

# Minimax Approximation of Representation Coefficients From Generalized Samples

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**Abstract**—Many sources of information are of analog or continuous-time nature. However, digital signal processing applications rely on discrete data. We consider the problem of approximating  $L_2$  inner products, i.e., representation coefficients of a continuous-time signal, from its generalized samples. Adopting a robust approach, we process these generalized samples in a minimax optimal sense. Specifically, we minimize the worst approximation error of the desired representation coefficients by proper processing of the given sample sequence. We then extend our results to criteria which incorporate smoothness constraints on the unknown function. Finally, we compare our methods with the piecewise-constant approximation technique, commonly used for this problem, and discuss the possible improvements by the suggested schemes.

**Index Terms**—Generalized sampling, interpolation, robust approximation, smoothness.

## I. INTRODUCTION

**S**IGNAL processing applications are concerned mainly with digital data, although the origin of many sources of information is analog. This is the case for speech and audio, optics, radar, sonar, biomedical signals, and more. In many cases, analysis of a continuous-time signal  $\mathbf{x}(t)$  is obtained by evaluating  $L_2$  inner-products  $\langle \mathbf{w}_n(t), \mathbf{x}(t) \rangle_{L_2}$  for a set of predetermined analysis functions  $\{\mathbf{w}_n(t)\}$ . For example, one may calculate a Gabor [1] or wavelet [2] representation of a signal.

Typically, the analysis functions  $\{\mathbf{w}_n(t)\}$  are analytically known. On the other hand, in many applications of digital signal processing, there is no knowledge of the continuous-time signal  $\mathbf{x}(t)$ , but only of its sample sequence. Our problem is to approximate the required  $L_2$  inner-products, by proper processing of the available samples.

In some cases the sampled version of a signal is sufficient to calculate the original function. A well known example is the classical Whittaker-Shannon sampling theorem. See also [3], [4] for additional shift invariant settings. If the analog input

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can be determined from the sample sequence, then the required representation coefficients can be calculated as well. Our main focus here is on situations where the knowledge of the continuous-time function is incomplete, so that only approximations of the continuous-time inner products can be obtained. A well known example is the initialization problem in wavelet analysis. To initialize the pyramid algorithm [5] we need the representation coefficients of the continuous-time function  $\mathbf{x}(t)$ , in the initial scale. Unfortunately, these coefficients are typically unavailable, and we only have the samples of  $\mathbf{x}(t)$ , obtained at the output of an anti-aliasing filter. A common practice in wavelet analysis is to assume that the available samples are the required representation coefficients. This false assumption is also known in the literature as the ‘wavelet crime’ [6]. In [7] the authors address this problem by suggesting a digital filter to process the available sample sequence, prior to applying the pyramid algorithm. In fact, it can be shown that their result is compatible with a special case of our derivations, presented in Section IV-B.

A common approach to cope with incomplete knowledge of  $\mathbf{x}(t)$  is to first interpolate the given samples using some synthesis functions. Then, the required  $L_2$  inner-products can be performed using the approximation (see, for example, [8]). Unfortunately, the best choice of the synthesis functions is not always clear. See [9] for error analysis, when approximations of a function are performed in a shift invariant setup.

Yet another approach to approximate an  $L_2$  inner-product is to perform numerical integration by a Riemann-type sum. Assuming ideal and uniform sampling, the convergence of such approximations was studied in [10]. The ideal and uniform sampling case was also considered in [11], [12]. In order to approximate a single representation coefficient  $\langle \mathbf{w}(t), \mathbf{x}(t) \rangle_{L_2}$ , it was suggested to calculate an  $l_2$  inner product  $\sum_n b[n] \mathbf{x}(nT)$  instead. The sequence  $b$  was determined by minimizing an upper bound on the approximation error. In practice, however, ideal sampling is impossible to implement. A more practical model considers generalized samples [4], [13]–[18], which are represented as the inner products of the signal with a set of sampling functions  $\{\mathbf{s}_n(t)\}$ . This sampling model is general enough to describe any linear and bounded acquisition device (Riesz representation theorem [19], [20]).

In this paper, we take an approach that is similar in spirit to the works in [7] and [16]. Given the generalized samples, we approximate the desired representation coefficients  $\{\langle \mathbf{w}_n(t), \mathbf{x}(t) \rangle_{L_2}\}$  in a minimax optimal sense. The solution we obtain can be interpreted as an interpolation of the given samples, followed by an application of the analysis functions  $\{\mathbf{w}_n(t)\}$ . The advantage of our framework is that the interpolation stage stems naturally from the setup of the problem,

rather than being pre-specified arbitrarily. Furthermore, the division of the algorithm into interpolation and analysis stages is more of conceptual rather than practical nature; both stages can be performed simultaneously, by digital processing of the available samples.

Our results extend [11] in several ways. First, by considering generalized samples our derivations are applicable to practical acquisition devices. Second, we show how to incorporate prior knowledge that the generalized samples stem from a smooth function. Third, our derivations are applicable to a series of representation coefficients. Finally, we analyze the performance of the suggested approach, giving sufficient conditions for it to outperform piecewise-constant approximations.

The outline of the paper is as follows. In Section II we describe the notations and the mathematical preliminaries. Section III discusses situations where the required  $L_2$  inner products can be evaluated exactly, and establishes a minimax approximation criterion when this is not the case. The minimax objective is solved in Section IV. In Section V we consider the problem of incorporating smoothness constraints. Specifically, if there is prior knowledge of the input to be smooth, then we show how to alter the minimax solution by recasting the problem in a proper Sobolev space [21], presenting [11] as a special setting of our derivations. Section VI discusses the relations between the errors due to the suggested minimax approach and approximations by a Riemann-type summation. We show the possible gain in performance by the proposed method and derive sufficient conditions for it to dominate the summation approach. Finally, in Section VII, we conclude with several simulations.

## II. NOTATIONS AND MATHEMATICAL PRELIMINARIES

We denote continuous-time signals by bold lowercase letters, omitting the time dependence, when possible. The elements of a sequence  $c \in l_2$  will be written with square brackets, e.g.,  $c[n]$ .  $\mathbf{X}^F(\omega) = \int \mathbf{x}(t)e^{-j\omega t}dt$  is the continuous-time Fourier transform of  $\mathbf{x}$  and  $C^f(\omega) = \sum_n c[n]e^{-j\omega n}$  is the ( $2\pi$  periodic) discrete-time Fourier transform (DTFT) of the sequence  $c$ . The operator  $P_{\mathcal{A}}$  represents the orthogonal projection onto a closed subspace  $\mathcal{A}$ , and  $\mathcal{A}^\perp$  is the orthogonal complement of  $\mathcal{A}$ . The Moore–Penrose pseudoinverse [22] and the adjoint of a bounded transformation  $T$  are written as  $T^\dagger$  and  $T^*$ , respectively.  $\Re$  stands for the real part.

Inner products and norms are denoted by  $\langle a, b \rangle_{\mathcal{H}}$  and  $\|a\|_{\mathcal{H}}$ , respectively. Here,  $\mathcal{H}$  stands for the Hilbert space involved. Usually, we will consider  $\mathcal{H}$  to be  $L_2, l_2$  or the order-one Sobolev space  $W_2^1$ , which will be discussed in detail in Section V. When the derivations are general enough to describe inner products and norms within any Hilbert space, we will omit the space subscript from the notations, i.e.,  $\langle f, g \rangle$  or  $\|f\|$ . All inner products are linear with respect to the second argument. For example,  $\langle \mathbf{x}, \mathbf{y} \rangle_{L_2} = \int_{-\infty}^{\infty} \bar{\mathbf{x}}(t)\mathbf{y}(t)dt$ .

An easy way to describe linear combinations and inner products is by utilizing set transformations. A set transformation  $V : l_2 \rightarrow \mathcal{H}$  corresponding to frame [23] vectors  $\{\mathbf{v}_n(t)\}$  is defined by  $Va = \sum_n a[n]\mathbf{v}_n(t)$  for all  $a \in l_2$ . From the definition of the adjoint, if  $a = V^*\mathbf{y}$ , then  $a[n] = \langle \mathbf{v}_n, \mathbf{y} \rangle$ .

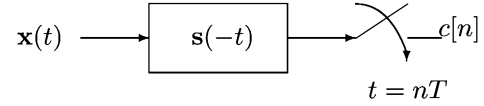


Fig. 1. Filtering with impulse response  $\mathbf{s}(-t)$  followed by ideal sampling. The sampling vectors are  $\{\mathbf{s}(t - nT)\}$ .

We define by  $S(W)$  the set transformation corresponding to the vectors  $\{\mathbf{s}_n\}(\{\mathbf{w}_n\})$ . Accordingly, the generalized samples  $c[n] = \langle \mathbf{s}_n, \mathbf{x} \rangle_{L_2}$  can be written as  $c = S^*\mathbf{x}$ , and the desired representation coefficients  $q[n] = \langle \mathbf{w}_n, \mathbf{x} \rangle_{L_2}$  by  $q = W^*\mathbf{x}$ . We define  $\mathcal{S}$  to be the sampling space, which is the closure of  $\text{span}\{\mathbf{s}_n\}$ . Similarly,  $\mathcal{W}$  is the analysis space, obtained by the closure of  $\text{span}\{\mathbf{w}_n\}$ .

To handle well posed problems, we assume that the sample sequence  $c$  and the desired representation coefficients  $q$  have finite energy, i.e.,  $c, q \in l_2$ . This will assure that for any bounded transformation  $G : l_2 \rightarrow l_2$  applied to the generalized samples  $c$ , the error sequence  $q - G(c)$  is in  $l_2$  as well. Accordingly, criteria that consider the  $l_2$  norm of the error sequence are well defined. One way to enforce  $c, q \in l_2$  is to require that  $\{\mathbf{s}_n\}$  and  $\{\mathbf{w}_n\}$  form frames [23] for  $\mathcal{S}$  and  $\mathcal{W}$ , respectively, which is an assumption made throughout this paper.

## III. PROBLEM FORMULATION

We are given the generalized samples of a continuous-time function  $\mathbf{x}(t)$ , modeled by

$$c[n] = \langle \mathbf{s}_n(t), \mathbf{x}(t) \rangle_{L_2}. \quad (1)$$

An example is an analog to digital converter which performs prefiltering prior to sampling, as shown in Fig. 1. In such a setting, the sampling vectors  $\{\mathbf{s}_n(t) = \mathbf{s}(t - nT)\}$  are shifted and mirrored versions of the prefilter impulse response [13].

We wish to evaluate a set of continuous-time inner products  $q$  defined by

$$q[n] = \langle \mathbf{w}_n, \mathbf{x} \rangle_{L_2} \quad (2)$$

where the analysis functions  $\{\mathbf{w}_n\}$  are analytically known. The input  $\mathbf{x}$  is known only through its generalized samples  $c$  of (1). Our goal is to approximate the required representation coefficients  $q$  by proper processing of the sample sequence  $c$ .

A natural question to be first considered is whether there is an unavoidable error due to our partial knowledge of  $\mathbf{x}(t)$ , or can we evaluate **exactly** the required  $L_2$  inner products, based on the generalized samples. The following theorem addresses this preliminary question.

*Theorem 1:* Let  $\mathbf{x}$  be an arbitrary function, satisfying  $c = S^*\mathbf{x}$ . It is possible to obtain the coefficients  $q = W^*\mathbf{x}$  by proper processing of the sample sequence  $c$  if and only if  $\mathcal{W} \subseteq \mathcal{S}$ . In this case,  $q = W^*S(S^*S)^\dagger c$ .  $\square$

*Proof:* See Appendix A.

In some cases, we may have additional prior knowledge on  $\mathbf{x}$ , such that not all signals in  $L_2$  should be considered. By restricting our attention to a proper subgroup, it is possible to obtain a zero error, even if  $\mathcal{W} \not\subseteq \mathcal{S}$ . This is true whenever the knowledge of  $\mathbf{x}$  allows us to determine a bijection (injective and

surjective transformation) between  $\mathbf{x}(t)$  and its samples. To illustrate this point, suppose that  $\mathbf{x} \in \mathcal{A}$ , where  $\mathcal{A}$  is a closed subspace of  $L_2$  satisfying the direct sum condition  $L_2 = \mathcal{A} \oplus \mathcal{S}^\perp$  (i.e.,  $L_2$  can be described by the sum set  $\{\mathbf{a} + \mathbf{v}; \mathbf{a} \in \mathcal{A}, \mathbf{v} \in \mathcal{S}^\perp\}$  with the property  $\mathcal{A} \cap \mathcal{S}^\perp = \{0\}$ ). Then, we can perfectly reconstruct  $\mathbf{x}$  from its generalized samples by

$$\mathbf{x} = A(S^*A)^\dagger c \quad (3)$$

where  $A$  is any bounded set transformation with range  $\mathcal{A}$  [16]. As a result, we can also perfectly evaluate the coefficients  $q = W^*\mathbf{x}$  by

$$q = W^*A(S^*A)^\dagger c. \quad (4)$$

Another example in which a bijection between the signal and its generalized samples exists is the finite innovation setting considered in [24].

Nevertheless, in the general case, the condition  $\mathcal{W} \subseteq \mathcal{S}$  may not be satisfied, or there may be no prior knowledge on  $\mathbf{x}(t)$ . Thus, the coefficients  $W^*\mathbf{x}$  cannot be computed exactly and instead must be approximated from the given samples  $c$ . A common approach is to perform Riemann-type sum approximations [10]:

$$\langle \mathbf{w}(t), \mathbf{x}(t) \rangle_{L_2} \approx T \sum_n c[n] \overline{\mathbf{w}(nT)} \quad (5)$$

if one implicitly assumes that the generalized samples of  $\mathbf{x}$  are close to the mean value of the input signal, within an interval of length  $T$ . However, this approximation is not generally optimal in any sense.

Alternatively, to obtain an optimal solution in the squared error sense we may approximate the continuous-time inner products by choosing a sequence  $d$  which minimizes the squared norm of the error vector  $e = W^*\mathbf{x} - d$ . Since  $\mathbf{x}$  satisfies  $c = S^*\mathbf{x}$ , by decomposing  $\mathbf{x}$  along  $\mathcal{S}$  and  $\mathcal{S}^\perp$  the error can be written as

$$e = W^*S(S^*S)^\dagger c + W^*P_{\mathcal{S}^\perp}\mathbf{x} - d \quad (6)$$

where we used  $P_{\mathcal{S}}\mathbf{x} = S(S^*S)^\dagger c$ . This leads to the following objective:

$$\min_d \|W^*S(S^*S)^\dagger c + W^*P_{\mathcal{S}^\perp}\mathbf{x} - d\|_{l_2}^2. \quad (7)$$

Unfortunately, the solution of (7) depends on  $P_{\mathcal{S}^\perp}\mathbf{x}$ , which is unknown. To eliminate the dependency on  $\mathbf{x}$ , we may instead consider a robust approach, where the sequence  $d$  is optimized for the worst possible input  $\mathbf{x}$ . Valid inputs must be consistent with the known samples, i.e., must satisfy  $c = S^*\mathbf{x}$ . Additionally, if the norm of the input is unbounded, then so is the error. Therefore, to define a well posed problem, we assume that  $\mathbf{x}$  is norm bounded by a positive constant  $L$ , so that the set of possible inputs is

$$\mathcal{D} = \{\mathbf{x} : \|\mathbf{x}\| \leq L, c = S^*\mathbf{x}\}. \quad (8)$$

We then consider the minimax objective

$$\min_d \max_{\mathbf{x} \in \mathcal{D}} \|W^*\mathbf{x} - d\|_{l_2}^2. \quad (9)$$

In the next sections, we derive a solution for  $d$ , and compare its performance with the piecewise-constant approximation approach given in (5).

#### IV. MINIMAX APPROXIMATION

The minimax problem of (9) is closely related to the generalized sampling problem considered in ([16, Theorem 3]). Relying on results obtained in that context leads to the following theorem.

*Theorem 2:* Consider the problem

$$\min_d \max_{c=S^*\mathbf{x}, \|\mathbf{x}\| \leq L} \|W^*\mathbf{x} - d\|_{l_2}^2$$

where  $W$  and  $S$  are bounded set transformations with range  $\mathcal{W}$  and  $\mathcal{S}$ , respectively. The (unique) solution is

$$d = W^*S(S^*S)^\dagger c. \quad (10)$$

Before going into the details of the proof, note that we have not specified the exact Hilbert space in which the bound  $\|\mathbf{x}\| \leq L$  and the inner products  $S^*\mathbf{x}, W^*\mathbf{x}$  are calculated, since the derivations are general enough to be applicable to any Hilbert space. In Section V we will show how smoothness constraints can be incorporated by applying Theorem 2 to different Hilbert spaces. Additionally, the upper norm bound  $L$  is not expressed in the solution (10). Thus, one only has to be sure that the signal has a finite norm, while its exact value is irrelevant to the computation of  $d$ . The value of  $L$  will be used, however, in Section VI for analyzing the performance of the proposed algorithm.

*Proof:* First we note that any  $\mathbf{x}$  in  $\mathcal{D}$  of (8) is of the form  $\mathbf{x} = S(S^*S)^\dagger c + \mathbf{v}$  for some  $\mathbf{v} \in \mathcal{G}$  where

$$\mathcal{G} = \{\mathbf{v} : \mathbf{v} \in \mathcal{S}^\perp, \|\mathbf{v}\| \leq L'\} \quad (11)$$

and

$$L' = \sqrt{L^2 - \|S(S^*S)^\dagger c\|_{l_2}^2}. \quad (12)$$

Thus,

$$\begin{aligned} & \max_{\mathbf{x} \in \mathcal{D}} \|W^*\mathbf{x} - d\|_{l_2}^2 \\ &= \max_{\mathbf{v} \in \mathcal{G}} \|W^*S(S^*S)^\dagger c - d + W^*\mathbf{v}\|_{l_2}^2 \\ &= \max_{\mathbf{v} \in \mathcal{G}} \|a_d + W^*\mathbf{v}\|_{l_2}^2 \\ &= \max_{\mathbf{v} \in \mathcal{G}} (\|a_d\|_{l_2}^2 + 2\Re\{\langle a_d, W^*\mathbf{v} \rangle_{l_2}\} + \|W^*\mathbf{v}\|_{l_2}^2) \end{aligned} \quad (13)$$

where we defined  $a_d = W^*S(S^*S)^\dagger c - d$ . As a result, the maximum in (13) is achieved when

$$\Re\{\langle a_d, W^*\mathbf{v} \rangle_{l_2}\} = |\langle a_d, W^*\mathbf{v} \rangle_{l_2}|. \quad (14)$$

Indeed, let  $\mathbf{v} \in \mathcal{G}$  be the vector for which the maximum is achieved. If  $\langle a_d, W^*\mathbf{v} \rangle_{l_2} = 0$  then (14) is trivially true. Otherwise, we can define

$$\mathbf{v}_2 = \frac{\langle W^*\mathbf{v}, a_d \rangle_{l_2}}{|\langle W^*\mathbf{v}, a_d \rangle_{l_2}|} \mathbf{v}. \quad (15)$$

Clearly,  $\|\mathbf{v}\| = \|\mathbf{v}_2\|$  and  $\mathbf{v}_2 \in \mathcal{G}$ . In addition,  $\|W^*\mathbf{v}\|_{l_2} = \|W^*\mathbf{v}_2\|_{l_2}$  and  $\langle a_d, W^*\mathbf{v}_2 \rangle_{l_2} = |\langle a_d, W^*\mathbf{v} \rangle_{l_2}|$  so that the ob-

jective in (13) at  $\mathbf{v}_2$  is larger than the objective at  $\mathbf{v}$  unless (14) is satisfied.

Combining (14) and (13), our problem becomes

$$\min_d \max_{\mathbf{v} \in \mathcal{G}} \left( \|a_d\|_{\ell_2}^2 + 2|\langle a_d, W^* \mathbf{v} \rangle_{\ell_2}| + \|W^* \mathbf{v}\|_{\ell_2}^2 \right). \quad (16)$$

Denoting the optimal objective value by  $A$ , and replacing the order of minimization and maximization, we get a lower bound

$$\begin{aligned} A &\geq \max_{\mathbf{v} \in \mathcal{G}} \min_d \left( \|a_d\|_{\ell_2}^2 + 2|\langle a_d, W^* \mathbf{v} \rangle_{\ell_2}| + \|W^* \mathbf{v}\|_{\ell_2}^2 \right) \\ &= \max_{\mathbf{v} \in \mathcal{G}} \|W^* \mathbf{v}\|_{\ell_2}^2 \end{aligned} \quad (17)$$

where we used the fact that  $\|a_d\|_{\ell_2}^2 + 2|\langle a_d, W^* \mathbf{v} \rangle_{\ell_2}| \geq 0$ , with equality for  $d$  of (10). Thus, for any choice of  $d$ ,

$$\min_d \max_{\mathbf{v} \in \mathcal{G}} \|a_d - W^* \mathbf{v}\|_{\ell_2}^2 \geq \max_{\mathbf{v} \in \mathcal{G}} \|W^* \mathbf{v}\|_{\ell_2}^2. \quad (18)$$

The proof then follows from the fact that  $d$  given by (10) achieves the lower bound (18). Uniqueness of  $d$  follows from (16), as the optimal solution must satisfy  $a_d = 0$ .  $\square$

Note that (10) resembles the solution of the Wiener–Hopf equations, where the Gramian matrix of the autocorrelations is first inverted (pseudoinverted), and the cross-correlation Gramian matrix is then applied. Another interesting interpretation of (10) is obtained by noticing that  $P_S \mathbf{x} = S(S^* S)^\dagger c$ . This leads to the following corollary.

*Corollary 1:* The solution (10) can be written as

$$d = W^* P_S \mathbf{x}. \quad (19)$$

This means that our robust approach first approximates the signal by its orthogonal projection onto the sampling space, and then applies the analysis functions  $\{\mathbf{w}_n\}$ . Thus, we can also conclude that the suggested approximation method results in zero error if  $\mathcal{W} \subseteq \mathcal{S}$  or if the prior knowledge  $\mathbf{x} \in \mathcal{S}$  exists. In fact, by identifying  $A$  of (4) with  $S$ , the solutions indeed coincide. Interestingly,  $P_S \mathbf{x}$  is the minimax approximation of  $\mathbf{x}$  over the set  $\mathcal{D}$  of (8), as incorporated in the following proposition.

*Proposition 1:* The unique solution of

$$\min_{\hat{\mathbf{x}}} \max_{\mathbf{x} \in \mathcal{D}} \|\mathbf{x} - \hat{\mathbf{x}}\|^2$$

with  $\mathcal{D}$  of (8) is  $\hat{\mathbf{x}} = P_S \mathbf{x}$ .

*Proof:* Projecting  $\mathbf{x} - \hat{\mathbf{x}}$  onto  $\mathcal{S}$  and  $\mathcal{S}^\perp$  we have

$$\|\mathbf{x} - \hat{\mathbf{x}}\|^2 = \|S(S^* S)^\dagger c - P_S \hat{\mathbf{x}}\|^2 + \|P_{\mathcal{S}^\perp} \mathbf{x} - P_{\mathcal{S}^\perp} \hat{\mathbf{x}}\|^2. \quad (20)$$

The maximization is then

$$\begin{aligned} \max_{\mathbf{x} \in \mathcal{D}} \|\mathbf{x} - \hat{\mathbf{x}}\|^2 &= \|e_S(\hat{\mathbf{x}})\|^2 + \max_{\mathbf{x} \in \mathcal{D}} \|P_{\mathcal{S}^\perp} \mathbf{x} - P_{\mathcal{S}^\perp} \hat{\mathbf{x}}\|^2 \\ &= \|e_S(\hat{\mathbf{x}})\|^2 + \max_{\mathbf{x} \in \mathcal{D}} \left( \|P_{\mathcal{S}^\perp} \mathbf{x}\|^2 \right. \\ &\quad \left. - 2\Re\langle P_{\mathcal{S}^\perp} \mathbf{x}, P_{\mathcal{S}^\perp} \hat{\mathbf{x}} \rangle + \|P_{\mathcal{S}^\perp} \hat{\mathbf{x}}\|^2 \right) \end{aligned} \quad (21)$$

where  $e_S(\hat{\mathbf{x}}) = S(S^* S)^\dagger c - P_S \hat{\mathbf{x}}$ . Similarly to the proof of Theorem 2, we can replace  $-2\Re\langle P_{\mathcal{S}^\perp} \mathbf{x}, P_{\mathcal{S}^\perp} \hat{\mathbf{x}} \rangle$  by its absolute value. The minimax objective is then lower bounded by

$$\begin{aligned} \min_{\hat{\mathbf{x}}} \left( \|e_S(\hat{\mathbf{x}})\|^2 + \max_{\mathbf{x} \in \mathcal{D}} \left( \|P_{\mathcal{S}^\perp} \mathbf{x}\|^2 \right. \right. \\ \left. \left. + 2|\langle P_{\mathcal{S}^\perp} \mathbf{x}, P_{\mathcal{S}^\perp} \hat{\mathbf{x}} \rangle| + \|P_{\mathcal{S}^\perp} \hat{\mathbf{x}}\|^2 \right) \right) \\ \geq \max_{\mathbf{x} \in \mathcal{D}} \|P_{\mathcal{S}^\perp} \mathbf{x}\|^2 \end{aligned} \quad (22)$$

where we used the fact that for all  $\hat{\mathbf{x}}$  we must have  $\|e_S(\hat{\mathbf{x}})\|^2 \geq 0$  and  $2|\langle P_{\mathcal{S}^\perp} \mathbf{x}, P_{\mathcal{S}^\perp} \hat{\mathbf{x}} \rangle| + \|P_{\mathcal{S}^\perp} \hat{\mathbf{x}}\|^2 \geq 0$ . The proof then follows by noticing that  $\hat{\mathbf{x}} = P_S \mathbf{x}$  is the minimizer which achieves this lower bound. Furthermore, it is unique, since from (22) the optimal solution must satisfy  $P_{\mathcal{S}^\perp} \hat{\mathbf{x}} = 0$  and  $e_S(\hat{\mathbf{x}}) = 0$ .  $\square$

We conclude that the problem of approximating the representation coefficients in a minimax sense could be split into two stages; first obtaining the minimax approximation of  $\mathbf{x}$  itself, and then applying the analysis operator  $W^*$  to the approximation.

### A. The Shift Invariant Case

The approximation (10) was derived for general sampling and analysis subspaces. An interesting special case of this setup is when in addition,  $\mathcal{S}$  and  $\mathcal{W}$  are real shift invariant (SI) subspaces, each spanned by shifts of length  $T$  of some fixed generating function [13], [16]. In this setting, as we will show, the approximation sequence  $d$  can be obtained by discrete-time filtering of the sample sequence  $c$ .

Let  $\mathbf{s}(t)$  and  $\mathbf{w}(t)$  be the real generators of  $\mathcal{S}$  and  $\mathcal{W}$ , respectively. Then, the SI subspaces are

$$\mathcal{S} = \left\{ \mathbf{f}(t) : \mathbf{f}(t) = \sum_n a[n] \mathbf{s}(t - nT), a \in \ell_2 \right\}; \quad (23)$$

$$\mathcal{W} = \left\{ \mathbf{f}(t) : \mathbf{f}(t) = \sum_n a[n] \mathbf{w}(t - nT), a \in \ell_2 \right\}. \quad (24)$$

In this SI case, the samples  $c[n]$ , which are given by

$$c[n] = \int \mathbf{s}(t - nT) \mathbf{x}(t) dt = \mathbf{x}(t) * \mathbf{s}(-t)_{t=nT} \quad (25)$$

correspond to ideal sampling at times  $t = nT$  of the output of a filter with an impulse response  $\mathbf{s}(-t)$  and  $\mathbf{x}(t)$  as its input (see Fig. 1). Here  $\mathbf{g}(t) * \mathbf{z}(t)$  denotes continuous-time convolution between the signals  $\mathbf{g}(t)$  and  $\mathbf{z}(t)$ , and  $\mathbf{y}(t)_{t=nT} = \mathbf{y}(nT)$ .

To ensure that the functions  $\{\mathbf{s}(t - nT)\}$  and  $\{\mathbf{w}(t - nT)\}$  form frames for  $\mathcal{S}$  and  $\mathcal{W}$ , respectively, a simple condition can be verified in the frequency domain [23], [25]

$$\begin{aligned} \alpha &\leq R_{\mathbf{w}, \mathbf{w}}^f(\omega) \leq \beta, \quad \omega \in \mathcal{I}_{\mathcal{W}}; \\ \gamma &\leq R_{\mathbf{s}, \mathbf{s}}^f(\omega) \leq \eta, \quad \omega \in \mathcal{I}_{\mathcal{S}} \end{aligned} \quad (26)$$

for some  $0 < \alpha \leq \beta < \infty$  and  $0 < \gamma \leq \eta < \infty$ . Here, we denote

$$R_{\mathbf{A}, \mathbf{B}}^f(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \overline{\mathbf{A}^F} \left( \frac{\omega + 2\pi k}{T} \right) \mathbf{B}^F \left( \frac{\omega + 2\pi k}{T} \right) \quad (27)$$

where  $\mathbf{W}^F(\omega)$ ,  $\mathbf{S}^F(\omega)$  are the continuous-time Fourier transforms of the generators  $\mathbf{s}(t)$ ,  $\mathbf{w}(t)$ , and  $\mathcal{I}_W, \mathcal{I}_S$  are the set of frequencies  $\omega$  for which  $R_{\mathbf{W}, \mathbf{W}}^f(\omega) \neq 0$  and  $R_{\mathbf{S}, \mathbf{S}}^f(\omega) \neq 0$ , respectively.

Letting  $S$  and  $W$  be the set transformations of  $\{\mathbf{s}(t - nT)\}$  and  $\{\mathbf{w}(t - nT)\}$ , respectively, it is easy to see that  $W^*Sa$  is equivalent to filtering the sequence  $\{a[n]\}$  by a discrete-time LTI filter with frequency response  $R_{\mathbf{W}, \mathbf{S}}^f(\omega)$ . Similarly, the pseudoinverse operator  $(S^*S)^\dagger$  corresponds to applying a filter with DTFT  $1/R_{\mathbf{S}, \mathbf{S}}^f(\omega)$  for  $\omega \in \mathcal{I}_S$  and zero otherwise. Therefore, the sequence  $d = W^*S(S^*S)^\dagger c$  can be obtained by filtering the sample sequence  $c$  with a digital filter

$$G^f(\omega) = \begin{cases} \frac{R_{\mathbf{W}, \mathbf{S}}^f(\omega)}{R_{\mathbf{S}, \mathbf{S}}^f(\omega)}, & \omega \in \mathcal{I}_S; \\ 0, & \omega \notin \mathcal{I}_S. \end{cases} \quad (28)$$

We point out that by a proper choice of the sampling and analysis functions, the filter (28) is compatible with the solution for the ‘‘wavelet crime’’ problem obtained in [7].

## V. IMPOSING SMOOTHNESS BY SOBOLEV SPACES

The objective in Theorem 2 considers functions within the set  $\mathcal{D} = \{\mathbf{x} : S^*\mathbf{x} = c, \|\mathbf{x}\| \leq L\}$ . However, sometimes we have prior knowledge that the input signal is ‘smooth’. By restricting the set of possible inputs to include only smooth functions, the performance of the robust objective may be improved.

Sobolev spaces are natural candidates to describe smoothness. For simplicity, our main discussion will concern the Sobolev space of order one [21].

*Definition 1:* The Sobolev space of order one  $W_2^1$  is the Hilbert space of finite energy functions which also have a finite energy first derivative. A possible choice of inner product in this space is

$$\langle \mathbf{a}, \mathbf{b} \rangle_{W_2^1} = \langle \mathbf{a}, \mathbf{b} \rangle_{L_2} + \langle \mathbf{a}^{(1)}, \mathbf{b}^{(1)} \rangle_{L_2} \quad (29)$$

where  $\mathbf{a}^{(1)}$  and  $\mathbf{b}^{(1)}$  stand for the first derivative of  $\mathbf{a}$  and  $\mathbf{b}$ , respectively.

If we have prior knowledge that  $\mathbf{x}$  and its first derivative are of finite energy (which in particular implies that  $\mathbf{x}$  is continuous), then we may consider the set of possible inputs to be

$$\tilde{\mathcal{D}} = \{\mathbf{x} : S^*\mathbf{x} = c, \|\mathbf{x}\|_{W_2^1} \leq \tilde{L}\} \quad (30)$$

where  $\tilde{L}$  is an upper bound on the  $W_2^1$  norm of  $\mathbf{x}$ . This leads to the following minimax objective:

$$\min_d \max_{\mathbf{x} \in \tilde{\mathcal{D}}} \|W^*\mathbf{x} - d\|_{L_2}^2. \quad (31)$$

To solve (31) we may use Theorem 2 as its derivations are general enough to be applicable to any Hilbert space. Note, however, that objective (31) contains mixed inner products and norms;  $S^*\mathbf{x}$  and  $W^*\mathbf{x}$  describe  $L_2$  inner products, while  $\|\mathbf{x}\|_{W_2^1} \leq \tilde{L}$  is a Sobolev norm constraint. Hence, we will first recast the whole problem into the order one Sobolev space, and then apply the results of Theorem 2.

To this end, note that  $W_2^1$  inner products (29) can be compactly written in the Fourier domain by

$$\langle \mathbf{s}, \mathbf{x} \rangle_{W_2^1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{S}^F(\omega)} \mathbf{X}^F(\omega) (1 + \omega^2) d\omega \quad (32)$$

where  $\mathbf{S}^F(\omega)$  and  $\mathbf{X}^F(\omega)$  are the Fourier transforms of  $\mathbf{s}(t)$  and  $\mathbf{x}(t)$ , respectively. As introduced in [11], we can use (32) to rewrite  $L_2$  inner products as  $W_2^1$  inner products. Specifically, for any  $\mathbf{a} \in L_2, \mathbf{b} \in W_2^1$

$$\langle \mathbf{a}(t), \mathbf{b}(t) \rangle_{L_2} = \langle \mathbf{a}(t) * \mathbf{u}(t), \mathbf{b}(t) \rangle_{W_2^1} \quad (33)$$

where  $*$  stands for the convolution operation and

$$\mathbf{u}(t) = \frac{1}{2} e^{-|t|} \quad (34)$$

is the inverse Fourier transform of  $1/(1 + \omega^2)$ .

Using (33), we can replace the  $L_2$  inner products  $W^*\mathbf{x}$  and  $S^*\mathbf{x}$  using their Sobolev counterparts, which leads to the following theorem.

*Theorem 4:* Consider the problem

$$\min_d \max_{\mathbf{x} \in \tilde{\mathcal{D}}} \|W^*\mathbf{x} - d\|_{L_2}^2 \quad (35)$$

where  $W$  and  $S$  are bounded set transformations with range  $\mathcal{W}$  and  $\mathcal{S}$ , respectively, and  $\tilde{\mathcal{D}}$  is given by (30). The (unique) solution is

$$d = \tilde{W}^* \tilde{S} (\tilde{S}^* \tilde{S})^\dagger c \quad (36)$$

where the inner products described by (36) are computed in the order one Sobolev space,  $\tilde{S}, \tilde{W}$  are the set transformations of  $\{\tilde{\mathbf{s}}_n = \mathbf{s}_n * \mathbf{u}\}$  and  $\{\tilde{\mathbf{w}}_n = \mathbf{w}_n * \mathbf{u}\}$ , respectively, and the function  $\mathbf{u}(t)$  is given by (34).

Before stating the proof, note that (36) describes  $W_2^1$  inner products. In practice, this means that the  $i, j$ th element of the matrix  $\tilde{W}^* \tilde{S}$  is

$$\langle \tilde{\mathbf{w}}_i, \tilde{\mathbf{s}}_j \rangle_{W_2^1} = \langle \mathbf{w}_i, \tilde{\mathbf{s}}_j \rangle_{L_2} = \langle \mathbf{w}_i, \mathbf{s}_j * \mathbf{u} \rangle_{L_2} \quad (37)$$

with  $\mathbf{u}$  as in (34). Similarly, the  $i, j$ th element of  $\tilde{S}^* \tilde{S}$  is

$$\langle \tilde{\mathbf{s}}_i, \tilde{\mathbf{s}}_j \rangle_{W_2^1} = \langle \mathbf{s}_i, \mathbf{s}_j * \mathbf{u} \rangle_{L_2}. \quad (38)$$

*Proof:* Using (33), we can rewrite (35) as

$$\min_d \max_{c = \tilde{S}^*\mathbf{x}, \|\mathbf{x}\|_{W_2^1} \leq \tilde{L}} \|\tilde{W}^*\mathbf{x} - d\|_{L_2}^2$$

where  $\tilde{S}$  and  $\tilde{W}$  are the set transformations of  $\{\tilde{\mathbf{s}}_n = \mathbf{s}_n * \mathbf{u}\}$  and  $\{\tilde{\mathbf{w}}_n = \mathbf{w}_n * \mathbf{u}\}$ , respectively. Since the derivations of Theorem 2 apply to any Hilbert space, the solution has the same form as in (10), resulting in (36).  $\square$

The result of Theorem 4 can be interpreted in several ways. Rewriting  $P_{\tilde{\mathcal{S}}} \mathbf{x} = \tilde{S} (\tilde{S}^* \tilde{S})^\dagger \tilde{S}^* \mathbf{x} = \tilde{S} (\tilde{S}^* \tilde{S})^\dagger c$  and using (37), we obtain the following corollary.

*Corollary 2:* Equation (36) can be written as

$$d = W^* P_{\tilde{\mathcal{S}}} \mathbf{x} \quad (39)$$

where  $\tilde{\mathcal{S}}$  is the closure of  $\text{span}\{\tilde{\mathbf{s}}_n\}$  and  $P_{\tilde{\mathcal{S}}}\mathbf{x}$  stands for the orthogonal projection of  $\mathbf{x}$ , in the  $W_2^1$  sense, onto  $\tilde{\mathcal{S}}$ . The operator  $W^*$  describes the usual  $L_2$  inner products with the analysis functions.

Note that we implicitly assume that the possibly infinite sum involved in the computation of  $P_{\tilde{\mathcal{S}}}\mathbf{x}$  is well defined. This can be assured if the functions  $\{\tilde{\mathbf{s}}_n\}$  form a frame for the closure of their span. We address this question in Appendix B.

Another interesting interpretation of Theorem 4 is evident by rewriting all the inner products in the  $L_2$  space. Combining (38) with (39) we obtain the following corollary.

*Corollary 3:* Equation (36) can be written as

$$d = W^* \tilde{\mathcal{S}}(S^* \tilde{\mathcal{S}})^\dagger S^* \mathbf{x} = W^* E_{\tilde{\mathcal{S}}, \mathcal{S}^\perp} \mathbf{x} \quad (40)$$

where  $E_{\tilde{\mathcal{S}}, \mathcal{S}^\perp}$  is the oblique projection operator [17], [26], in the  $L_2$  sense, with range space  $\tilde{\mathcal{S}}$  and null space  $\mathcal{S}^\perp$ .

In analogy to Proposition 1 it can be shown that  $P_{\tilde{\mathcal{S}}}\mathbf{x} = E_{\tilde{\mathcal{S}}, \mathcal{S}^\perp} \mathbf{x}$  is the unique solution of

$$\arg \min_{\hat{\mathbf{x}}} \max_{\mathbf{x} \in \tilde{\mathcal{D}}} \|\mathbf{x} - \hat{\mathbf{x}}\|_{W_2^1}^2.$$

Corollary 3 implies that the problem of Theorem 4 could be split into two parts; first obtaining the Sobolev minimax approximation of  $\mathbf{x}$  itself (which is an oblique projection in the  $L_2$  space), and only then applying the analysis operator  $W^*$  to that approximation.

In this section we have considered the Sobolev space of order one. It is possible to extend the derivations to higher order Sobolev spaces, if a sufficient degree of smoothness is known to be present. The order  $r$  Sobolev space  $W_2^r$  is composed of finite energy functions with  $r$  finite energy derivatives;  $W_2^r$  inner products can be written as  $\langle \mathbf{a}, \mathbf{b} \rangle_{W_2^r} = \sum_{k=0}^r \langle \mathbf{a}^{(k)}, \mathbf{b}^{(k)} \rangle_{L_2}$ , where the  $(k)$  superscript stands for the  $k$ th derivative. Thus, we can obtain similar results, which only require the replacement of the function  $\mathbf{u}(t)$  in (34) with the inverse Fourier transform of  $1/(1 + \omega^2 + \omega^4 \dots + \omega^{2r})$ .

As a concluding remark, we note that our solution takes the form of applying the analysis functions  $\{\mathbf{w}_n\}$  to  $\hat{\mathbf{x}} = P_{\tilde{\mathcal{S}}}\mathbf{x} = E_{\tilde{\mathcal{S}}, \mathcal{S}^\perp} \mathbf{x}$ , which is the minimax approximation of  $\mathbf{x}$  within the space  $\tilde{\mathcal{S}}$ . This space is determined by the sampling functions  $\{\mathbf{s}_n\}$  and the smoothness constraint (manifested by  $\mathbf{u}$ ). Thus, we have obtained a nice counterpart to methods that **arbitrarily** choose the interpolation space.

#### A. Smoothness and the Shift Invariant Case

In the special case where  $\mathcal{S}$  and  $\mathcal{W}$  are real SI subspaces, as in (23) and (24), the sample sequence  $c$  may be processed by a digital filter in order to obtain the minimax approximation (36). Let  $\mathbf{u}(t)$  of (34), be the inverse Fourier transform of  $1/(1 + \omega^2)$  and define  $\tilde{\mathbf{s}} = \mathbf{s} * \mathbf{u}$ . Then, the frequency response of the minimax filter of Theorem 4 takes on the form

$$G^f(\omega) = \begin{cases} \frac{R_{\mathbf{W}, \tilde{\mathbf{S}}}^f(\omega)}{R_{\tilde{\mathbf{S}}, \tilde{\mathbf{S}}}^f(\omega)}, & \omega \in \mathcal{I}_{\mathcal{S}}; \\ 0, & \omega \notin \mathcal{I}_{\mathcal{S}} \end{cases} \quad (41)$$

where  $\tilde{\mathbf{S}}^F(\omega) = \mathbf{S}^F(\omega)/(1 + \omega^2)$  is the Fourier transform of  $\tilde{\mathbf{s}}(t)$ ,  $\mathcal{I}_{\mathcal{S}}$  defines the support of  $R_{\tilde{\mathbf{S}}, \tilde{\mathbf{S}}}^f(\omega)$ , and  $R_{\mathbf{W}, \tilde{\mathbf{S}}}^f(\omega)$ ,  $R_{\tilde{\mathbf{S}}, \tilde{\mathbf{S}}}^f(\omega)$  are defined according to (27).

#### B. Extension of the Ideal Sampling Results

In this section, we show how Theorem 4 extends the results of [11], [12]. In these works it was assumed that a single representation coefficient  $\langle \mathbf{w}(t), \mathbf{x}(t) \rangle_{L_2}$  is to be approximated by linearly processing the ideal sample sequence  $\{\mathbf{x}(nT)\}$  of some function  $\mathbf{x} \in W_2^1$ . Denoting by  $S_T$  the ideal sampling operator

$$S_T \mathbf{x} = \{\mathbf{x}(nT)\} \quad (42)$$

the processing is performed by calculating an  $l_2$  inner product  $\langle b, S_T \mathbf{x} \rangle_{L_2}$  with some sequence  $b$ . To determine  $b$ , the approximation error was upper bounded by

$$|\langle \mathbf{w}(t), \mathbf{x}(t) \rangle_{L_2} - \langle b, S_T \mathbf{x} \rangle_{L_2}| \leq B \|\mathbf{x}\|_{W_2^1}$$

where  $B$  is a constant that depends on  $T, b, \mathbf{w}$  and the function  $\mathbf{u}(t)$  of (34). Then,  $B$  was minimized with respect to  $b$ .

Reinterpreting the derivations in [11], the approximation problem of ([11, Theorem 3]) can be restated as a minimax objective

$$\min_b \max_{\|\mathbf{x}\|_{W_2^1} \leq \tilde{L}} |\langle \mathbf{w}, \mathbf{x} \rangle_{L_2} - \langle b, S_T \mathbf{x} \rangle_{L_2}| \quad (43)$$

where  $\tilde{L}$  is some (finite) upper bound on the  $W_2^1$  norm of  $\mathbf{x}$ . In [11] it is found that the optimal sequence  $b$  satisfies

$$\sum_n \mathbf{u}(t - nT) b[n] = P_{\mathcal{U}}(\mathbf{w} * \mathbf{u}) \quad (44)$$

where  $P_{\mathcal{U}}$  is the orthogonal projection, in the  $W_2^1$  sense, onto  $\mathcal{U}$ , which is the closure of  $\text{span}\{\mathbf{u}(t - nT)\}$ .

We now show that this result of [11] is a special case of Theorem 4. First, define  $U$  to be the set transformation of the function set  $\{\mathbf{u}(t - nT)\}$ . It is not hard to show that on  $W_2^1$ ,  $U$  is the adjoint of the ideal sampling operator  $S_T$ , i.e., using operator notations

$$S_T \mathbf{x} = U^* \mathbf{x}.$$

We note that  $U$  (as well as  $U^* = S_T$ ) is a well-defined bounded operator in  $W_2^1$  ([9, App. C]). Additionally, the single representation coefficient  $\langle \mathbf{w}, \mathbf{x} \rangle_{L_2}$  can be written as the order one Sobolev inner product  $\tilde{W}^* \mathbf{x}$ , with  $\tilde{W}$  being the set transform of  $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) * \mathbf{u}(t)$ . Identifying  $\tilde{\mathcal{S}}$  with  $U$ , we have from (36)

$$\begin{aligned} d &= \tilde{W}^* U (U^* U)^\dagger c = \langle \tilde{\mathbf{w}}, U (U^* U)^\dagger c \rangle_{W_2^1} \\ &= \langle (U^* U)^\dagger U^* \tilde{\mathbf{w}}, c \rangle_{l_2} = \langle b, c \rangle_{l_2}, \end{aligned} \quad (45)$$

where we denote  $b = (U^* U)^\dagger U^* \tilde{\mathbf{w}}$ . As a result,  $Ub$  is exactly the orthogonal projection of  $\tilde{\mathbf{w}} = \mathbf{w} * \mathbf{u}$  onto the space  $\mathcal{U}$ , which is compatible with (44).

## VI. ERROR ANALYSIS

In this section we investigate the error resulting from the minimax method. We then derive sufficient conditions for our method to outperform the sum approximation (5). Although we use the  $S$  and  $W$  operators (as opposed to their Sobolev counterparts  $\tilde{S}, \tilde{W}$ ), all derivations are applicable to Sobolev spaces by considering the appropriate inner products.

Let

$$e_{\text{mx}} = W^* \mathbf{x} - d = W^* \mathbf{x} - W^* S(S^* S)^\dagger c \quad (46)$$

be the error sequence due to the minimax approach. Using (19) we can express the error as

$$e_{\text{mx}} = W^* P_{S^\perp} \mathbf{x}. \quad (47)$$

Define  $e_{\text{sum}}$  to be the error sequence due to the sum approximation method (5). The  $n$ th element of  $e_{\text{sum}}$  satisfies

$$e_{\text{sum}}[n] = \langle \mathbf{w}_n, \mathbf{x} \rangle_{L_2} - T \langle S_T \mathbf{w}_n, c \rangle_{l_2} \quad (48)$$

where  $S_T$  is the ideal sampling operator (42). Note that to make (48) well defined, we implicitly assume  $\mathbf{w}_n \in W_2^1$ . We also note that  $e_{\text{mx}}$  and  $e_{\text{sum}}$  depend on the input signal  $\mathbf{x}$ . However, to simplify the exposition, we omit this dependence from the notations.

We now examine the conditions which will assure that  $\|e_{\text{mx}}\|_{l_2}^2 \leq \|e_{\text{sum}}\|_{l_2}^2$  for all possible inputs. In the following lemma we first introduce tight bounds for the difference  $\|e_{\text{sum}}\|_{l_2}^2 - \|e_{\text{mx}}\|_{l_2}^2$ . Clearly, if the difference is positive, then the minimax method is preferable to the sum approximation method, and vice versa.

*Lemma 1:* Let  $\mathbf{x} \in \mathcal{D} = \{\mathbf{x} : \|\mathbf{x}\|_{L_2} \leq L, c = S^* \mathbf{x}\}$ . Then

$$B_L \leq \|e_{\text{sum}}\|_{l_2}^2 - \|e_{\text{mx}}\|_{l_2}^2 \leq B_H \quad (49)$$

where the bounds are tight. Here

$$\begin{aligned} B_L &= \|a\|_{l_2}^2 - 2|\langle a, e_{\text{mx}} \rangle_{l_2}| \\ B_H &= \|a\|_{l_2}^2 + 2|\langle a, e_{\text{mx}} \rangle_{l_2}| \end{aligned} \quad (50)$$

and

$$a[n] = \langle \mathbf{w}_n, S(S^* S)^\dagger c \rangle_{L_2} - T \langle S_T \mathbf{w}_n, c \rangle_{l_2}. \quad (51)$$

*Proof:* Using (46) and (48) we can relate the two error sequences by

$$e_{\text{sum}} = a + e_{\text{mx}} \quad (52)$$

with  $a$  given by (51). Note that since the sample sequence  $c$  is available, and so are  $T, S$  and  $W$ , the sequence  $a$  is known as well. Furthermore,  $a \in l_2$ . The latter is evident by rewriting  $a = W^*(S(S^* S)^\dagger - T S_T^*)c$ . Since  $c \in l_2$ , it is sufficient to show that  $W^*(S(S^* S)^\dagger - T S_T^*)$  is a bounded operator. Indeed, since  $S_T$  is bounded on  $W_2^1$  ([9, App. C]), so is  $S_T^*$ . Additionally,  $S(S^* S)^\dagger$  and  $W^*$  are bounded due to the frame assumptions. Taking the squared norm of both sides of (52) and rearranging terms, we get

$$\|e_{\text{sum}}\|_{l_2}^2 - \|e_{\text{mx}}\|_{l_2}^2 = \|a\|_{l_2}^2 + 2\Re\{\langle a, e_{\text{mx}} \rangle_{l_2}\}.$$

The bounds (50) then follow from

$$-|\langle a, e_{\text{mx}} \rangle_{l_2}| \leq \Re\{\langle a, e_{\text{mx}} \rangle_{l_2}\} \leq |\langle a, e_{\text{mx}} \rangle_{l_2}|.$$

We now show that the bounds are tight. Assume to the contrary that for all  $\mathbf{x} \in \mathcal{D}$

$$\Re\{\langle a, e_{\text{mx}} \rangle_{l_2}\} < |\langle a, e_{\text{mx}} \rangle_{l_2}|.$$

Define

$$\mathbf{x}_2 = S(S^* S)^\dagger c + \frac{\langle W^* P_{S^\perp} \mathbf{x}, a \rangle_{l_2}}{|\langle W^* P_{S^\perp} \mathbf{x}, a \rangle_{l_2}|} P_{S^\perp} \mathbf{x}$$

using some  $\mathbf{x} \in \mathcal{D}$ . Clearly  $\mathbf{x}_2 \in \mathcal{D}$ . However,  $e_{\text{mx}} = W^* P_{S^\perp} \mathbf{x}_2$  satisfies

$$\Re\{\langle a, W^* P_{S^\perp} \mathbf{x}_2 \rangle_{l_2}\} = |\langle a, W^* P_{S^\perp} \mathbf{x}_2 \rangle_{l_2}|$$

thus, contradicting our initial assumption. The proof of tightness for the lower bound is similar.  $\square$

Since the tight upper bound  $B_H$  is nonnegative for all choices of  $e_{\text{mx}}$ , we conclude that the sum approximation method **cannot** outperform the proposed minimax approach, for **all** possible inputs. On the other hand, in some cases, it is possible to have better performance by the minimax approach, for all possible inputs. To assure this, the lower bound  $B_L$  must be positive. In the following lemma, we provide a tight upper bound on  $\|e_{\text{mx}}\|_{l_2}$  assuming that the set  $\{\mathbf{w}_n(t)\}$  is orthonormal. Using this bound, we then state a sufficient condition for the minimax method to outperform the standard sum approximation approach for all  $\mathbf{x} \in \mathcal{D}$ .

*Lemma 2:* Let  $\{\mathbf{w}_n(t)\}$  be an orthonormal set, and let  $\mathbf{x} \in \mathcal{D}$ . Then

$$\|e_{\text{mx}}\|_{l_2} \leq B_{\text{mx}} = \sqrt{1 - \cos^2(\mathcal{W}, \mathcal{S})} L' \quad (53)$$

where

$$L' = \sqrt{L^2 - \|S^*(S^* S)^\dagger c\|^2} \quad (54)$$

is the norm of  $P_{S^\perp} \mathbf{x}$  and  $\cos(\mathcal{W}, \mathcal{S}) = \inf_{\mathbf{y} \in \mathcal{W}, \|\mathbf{y}\|=1} \|P_{\mathcal{S}} \mathbf{y}\|$ .

Before giving the proof, we mention that  $\cos(\mathcal{W}, \mathcal{S})$  is related to the largest angle [13], [16] between the spaces. An explicit expression for  $\cos(\mathcal{W}, \mathcal{S})$  in the case of SI spaces is given in [13].

*Proof:* From the definition of  $e_{\text{mx}}$

$$\begin{aligned} \|e_{\text{mx}}\|_{l_2}^2 &= \langle W^* P_{S^\perp} \mathbf{x}, W^* P_{S^\perp} \mathbf{x} \rangle_{l_2} \\ &= \langle P_{S^\perp} \mathbf{x}, W W^* P_{S^\perp} \mathbf{x} \rangle \\ &= \langle P_{S^\perp} \mathbf{x}, P_{\mathcal{W}} P_{S^\perp} \mathbf{x} \rangle \\ &= \langle P_{\mathcal{W}} P_{S^\perp} \mathbf{x}, P_{\mathcal{W}} P_{S^\perp} \mathbf{x} \rangle \\ &= \|P_{\mathcal{W}} P_{S^\perp} \mathbf{x}\|^2 \end{aligned} \quad (55)$$

where we utilized the orthonormality of the analysis set  $\{\mathbf{w}_n(t)\}$  to write  $W W^* = P_{\mathcal{W}}$ . For any  $\mathbf{x} \in \mathcal{D}$  we have  $\|P_{S^\perp} \mathbf{x}\| \leq L'$ , where  $L'$  is given by (54). Thus, we can bound

$$\|P_{\mathcal{W}} P_{S^\perp} \mathbf{x}\| \leq \sin(\mathcal{S}^\perp, \mathcal{W}^\perp) L'$$

$$\begin{array}{c}
 \begin{array}{c} | \quad \|e_{sum}\| - \|e_{mx}\| \quad | \\ \hline B_L \qquad \qquad \qquad B_H \geq 0 \end{array} \quad (\text{Lemma 1}) \\
 \|e_{mx}\| \leq B_{mx} \quad (\text{Lemma 2}) \\
 B_{mx} \leq \|a\|_{l_2} / 2 \Rightarrow B_L \geq 0 \quad (\text{Corollary 4})
 \end{array}$$

Fig. 2. Regions of  $\|e_{sum}\| - \|e_{mx}\|$  for the case where  $\{\mathbf{w}_n(t)\}$  is an orthonormal set. If the maximal norm of the minimax error (53) is smaller than  $\|a\|_{l_2}/2$ , then the minimax approach is superior to the sum method, for all possible inputs.

where  $\sin(\mathcal{S}^\perp, \mathcal{W}^\perp) = \sup_{\mathbf{y} \in \mathcal{S}^\perp, \|\mathbf{y}\|=1} \|P_{\mathcal{W}} \mathbf{y}\|$ . From [13], [27]  $\sin(\mathcal{S}^\perp, \mathcal{W}^\perp) = \sqrt{1 - \cos^2(\mathcal{W}, \mathcal{S})}$  and the proof follows.  $\square$

*Corollary 4:* Let  $\{\mathbf{w}_n(t)\}$  be an orthonormal set. A sufficient condition for the minimax method to outperform the sum approach for all  $\mathbf{x} \in \mathcal{D}$  is  $\|a\|_{l_2} \geq 2B_{mx}$ , where  $a$  and  $B_{mx}$  are given by (51) and (53), respectively.

*Proof:* Using Lemma 1, the Cauchy–Schwartz inequality and Lemma 2, we have

$$B_L \geq \|a\|_{l_2}^2 - 2\|a\|_{l_2}\|e_{mx}\|_{l_2} \geq \|a\|_{l_2}^2 - 2\|a\|_{l_2}B_{mx},$$

from which the proof follows.  $\square$

The error analysis is summarized in Fig. 2.

Another interesting case, which is easy to evaluate, is when a single representation coefficient  $\langle \mathbf{w}, \mathbf{x} \rangle$  is to be approximated. In this setting,  $e_{mx}$ ,  $e_{sum}$  and  $a$  are all scalars. It can then be shown that

$$\begin{aligned}
 |e_{mx}| &\leq B_{mx} = L' \|P_{\mathcal{S}^\perp} \mathbf{w}\| \\
 |e_{sum}| &\leq B_{sum} = |a| + B_{mx}
 \end{aligned} \quad (56)$$

where the bounds are tight. Furthermore, the input that achieves both upper bounds is

$$\mathbf{x}_{worst} = S(S^*S)^\dagger c + \frac{aL'}{\|a\| \|P_{\mathcal{S}^\perp} \mathbf{w}\|} P_{\mathcal{S}^\perp} \mathbf{w}. \quad (57)$$

A sufficient condition for the minimax method to outperform the sum approach becomes

$$|a| \geq 2L' \|P_{\mathcal{S}^\perp} \mathbf{w}\|. \quad (58)$$

The proof is provided in Appendix C.

To conclude, when the spaces  $\mathcal{W}$  and  $\mathcal{S}$  are *close* (such that  $\cos(\mathcal{W}, \mathcal{S})$  is close to one), or when most of the signal's energy lies within the sampling space  $\mathcal{S}$  (such that  $L'$  is small), then the minimax method will outperform the standard approach. Similarly, for large sampling intervals  $T$ ,  $\|a\|_{l_2}$  can become large enough, assuring better performance by the minimax method.

## VII. SIMULATIONS

In this section, we simulate an example of approximating a single representation coefficient  $\langle \mathbf{w}, \mathbf{x} \rangle_{L_2}$ . The analysis function  $\mathbf{w}(t)$  is a modulated and normalized Gaussian

$$\mathbf{w}(t) = \alpha e^{-t^2/2} \cos(4\pi t) \quad (59)$$

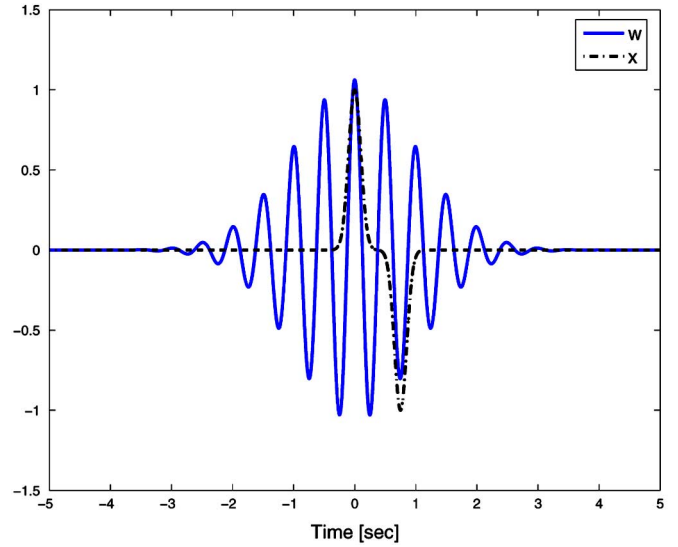


Fig. 3. The analysis function  $\mathbf{w}(t)$  and the input signal  $\mathbf{x}(t)$ .

with  $\alpha$  chosen such that  $\|\mathbf{w}\|_{L_2} = 1$ . The input  $\mathbf{x}$  is set to be

$$\mathbf{x}(t) = e^{-50t^2} - e^{-50(t-0.75)^2} \quad (60)$$

i.e., it is composed of two Gaussians, synchronized with the analysis function  $\mathbf{w}(t)$  (see Fig. 3). For this example  $\langle \mathbf{w}, \mathbf{x} \rangle_{L_2} \approx 0.2$ . We will consider two separate sampling schemes: zero-order-hold (ZOH) and resistor-capacitor (RC). In both schemes, we approximate the single representation coefficient  $\langle \mathbf{w}, \mathbf{x} \rangle_{L_2}$  based on the generalized samples.

### A. ZOH Sampling

Assume that the generalized samples of  $\mathbf{x}(t)$  are obtained by averaging the value of  $\mathbf{x}(t)$  within a small interval of length  $\Delta$ , i.e.

$$c[n] = \frac{1}{\Delta} \int_{nT}^{nT+\Delta} \mathbf{x}(t) dt. \quad (61)$$

In this setting, the  $n$ th sampling vector  $\mathbf{s}_n(t)$  of (1) is

$$\mathbf{s}_n(t) = \begin{cases} 1/\Delta, & t \in [nT, nT + \Delta]; \\ 0, & \text{otherwise.} \end{cases} \quad (62)$$

By processing the generalized samples  $\{c[n] = \langle \mathbf{s}_n, \mathbf{x} \rangle_{L_2}\}$  using the transformation (10), we obtain the minimax approximation of  $\langle \mathbf{w}, \mathbf{x} \rangle_{L_2}$ . The approximation can be obtained in the  $L_2$  space, or transformed into a proper Sobolev space using (36), when smoothness is of concern. Note that the input signal of the example (60) indeed satisfies  $\mathbf{x} \in W_2^1$ . Subsequently, as we will show, the minimax solution with the smoothness constraint outperforms the standard minimax method.

Interpreting the minimax solutions as the application of the analysis operator  $W^*$  to the approximates  $P_{\mathcal{S}} \mathbf{x}$  and  $E_{\mathcal{S}, \mathcal{S}^\perp} \mathbf{x}$  [(19) and (40), respectively], it is interesting to observe the signal approximations. Fig. 4 depicts the generator functions  $\mathbf{s}_0(t) \in \mathcal{S}$  and  $\tilde{\mathbf{s}}_0(t) \in \tilde{\mathcal{S}}$  for  $\Delta = 0.05$ . In Fig. 5 we plot a section of  $\mathbf{x}$  with its projections onto the appropriate sampling spaces. The parameters  $T$  and  $\Delta$  were set to 0.1[sec] and



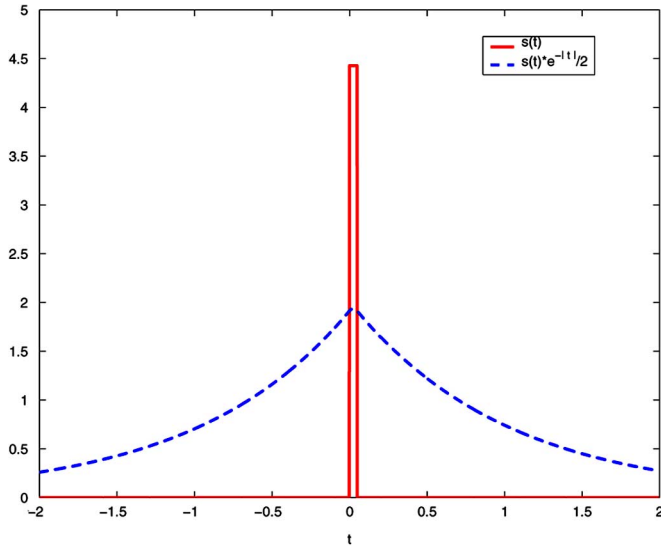


Fig. 4. ZOH sampling with  $\Delta = 0.05$ . Shown are  $s_0(t)$  (62) and  $\bar{s}_0(t) = s_0(t) * e^{-|t|}/2$ , which are the generators for the shift invariant spaces  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$ , respectively. For presentation purposes the plots are scaled.

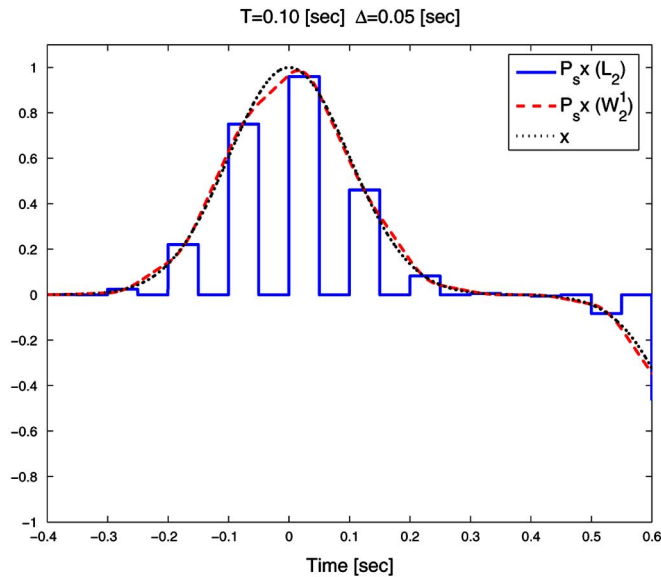


Fig. 5. A section of  $\mathbf{x}$  and its approximation in the sampling space. The  $L_2$  orthogonal projection onto  $\mathcal{S}$  yields rectangular pulses. The oblique projection onto  $\tilde{\mathcal{S}}$  yields a smooth function, which is very close to the original input.

0.05[sec], respectively. In this example, the space  $\tilde{\mathcal{S}}$  captures most of the signal's energy. Indeed, as can be seen in Fig. 5, the approximation  $E_{\tilde{\mathcal{S}}, \mathcal{S}^\perp} \mathbf{x}$  is very close to the original input.

For comparison, we also processed the samples using the standard sum approach (5). In Fig. 6, we present the errors for the input (60) using several choices of  $T$ . The minimax solution is optimized for the worst possible input within the considered set, which is different than (60). As a result, for some sampling intervals, the suggested robust solutions are better, while for others they are outperformed by the sum approximation.

It is also of interest to examine the signals that cause the highest value of the cost function. In Fig. 7, we plot these worst

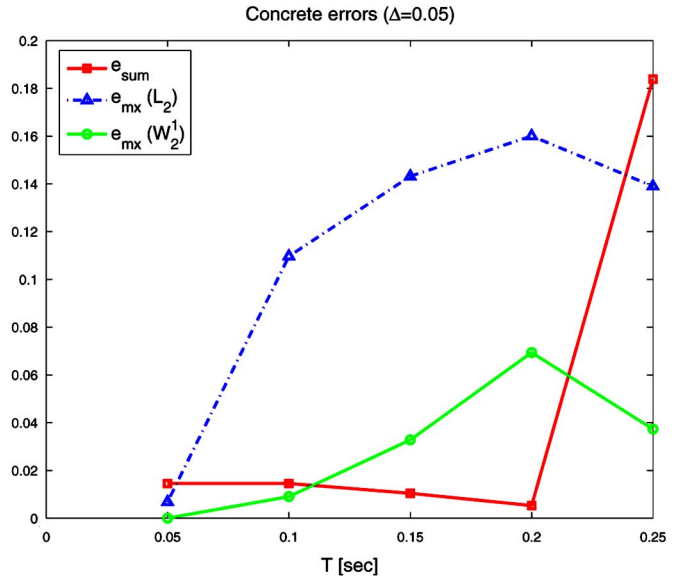


Fig. 6. The errors as a function of  $T$  for the specific input (60).

inputs. In both cases, the worst possible input is calculated according to (57), and is given by a projection of  $\mathbf{x}$  onto the sampling space, and a vector in  $\mathcal{S}^\perp$ , which has the smallest angle with the analysis function  $\mathbf{w}(t)$ . As can be seen in Fig. 7(a), the worst possible input in the set  $\mathcal{D} = \{\mathbf{x} : \|\mathbf{x}\|_{L_2} \leq L, c = \mathcal{S}^* \mathbf{x}\}$  is a highly nonsmooth function. This input is indeed possible in the  $L_2$  space, but it is not likely to appear if we know the signal to be smooth. If we consider only order one Sobolev functions, then the worst input is a smooth function and is much closer to the original input, as depicted in Fig. 7(b). The exceptionally good results of Fig. 7(b) are due to the fact that for this example, most of the signal's energy lies within the space  $\tilde{\mathcal{S}}$  (alternatively,  $L'$  is small). As a result, the approximation  $P_{\tilde{\mathcal{S}}} \mathbf{x} = E_{\tilde{\mathcal{S}}, \mathcal{S}^\perp} \mathbf{x}$  describes well the original input.

Note that in all cases, the worst inputs look the same for the acquisition device, as they both produce the same generalized samples. To illustrate the last point, in Fig. 8 we plot a section of  $\mathbf{x}$  and the worst possible inputs (for the  $L_2$  and the  $W_2^1$  sets). In addition, we present the orthogonal projection  $P_{\tilde{\mathcal{S}}} \mathbf{x}$ , in the  $L_2$  sense, which is composed of rectangular pulses describing the integration zones due to the sampling functions (62). As can be seen, all signals yield the same generalized samples, as they all have the same area within the rectangular pulses.

In Fig. 9 we plot the upper bounds of the performance for the different approximation methods. The upper two curves are due to (56). If in addition the input is known to be smooth, then we can perform all the inner products and norms in the order one Sobolev space. As a result, the value of the upper bound  $B_{\text{mx}}$  changes, and so does  $B_{\text{sum}}$  (the lower two curves of Fig. 9). The upper bounds are obtained by the worst possible inputs plotted in Fig. 7. Specifically, the signal of Fig. 7(b) achieves the lower two curves of Fig. 9 (with the lowest curve for the minimax method with the smoothness constraints, and the one above it for the sum approach). Similarly, when smoothness is not of concern, the signal of Fig. 7(a) achieves the top two error bound curves of Fig. 9 (with the higher curve for the sum approximation).

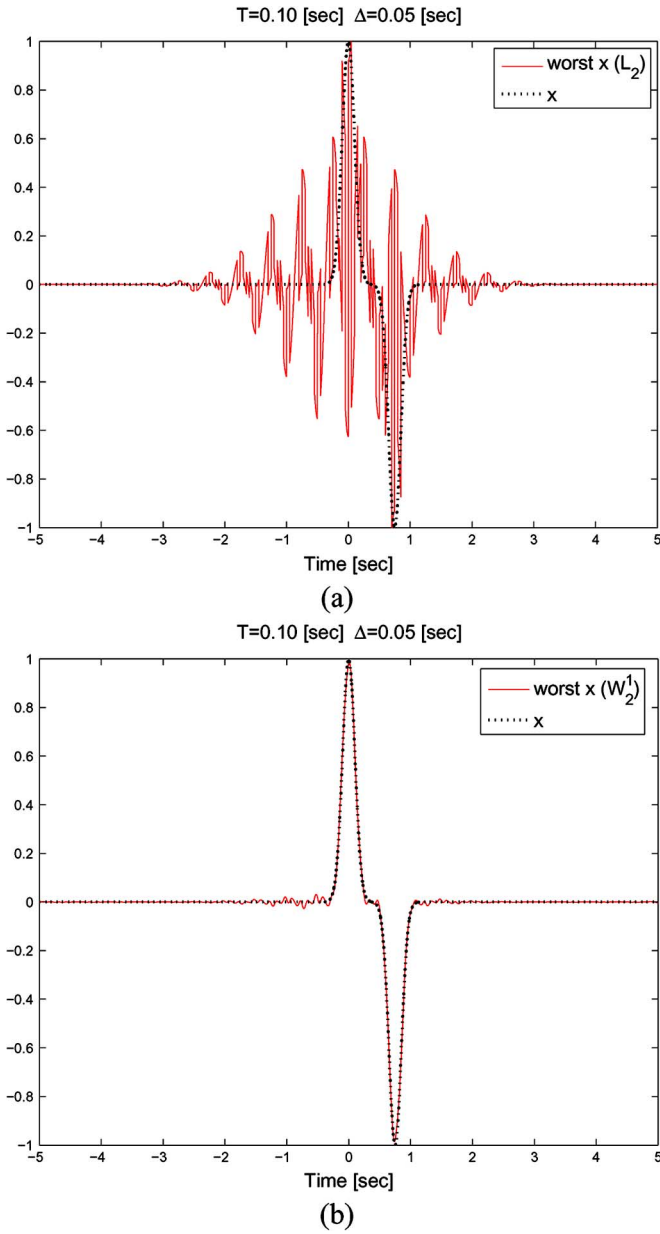


Fig. 7. The original input and the worst possible counterpart in (a)  $L_2$ . (b) Sobolev space of order one.

**B. RC Sampling**

Suppose now that the acquisition device is a low-pass RC circuit, followed by an ideal sampler with interval  $T$  (Fig. 10). The frequency response of the acquisition filter is given by  $1/(1 + j\omega RC)$ , and the  $n$ th sampling vector is a shifted and mirrored version of the impulse response

$$s_n(t) = \begin{cases} (RC)^{-1} e^{-\frac{t-nT}{RC}}, & t \leq nT; \\ 0, & \text{otherwise.} \end{cases} \quad (63)$$

Fig. 11 is similar to Fig. 7, when using the RC circuit sampling function (63) with  $RC = 0.5$ . Here as well, the sampling functions possess discontinuities, giving rise to a nonsmooth worst-case function, as shown in Fig. 11(a). When we expect the input to be smooth, the minimax objective with the smoothness constraint can be used. For such a criterion, the worst-case input function behaves accordingly [Fig. 11(b)].

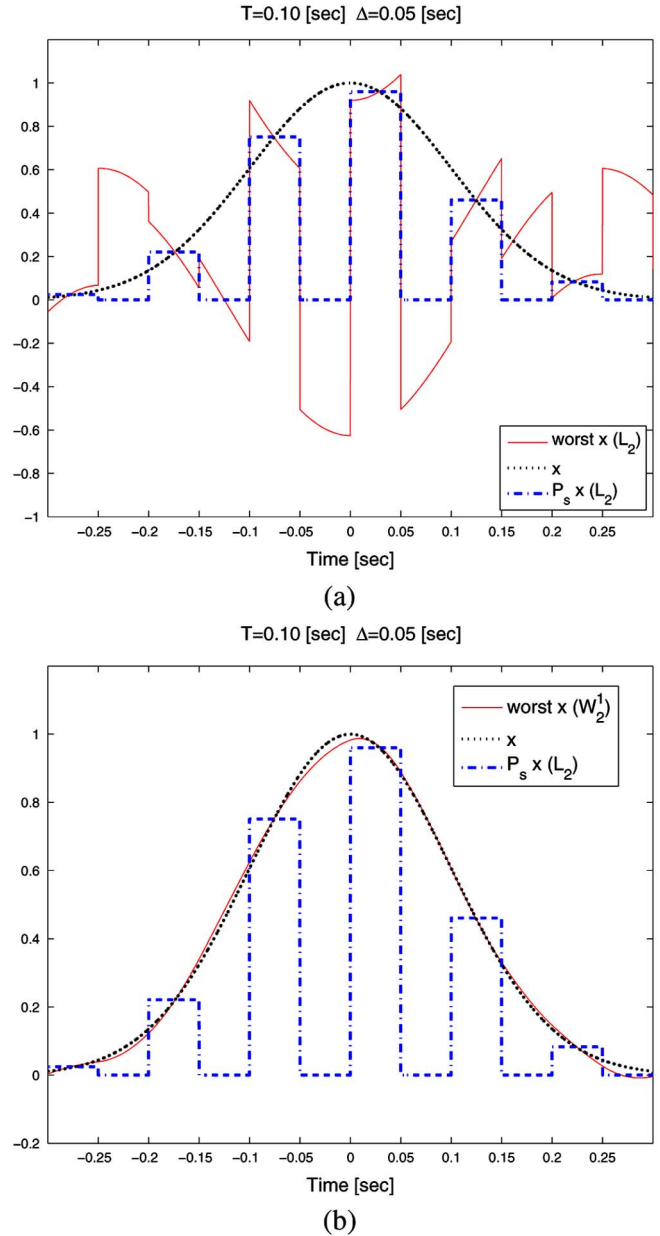


Fig. 8. A section of  $x$  and the worst possible counterpart in (a)  $L_2$ . (b) Sobolev space of order one. Both are plotted against  $P_s x$  to describe the integration zones.

Fig. 12 shows the approximation error for the input  $x(t)$  of (60). Since  $x(t)$  is a smooth function, imposing the smoothness constraint indeed improves the performance of the minimax methods. Here as well, the proposed robust criteria do not always outperform the Riemann sum approximation [Fig. 12(a)]. However, by considering the worst possible input, the superiority of the minimax methods is guaranteed. In Fig. 13 we show the upper error bounds for several values of  $T$  and  $RC$ . As expected, the robust approaches outperform the sum counterpart. Additionally, when we restrict the set of possible inputs to order one Sobolev functions, the worst case errors are smaller. As with the previous simulation, the presented error bounds are tight. For example, the worst case inputs of Fig. 11 achieve the error bounds of Fig. 13(b).

As a final remark, note that the worst-case signal (57) depends on the sampling and analysis functions. Therefore, when either

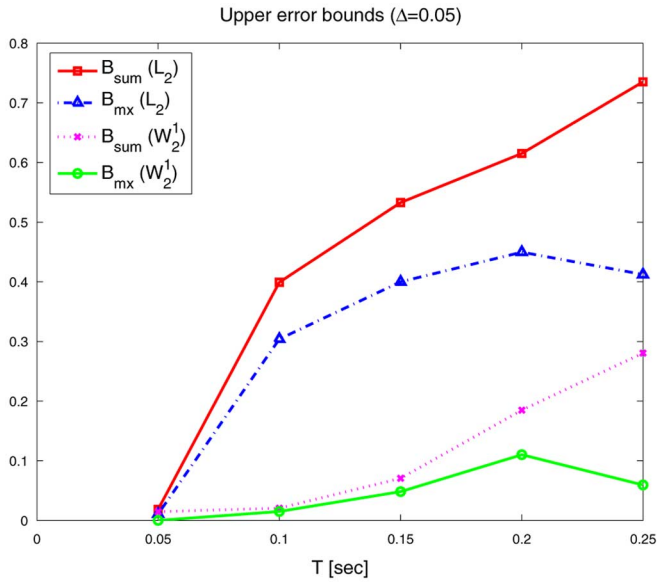


Fig. 9. Upper error bounds according to (56). The sampling functions are given by (62).

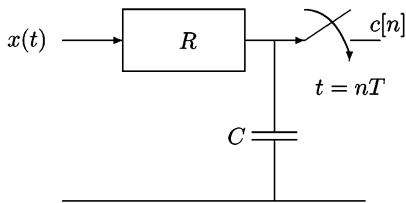


Fig. 10. An RC circuit, followed by ideal sampler, serves as the acquisition device.

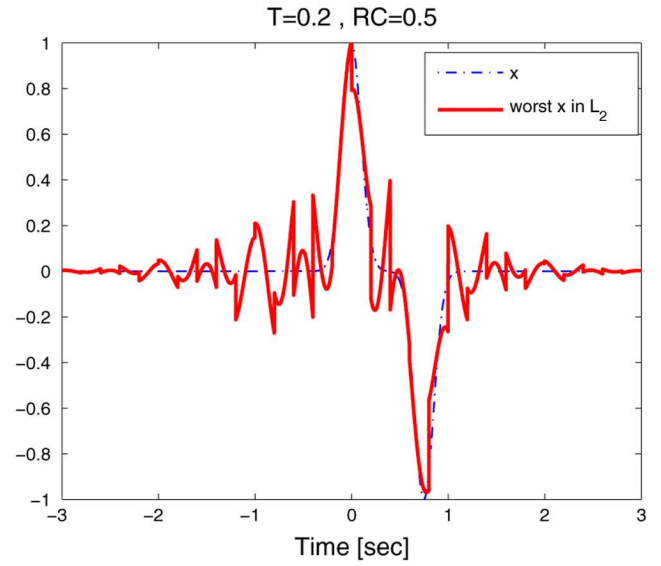
of them is nonsmooth, the worst-case function might be nonsmooth as well, being the sum of functions with discontinuities. As a result, if we have prior knowledge that the input  $\mathbf{x}(t)$  is smooth, then it is recommended to implement the minimax solution with the smoothness constraint.

VIII. SUMMARY

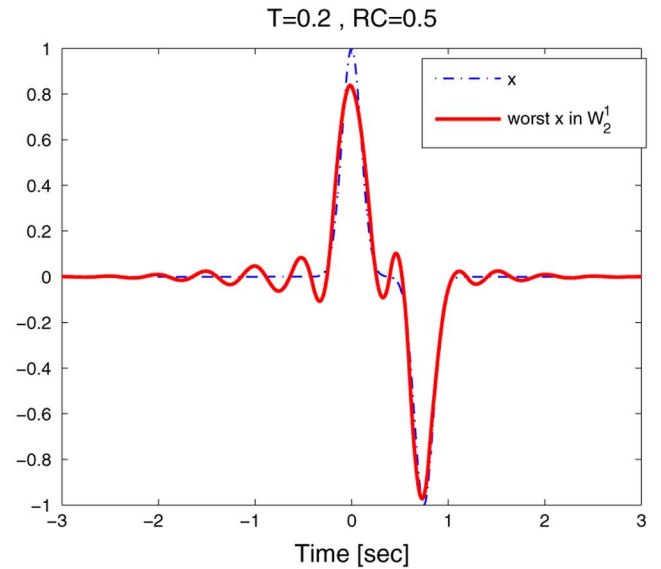
A minimax approach has been introduced for approximating inner-product calculations within the continuous-time domain, while having the generalized samples of the signal as the only available data. We have shown that if the input signal is known to be a smooth function, then a smoothness constraint can be incorporated into the minimax criterion. The latter was achieved by recasting the problem into a proper Sobolev space. A comparison of our proposed robust methods with a piecewise-constant approximation has been presented. Error bounds for the different methods were derived, showing the possible improvement by the minimax methods. The derivations presented herein extend recent results concerning the ideal sampling case, allowing for practical acquisition devices to be incorporated.

APPENDIX A  
PROOF OF THEOREM 1

In this appendix, we show that for a general  $\mathbf{x} \in \mathcal{H}$ , satisfying  $c = S^* \mathbf{x}$ , it is possible to obtain the required inner products  $q = W^* \mathbf{x}$  if and only if  $\mathcal{W} \subseteq \mathcal{S}$ . The proof is similar to the



(a)



(b)

Fig. 11. The original input and the worst possible counterpart in (a)  $L_2$ . (b) Sobolev space of order one. The sampling functions are given by (63).

proof of a sampling problem, considered in ([16, Sec. 3]). For completeness, we detail the derivations here.

Assume  $\mathcal{W} \subseteq \mathcal{S}$  and let  $d = Gc$  where

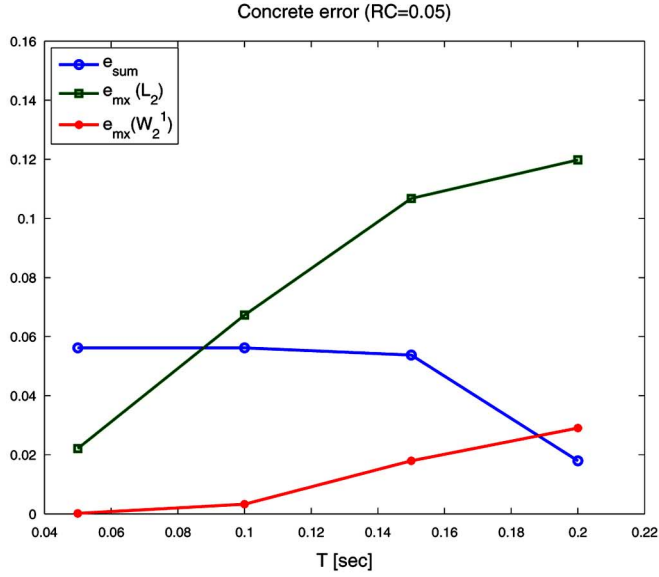
$$G = W^* S(S^* S)^\dagger. \tag{64}$$

We now show that  $d = W^* \mathbf{x}$ . Indeed, since for any function  $\mathbf{f}$ ,  $W^* \mathbf{f} = W^* P_{\mathcal{W}} \mathbf{f}$  we have  $Gc = W^* P_{\mathcal{W}} S(S^* S)^\dagger c$ . Substituting  $c = S^* \mathbf{x}$

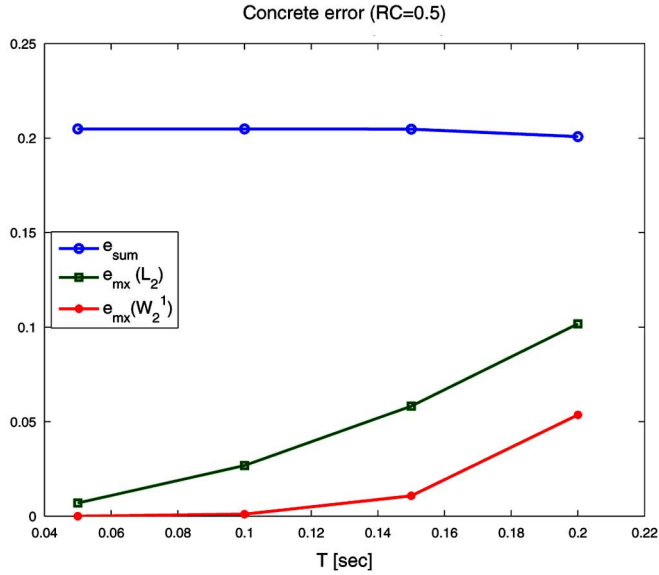
$$d = Gc = W^* P_{\mathcal{W}} P_{\mathcal{S}} \mathbf{x} = W^* P_{\mathcal{W}} \mathbf{x} = W^* \mathbf{x} \tag{65}$$

where we used the fact that  $P_{\mathcal{W}} P_{\mathcal{S}} = P_{\mathcal{W}}$  since  $\mathcal{W} \subseteq \mathcal{S}$ .

Now, assume that  $\mathcal{W} \not\subseteq \mathcal{S}$  and suppose that there exists a transformation  $d = G(c)$  achieving  $d = W^* \mathbf{x}$ . Consider the signal  $\mathbf{x}$  defined by  $\mathbf{x} = \mathbf{x}_{\mathcal{S}^\perp} + \mathbf{x}_{\mathcal{W}}$  where  $\mathbf{x}_{\mathcal{S}^\perp}$  is in  $\mathcal{S}^\perp$  but not in  $\mathcal{W}^\perp$  (such a function always exists since  $\mathcal{W} \not\subseteq \mathcal{S}$ ) and



(a)



(b)

Fig. 12. Concrete approximation errors for the input (60) processed by the RC circuit. (a) RC = 0.05. (b) RC = 0.5.

$\mathbf{x}_{\mathcal{W}} \in \mathcal{W}$ . For this choice,  $c = S^* \mathbf{x} = S^* \mathbf{x}_{\mathcal{W}}$  but  $W^* \mathbf{x} - W^* \mathbf{x}_{\mathcal{W}} = W^* \mathbf{x}_{\mathcal{S}^\perp} \neq 0$ . Since we assumed  $W^* \mathbf{x} = G(S^* \mathbf{x})$  and  $W^* \mathbf{x}_{\mathcal{W}} = G(S^* \mathbf{x}_{\mathcal{W}})$  we also have

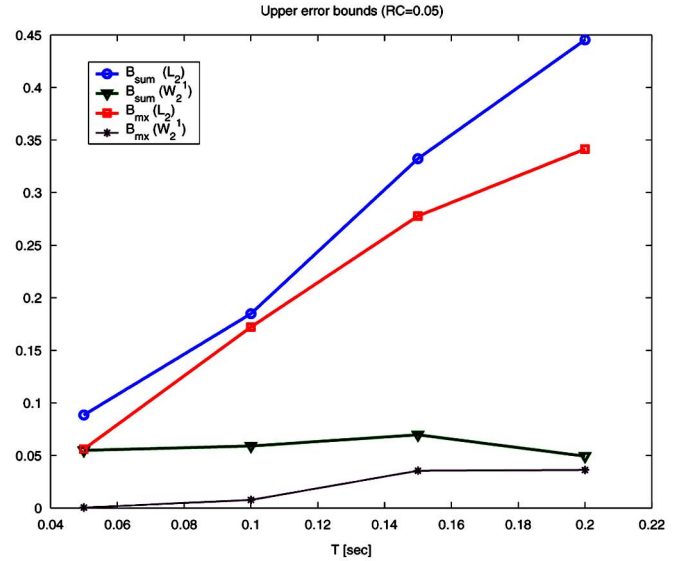
$$W^* \mathbf{x}_{\mathcal{S}^\perp} = G(S^* \mathbf{x}) - G(S^* \mathbf{x}_{\mathcal{W}}) = 0 \quad (66)$$

which implies that  $\mathbf{x}_{\mathcal{S}^\perp} \in \mathcal{W}^\perp$ , contradicting our assumption.

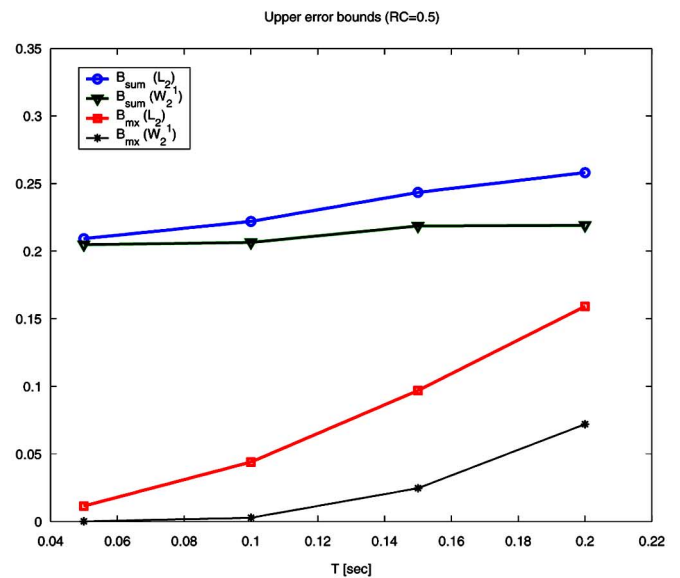
## APPENDIX B

### FRAME CONDITION IN THE SOBOLEV SPACE

In this appendix we address the following question: Assuming that the sampling functions  $\{s_n(t)\}$  constitute a frame for  $\mathcal{S} \subseteq L_2$ , do the modified functions  $\{\tilde{s}_n(t) = s_n(t) * \mathbf{u}(t)\}$  constitute a frame for  $\tilde{\mathcal{S}} \subseteq W_2^1$ ? As we will show, this is not always the case, but we give a sufficient condition for this to hold.



(a)



(b)

Fig. 13. Upper error bounds. The sampling functions are given by (63) with (a) RC = 0.05. (b) RC = 0.5.

Since the sampling functions  $\{s_n(t)\}$  form a frame for the closure of their span, there exist  $0 < A, B < \infty$  such that  $\forall \mathbf{x} \in \mathcal{S}$

$$A \|\mathbf{x}\|_{L_2}^2 \leq \sum_n |\langle s_n, \mathbf{x} \rangle_{L_2}|^2 \leq B \|\mathbf{x}\|_{L_2}^2. \quad (67)$$

Defining the modified functions to be  $\{\tilde{s}_n(t) = s_n(t) * \mathbf{u}(t)\}$ , where  $\mathbf{u}(t)$  is given by (34), and setting  $\tilde{\mathcal{S}} \subseteq W_2^1$  to be the closure of  $\text{span}\{\tilde{s}_n\}$ , we wish to examine whether for all  $\mathbf{y} \in \tilde{\mathcal{S}}$  there are constants  $0 < \tilde{A}, \tilde{B} < \infty$  such that

$$\tilde{A} \|\mathbf{y}\|_{W_2^1}^2 \leq \sum_n |\langle \tilde{s}_n, \mathbf{y} \rangle_{W_2^1}|^2 \leq \tilde{B} \|\mathbf{y}\|_{W_2^1}^2. \quad (68)$$

If the number of sampling functions is finite, then (68) always holds, as any finite set of functions which spans  $\tilde{\mathcal{S}}$  is a frame for  $\tilde{\mathcal{S}}$ . However, in the infinite dimensional case, this is no longer true.

We first show that the upper bound in (68) is always satisfied with  $\tilde{B} = B$ . To see this, let  $\mathbf{y} \in \tilde{\mathcal{S}}$ . Recalling that for any  $\mathbf{y} \in W_2^1$ , we have  $\langle \mathbf{s}_n, \mathbf{y} \rangle_{L_2} = \langle \tilde{\mathbf{s}}_n, \mathbf{y} \rangle_{W_2^1}$ , we can rewrite the middle term of (68) as  $\sum_n |\langle \mathbf{s}_n, \mathbf{y} \rangle_{L_2}|^2$ . Since  $\{\mathbf{s}_n\}$  is a frame for  $\mathcal{S}$ ,

$$A\|P_{\mathcal{S}}\mathbf{y}\|_{L_2}^2 \leq \sum_n |\langle \mathbf{s}_n, \mathbf{y} \rangle_{L_2}|^2 \leq B\|P_{\mathcal{S}}\mathbf{y}\|_{L_2}^2.$$

Using

$$B\|P_{\mathcal{S}}\mathbf{y}\|_{L_2}^2 \leq B\|\mathbf{y}\|_{L_2}^2 \leq B\|\mathbf{y}\|_{W_2^1}^2$$

we conclude that with  $\tilde{B} = B$ , the upper bound in (68) is always satisfied.

Unfortunately, satisfying the lower bound of (68) is not always possible. As an example, consider the case where for each  $n$ ,  $\mathbf{s}_n(t)$  has the Fourier transform

$$\mathbf{S}_n^F(\omega) = \begin{cases} 1, & n < \omega \leq n+1 \\ 0, & \text{otherwise.} \end{cases}$$

This is an orthonormal set of sampling functions, and hence it is a tight frame for  $\mathcal{S}$  with frame bounds  $A = B = 1$ . However, there is no strictly positive lower bound  $\tilde{A}$  satisfying the left hand side inequality of (68); notice that for the choice  $\mathbf{y}(t) = \tilde{\mathbf{s}}_n(t)$

$$\|\mathbf{y}\|_{W_2^1}^2 = \frac{1}{2\pi} \int_n^{n+1} \frac{1}{1+\omega^2} d\omega.$$

Defining  $b_n = \|\mathbf{y}\|_{W_2^1}^2$  we have from the orthonormality of the expansion

$$\sum_n \left| \langle \tilde{\mathbf{s}}_n, \mathbf{y} \rangle_{W_2^1} \right|^2 = b_n^2. \quad (69)$$

Recalling (68), its left hand side should satisfy  $\tilde{A}b_n \leq b_n^2$  for this particular example. However, by increasing  $n$  we can construct a sequence  $\{b_n\}$ , which is strictly positive, and converges to zero. Thus, for this example we must have  $\tilde{A} = 0$ .

Nevertheless, assuming that the sampling functions have a shift invariant structure, i.e., that for each  $n$ ,  $\mathbf{s}_n(t) = \mathbf{s}(t-nT)$  (and naturally also  $\tilde{\mathbf{s}}_n(t) = \tilde{\mathbf{s}}(t-nT)$ ), we can state a sufficient condition that will assure the existence of a strictly positive lower bound  $\tilde{A}$ .

*Proposition 2:* Let  $\{s(t-nT)\}$  be a frame for  $\mathcal{S}$ . If the partial sums

$$\hat{R}_{\tilde{\mathcal{S}},\mathcal{S}}^f(K,\omega) = \frac{1}{T} \sum_{k=-K}^K \left| \mathbf{S}^F \left( \frac{\omega+2\pi k}{T} \right) \right|^2 \quad (70)$$

converge uniformly, then  $\{\tilde{\mathbf{s}}(t-nT)\}$  is a frame for  $\tilde{\mathcal{S}}$ .

*Proof:* Relying on known results for the shift invariant setup [23], the frame condition for  $\tilde{\mathcal{S}}$  is satisfied if

$$0 < \tilde{A} \leq R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) \leq \tilde{B} < \infty, \quad \omega \in \mathcal{I}_{\tilde{\mathcal{S}}} \quad (71)$$

where

$$R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \left| \mathbf{S}^F \left( \frac{\omega+2\pi k}{T} \right) \right|^2 \frac{1}{1 + \left( \frac{\omega+2\pi k}{T} \right)^2} \quad (72)$$

is the DTFT of the correlation function  $r_{\tilde{\mathbf{s}},\mathbf{s}}[n] = \langle \tilde{\mathbf{s}}(t-nT), \mathbf{s}(t) \rangle_{L_2}$  and  $\mathcal{I}_{\tilde{\mathcal{S}}}$  are the set of frequencies  $\omega$  for which  $R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) \neq 0$ . We now show that if the partial sums (70) converge uniformly, then the lower bound in (71) is satisfied. Indeed, choose  $\epsilon = A/2$ , where  $A$  is the lower frame bound of  $\mathcal{S}$ . Then, there is an index  $K_\epsilon$ , such that the partial sum  $\hat{R}_{\tilde{\mathcal{S}},\mathcal{S}}^f(K_\epsilon, \omega) = (1)/(T) \sum_{k=-K_\epsilon}^{K_\epsilon} |\mathbf{S}^F((\omega+2\pi k)/T)|^2$  satisfies

$$R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) - \hat{R}_{\tilde{\mathcal{S}},\mathcal{S}}^f(K_\epsilon, \omega) < \epsilon \quad (73)$$

for all  $\omega \in \mathcal{I}_{\tilde{\mathcal{S}}}$ . Therefore

$$\begin{aligned} R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) &\geq \frac{1}{T} \sum_{k=-K_\epsilon}^{K_\epsilon} \left| \mathbf{S}^F \left( \frac{\omega+2\pi k}{T} \right) \right|^2 \frac{1}{1 + \left( \frac{\omega+2\pi k}{T} \right)^2} \\ &\geq \frac{1}{T} \sum_{k=-K_\epsilon}^{K_\epsilon} \left| \mathbf{S}^F \left( \frac{\omega+2\pi k}{T} \right) \right|^2 D(K_\epsilon) \\ &= \hat{R}_{\tilde{\mathcal{S}},\mathcal{S}}^f(K_\epsilon, \omega) D(K_\epsilon) \end{aligned}$$

where we define the strictly positive constant  $D(K_\epsilon) = (1 + ((2\pi + 2\pi K_\epsilon)/T)^2)^{-1}$ . Combining with (73),

$$R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) \geq \left( R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) - \epsilon \right) D(K_\epsilon) \geq \frac{A}{2} D(K_\epsilon)$$

where we used  $\epsilon = A/2$  and the frame bound  $R_{\tilde{\mathcal{S}},\mathcal{S}}^f(\omega) > A$ ,  $\forall \omega \in \mathcal{I}_{\tilde{\mathcal{S}}}$  of  $\tilde{\mathcal{S}}$  in the last inequality.  $\square$

## APPENDIX C

### ERROR BOUNDS FOR THE SCALAR CASE

In this appendix, we prove (56), (57), and (58).

To prove (56), note that for a single representation coefficient we have

$$|e_{\text{mx}}| = |\langle \mathbf{w}, P_{\mathcal{S}^\perp} x \rangle| = |\langle P_{\mathcal{S}^\perp} \mathbf{w}, P_{\mathcal{S}^\perp} x \rangle| \leq \|P_{\mathcal{S}^\perp} \mathbf{w}\| L' \quad (74)$$

where we used the Cauchy-Schwartz inequality and the norm constraint  $\|P_{\mathcal{S}^\perp} x\| \leq L'$ , with  $L'$  given by (54). The bound is tight, since

$$\mathbf{x} = S(S^*S)^\dagger c + \frac{P_{\mathcal{S}^\perp} \mathbf{w}}{\|P_{\mathcal{S}^\perp} \mathbf{w}\|} L' \quad (75)$$

is a valid input which achieves (74) with equality. Similarly, we can bound the error due to the sum method. Using (52),

$$|e_{\text{sum}}| \leq |a| + |e_{\text{mx}}| \leq |a| + \|P_{\mathcal{S}^\perp} \mathbf{w}\| L'. \quad (76)$$

This upper bound is obtained by setting  $\mathbf{x} = \mathbf{x}_{\text{worst}}$  as in (57). In fact, the signal  $\mathbf{x}_{\text{worst}}$  of (57) also achieves the upper bound in (74). Thus, there is a valid input which makes both the sum and the minimax methods to operate as worst as possible.

To prove (58), we must find a sufficient condition that ensures that the lower bound  $B_L$  of (50) is positive. Using (50) and (74) we have that  $B_L \geq |a|^2 - 2|a|\|P_{\mathcal{S}^\perp} \mathbf{w}\| L'$  from which (58) follows.

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