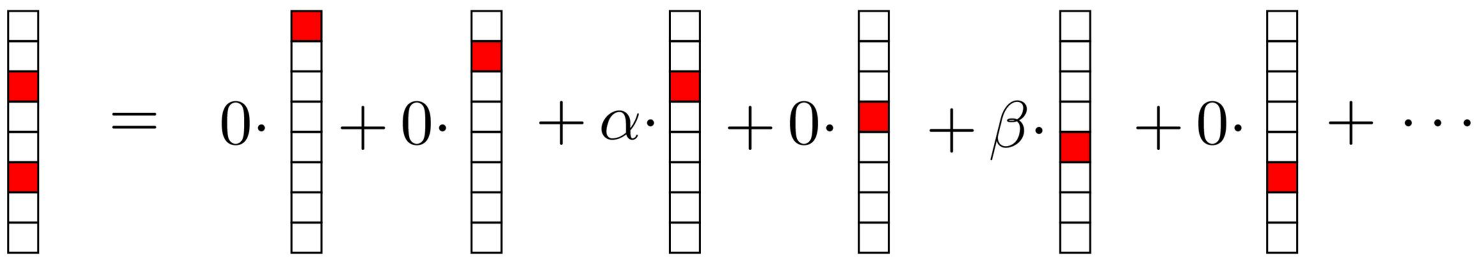


Structured Model

(Eldar and Mishali, 08)

$\mathcal{U} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_k$ where $\mathcal{A}_i$ is selected from a given set $\{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$

Examples:



- Standard CS: $\mathcal{A}_i$ is spaned by e_i . The coeffcients are <u>scalars</u>.

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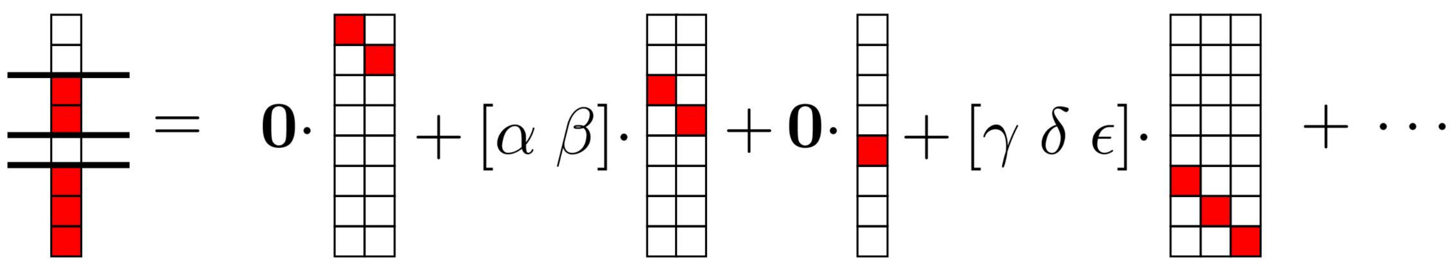
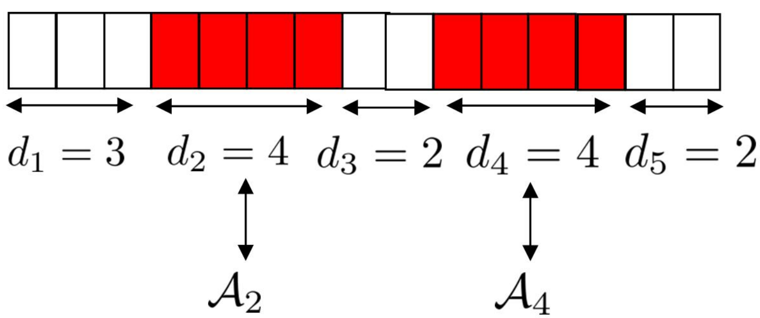


- Standard CS: $\mathcal{A}_i$ is spaned by e_i . The coeffcients are <u>scalars</u>.
- Block sparsity: $\mathcal{A}_i$ is spaned by *d* columns of the identity *I*.
  The coeffecients are <u>vectors.</u>
- Multiband signals: $\mathcal{A}_i$ is a frequency bin of width *B*.
  The coeffecients are <u>sequences</u>.

Key Result

Any structured union problem can be translated into block sparsity!

- Define a basis for each $\mathcal{A}_i$
- Any $x \in \mathcal{A}_i$ has a representation in terms of a vector $c[i]$ of length $d_i = \dim(\mathcal{A}_i)$
- If $\mathcal{A}_i$ is not in the sum then $c[i] = 0$
- x in the union is represented by:



- Samples $y_i = \langle a_i, x \rangle$ are equivalent to $y = Dc$

May be continuous

Finite vector

Block Sparsity

$$y = Dc \qquad c \text{ - block sparse}$$

- Convex optimization:

$$\min \sum_{i=1}^{m} \|c[i]\|_2 \quad \text{s.t.} \;\; y = Dc$$

minimize number of blocks with non-zero energy

- Subspace matching pursuit:
  choose block that best matches the residual

Both recover c under suitable conditions

Convex Relaxation

- l_1 - optimality is based on RIP
- Extend to block-RIP

$$(1-\delta_k)\|c\|_2^2 \;\le\; \|Dc\|_2^2 \;\le (1+\delta_k)\|c\|_2^2$$

For every block-k sparse c over $\mathcal{I} = \{d_1, ..., d_m\}$

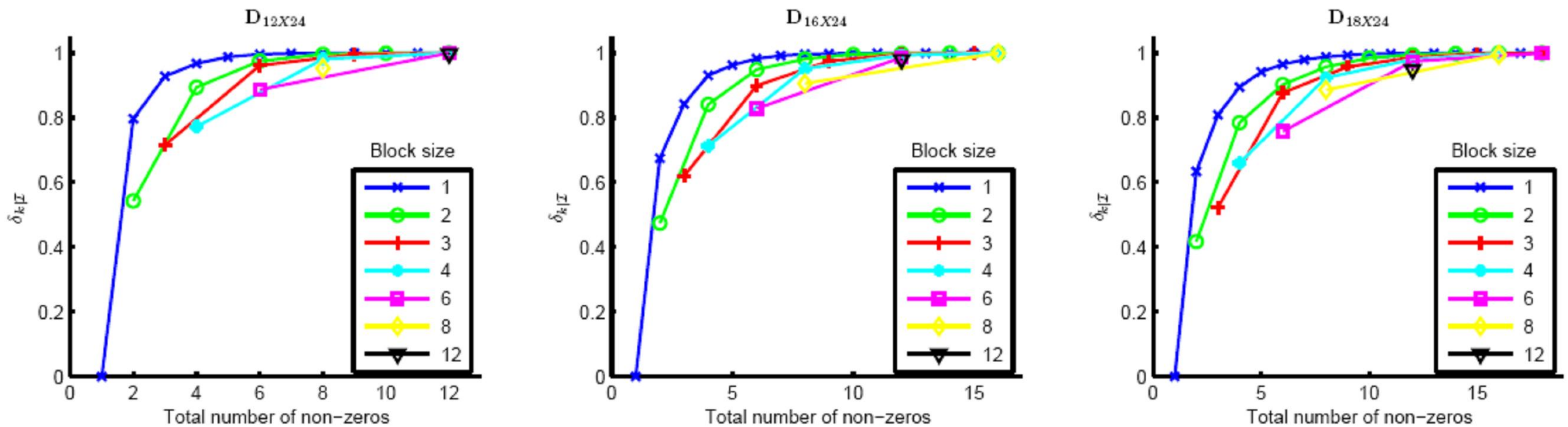
Theorem:

If $\delta_{2k} < \sqrt{2} - 1$ then the convex relaxation is exact

(Eldar and Mishali, 08)

Block-RIP

- Block RIP constant is typically smaller than standard RIP



(a) (b) (c)

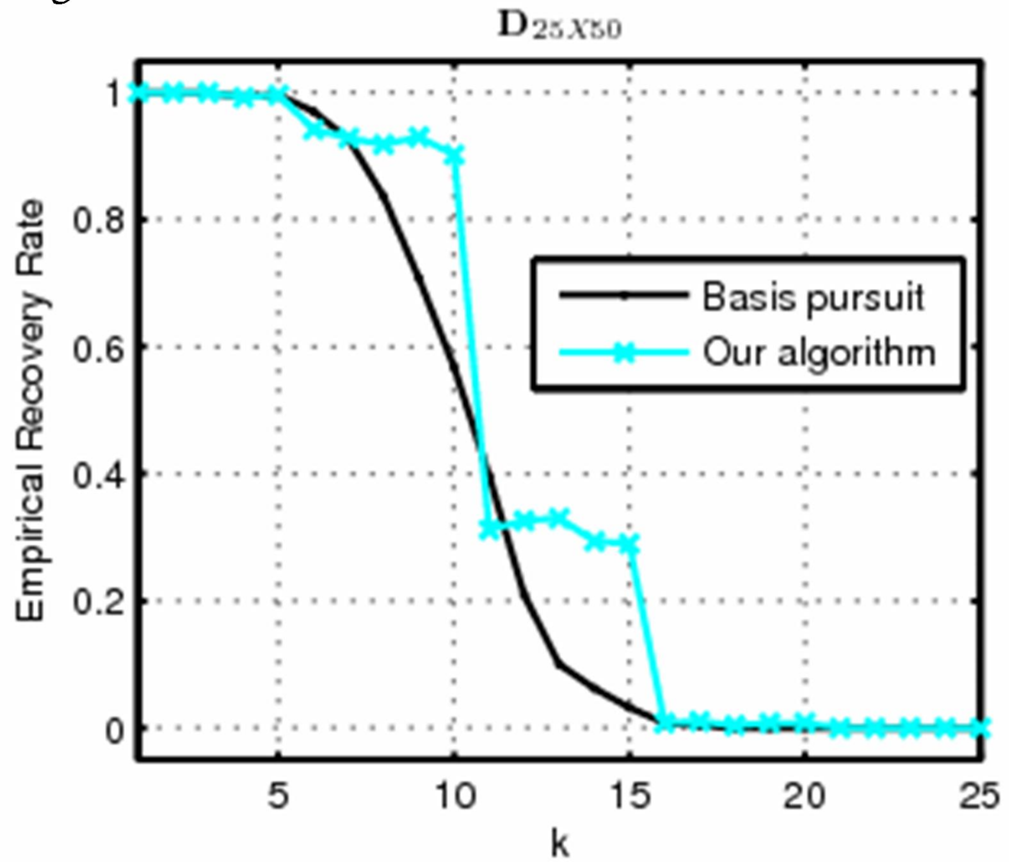
- Block RIP condition satisfied with high probability if

$$n \approx k(\log(m/k) + d)$$

(conventional RIP ➡ $n \approx k(d\log(m/k) + d)$)

Example

Block sparsity = 5



Our algorithm improves recovery over standard basis pursuit

Robust Recovery

- Suppose c is approximately block sparse and measurements are noisy

$$y = Dc + z \qquad \|z\| \leq \varepsilon$$

- $\min \sum_{i=1}^{m} \|c[i]\|_2 \quad \text{s.t.} \quad \|y - Dc\| \leq \varepsilon$

Theorem:

If $\delta_{2k} < \sqrt{2} - 1$ then
$\|c_0 - c'\|_2 \leq \alpha \|c_0 - c^k\| + \beta\varepsilon$

(Eldar and Mishali, 08)

c_0 - true vector

c' - algorithm output

c^k - best block approximation

α, β - known constants

Block Coherence

(Eldar and Bolcskei, 08)

- Standard coherence: $\mu = \max_{i \neq j} \langle d_i, d_j \rangle$
- Block coherence:

$$\mu_B = \max_{i \neq j} \frac{1}{d} \rho(D^H[i] D[j])$$

$\rho(A)$ - largest singular value

d - block length

$$D = (\underbrace{D[1]}_{d \text{ columns}} \; D[2] \ldots D[m])$$

- Properties:
  - $0 \leq \mu_B \leq 1$
  - $\mu_B \leq \mu \leftarrow$ Improved recovery results
  - Operational meaning: uncertainty relation

Recovery Conditions

Theorem:

A block sparse c can be recovered from $y = Dc$ using convex relaxation if $kd < \frac{1}{2}({\mu_B}^{-1} + d)$

(Eldar and Bolcskei, 08)

- If block structure is ignored then the condition becomes $kd < \frac{1}{2}(\mu^{-1} + 1)$

$$\mu^{-1} \leq \mu_B^{-1}$$
$$1 \leq d$$

- Same conditions ensure recovery with subspace OMP

Sparsity vs. Union Sparsity

Standard Sparsity		Union Sparsity
K nonzero elements	**Model**	K nonzero blocks/subspaces
Optimization: l_1 Greedy: OMP	**Algorithms**	Optimization: mixed l_2 / l_1 Greedy: subspace OMP
Small RIP Small coherence	**Equivalence**	Small block RIP Small block coherence

Sparsity vs. Union Sparsity

Standard Sparsity		Union Sparsity
K nonzero elements	**Model**	K nonzero blocks/subspaces
Optimization: l_1 Greedy: OMP	**Algorithms**	Optimization: mixed l_2 / l_1 Greedy: subspace OMP
Small RIP Small coherence	**Equivalence**	Small block RIP Small block coherence

Advantage of Union Sparsity:

- Block coherence and block RIP are smaller than coherence and RIP
  - $\longrightarrow$ weaker equivalence conditions
- Empirical performance improvement

Can Treat an Infinite Union !

(Gedalyahu and Eldar, 09)

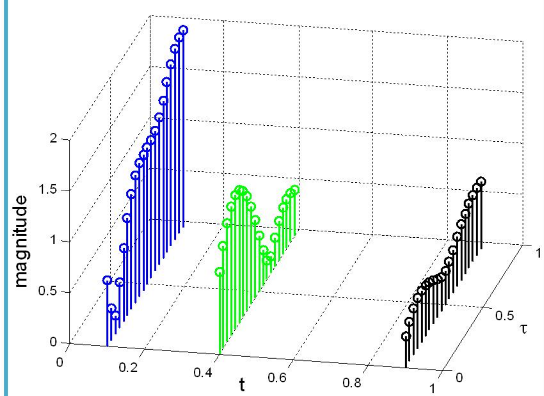
Application: Multipath Identification

Transmitted Signal

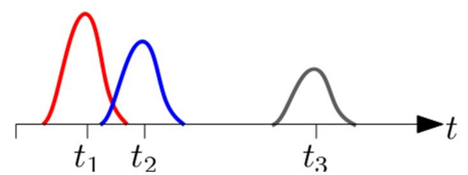
$$x_T(t) = \sum_{n\in\mathbb{Z}} g(t - nT)$$

Probing rate

Multipath Medium



Received Signal



$$x_R(t) = \sum_{k=1}^{K}\sum_{n\in\mathbb{Z}} a_k[n]\, g(t - t_k - nT)$$

kth path's time-variant gain coefficient

kth path's propagation delay

Infinite union of infinite dimensional subspaces

Known pulse shape $g(t)$ $\longrightarrow$ Structure !

Conclusion

- Efficient recovery for structured union of subspace
- Equivalence and stability using block RIP
- Equivalence using block coherence
- First step: future work should explore other structures

Theory of CS can be extended to subspaces

References

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Thank you!