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Orthonormal shift-invariant adaptive local trigonometric decomposition

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Abstract

In this paper, an extended library of smooth local trigonometric bases is defined, and an appropriate fast "best-basis" search algorithm is introduced. When compared with the standard local cosine decomposition (LCD), the proposed algorithm is advantageous in three respects. First, it leads to a best-basis expansion that is shift-invariant. Second, the resulting representation is characterized by a lower information cost. Third, the polarity of the folding operator is adapted to the parity properties of the segmented signal at the end-points. The shift invariance stems from an adaptive relative shift of expansions in distinct resolution levels. We show that at any resolution level ℓ it suffices to examine and select one of two relative shift options – a zero shift or a $2^{-\ell-1}$ shift. A variable folding operator, whose polarity is locally adapted to the parity properties of the signal, further enhances the representation. The computational complexity is manageable and comparable to that of the LCD. © 1997 Elsevier Science B.V.

Zusammenfassung

In diesem Beitrag wird eine erweiterte Bibliothek von glatten trigonometrischen Ortsbasisfunktionen definiert und ein geeignetes schnelles Suchverfahren zum Auffinden der "besten Basis" eingeführt. Verglichen mit der standardmäßigen Ortszerlegung mit Kosinusfunktionen (LCD) ist der vorgeschlagene Algorithmus in dreifacher Hinsicht vorteilhaft: (1) Er führt auf der Grundlage der besten Basisfunktionen zu der Entwicklung, die verschiebungsinvariant ist. (2) Charakteristisch für die sich ergebende Darstellung ist der geringere Informationsaufwand. (3) Die Polarität des Faltungsoperators ist an die Übereinstimmung der Signalsegmente an den jeweiligen Endpunkten angepaßt. Die Verschiebungsinvarianz beruht auf einer adaptiven relativen Verschiebung der Entwicklung für die verschiedenen Auflösungsgrade. Wir zeigen, daß es für jeden Auflösungsgrad ℓ genügt, genau eine von zwei relativen Verschiebungsoptionen auszuwählen: Verschiebung um Null oder um $2^{-\ell-1}$. Zusätzliche Verbesserungen erhält man durch einen variablen Faltungsoperator, dessen Polarität örtlich an die Gleichheitseigenschaften des Signals adaptiert wird. Der Rechenaufwand ist leicht zu handhaben und entspricht etwa dem der LCD. © 1997 Elsevier Science B.V.

Résumé

Dans cet article est définie une librairie extensive des bases trigonométriques adoucies, et un algorithme approprié de recherche de "meilleure base" rapide est introduit. Quand on le compare à la décomposition locale en cosinus (DLC) classique, l'algorithme proposé présente 3 avantages. Tout d'abord, il conduit à une expansion selon la meilleure base qui est invariante en translation. De plus, la représentation résultante est characterisée par un coût d'information inférieur.

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Enfin, la polarité de l'opérateur de repliement est adaptée aux propriétés de parité du signal segmenté à ses extrémités. L'invariance en translation provient d'un changement relatif adaptatif des expansions à des niveaux de résolution distincts. Nous montrons qu'à chaque niveau de résolution ℓ il suffit d'examiner et de sélectionner l'une des deux options de déplacement relatifs, un déplacement de 0 ou un déplacement de $2^{-\ell-1}$. Un opérateur de repliement variable, dont la polarité est adaptée localement aux propriétés de parité du signal, rehausse encore la représentation. La complexité informatique reste acceptable et est comparable à celle de la DLC.

Keywords: Shift-invariant; Best-basis; Time frequency; Lapped transform; Algorithm

1. Introduction

Adaptive representations in libraries of bases have been widely used in recent years. Instead of representing a prescribed signal in a fixed basis, it is useful to choose a suitable basis that facilitates a desired application, such as compression [45, 59, 31, 1, 56], identification and classification [8, 52, 53] or noise removal (denoising) [25, 21, 15, 33]. In general, for a given signal we are looking for a basis which matches well the signal, in the sense that only relatively few coefficients in the expansion are dominant, while the remaining coefficients are small and their total contribution is negligible. How specifically suitable a basis is, depends on the problem at hand. For signal compression, as an example, a preferable basis accelerates the descending rate of the coefficients' amplitudes, when sorted in a decreasing magnitude order, whereas for classification, we select a basis which most discriminates between given classes. Such a basis reduces the dimensionality of the problem and emphasizes the dissimilarity between distinct classes [51].

Practical 'best basis' search procedures are necessarily confined to finite-size libraries. Such libraries are not only required to be flexible and versatile enough to describe various local features of signals, but also need to be aptly organized in a structure that facilitates a fast search algorithm for the 'best basis'. Coifman and Meyer [16, 20, 45] were the first to introduce libraries of orthonormal bases whose elements are localized in time-frequency plane and structured into a binary tree where the best basis can be efficiently searched for. One of the libraries, a library of local trigonometric bases, consists of sines or cosines multiplied by smooth window functions. Their localization properties depend on the steepness of the ascending and descending parts of the window functions [26]. Another library, a library of wavelet packet bases, comprises basis functions which are translations and dilations of wavelet packets, and their localization properties in time-frequency plane depend on those of the 'mother wavelet' [18, 29]. Both libraries are naturally organized in binary trees whose nodes represent subspaces that are orthogonally split into children nodes [17]. Accordingly, the basis functions of a parent node can be replaced by the collection of basis functions that correspond to the children nodes. This flexibility in choosing a basis for each subspace implies adaptive representations, by a recursive comparison between the information costs of parent nodes and their children nodes.

Selecting a desirable information cost functional is clearly application dependent [51, 55, 59]. Entropy, for example, may be used to measure effectively the energy concentration of the generated nodes [21, 32, 58]. Statistical analysis of the best-basis coefficients may provide a characteristic time-frequency signature of the signal, potentially useful in simplifying identification and classification applications [8, 35]. A serious deficiency of this approach is the lack of shift invariance. Both the wavelet packet decomposition (WPD) and local cosine decomposition (LCD) of Coifman and Wickerhauser [19], as well as the extended algorithms, proposed by Herley et al. [27, 28], are sensitive to the signal location with respect to the chosen time origin.

Shift-invariant multiresolution representations exist. However, some methods either entail high oversampling rates (e.g. in [50, 6, 7, 34, 49] no down-sampling with the changing scale is allowed) or immense computational complexity (e.g., the *matching pursuit* algorithm [40, 22]). In some other methods, the resulting representations are non-unique and involve approximate signal reconstructions, as is the case for zero-crossing or local maxima methods [37, 30, 38, 39, 5]. Another approach has given up obtaining shift-invariance and settled for a less restrictive property named *shiftability* [54, 3], which is accomplished by imposing limiting conditions on the scaling function [57, 3, 4].

Recently, several authors proposed independently to extend the library of bases, in which the best representations are searched for, by introducing additional degrees of freedom that adjust the time localization of the basis functions [47, 11, 12, 23, 36]. It was proved that the proposed modifications of the wavelet transform and wavelet packet decomposition lead to orthonormal best-basis representations which are shift-invariant and characterized by lower information costs. The principal idea is to adapt the down-sampling when expanding each parent node. That is, following the low-pass and high-pass filtering, retain either all the odd samples or all the even samples, according to the choice which minimizes the cost function.

In this work, which is summarized in [13], we present an analogous generalized procedure for the library of smooth local trigonometric bases. The strategy in obtaining shift-invariance is likewise based on extending the library to include all their shifted versions, organizing it in a tree structure and providing an efficient 'best-basis' search algorithm. An additional degree of freedom that incorporates an adaptive folding operator into the best decomposition tree, further enhances the resultant representation [14].

The shift invariance stems from a relative shift between expansions in distinct resolution levels. It is shown that at any resolution level ℓ it suffices to examine and select one of two relative shift options – a zero shift or a $2^{-\ell-1}$ shift. The choice between these two options, enabled by the extended library, is made in accordance with minimizing the information cost. Hence, the attained representation is not only shift-invariant, but also characterized by a lower information cost when compared to the LCD. Its quality is further enhanced by applying an adaptive-polarity folding operator which splits the prescribed signal and 'folds' *adaptively* overlapping parts back into the segments. The polarity of the folding operation is locally adapted to the signal at the finest resolution level, and a recursive process is carried out towards the coarsest resolution level merging segments where beneficial. Each segment of the signal is then represented by a trigonometric basis which possesses the same parity properties at the end-points.

The computational complexity of the proposed algorithm, namely executing *shift-invariant adapted-polarity local trigonometric decomposition* (SIAP-LTD), is $O[N(L+2^{\log_2 N-L+1})\log_2 N]$, where N denotes the length of the signal and $L+1$ is the number of resolution levels ($L \leqslant \log_2 N$). This complexity is comparable to that of the LCD ($O(NL\log_2 N)$) [19] with the benefits of shift invariance and a higher quality (lower 'information cost') 'best basis'. To demonstrate the shift-invariant properties of SIAP-LTD, compared to LCD which lacks this feature, we refer to the expansions of the signals $g(t)$ and $g(t-5\cdot 2^{-7})$ (Fig. 1). These signals contain

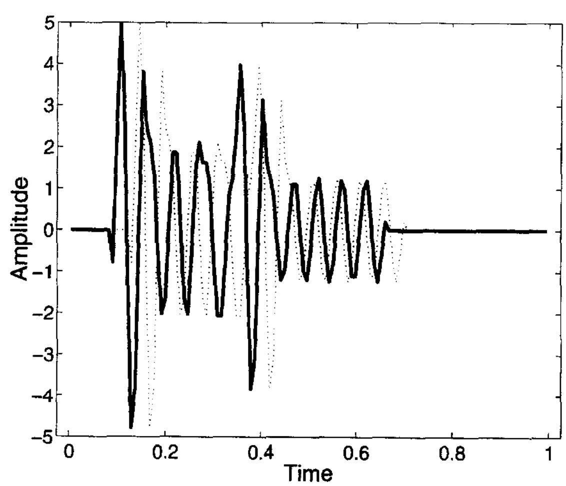

Fig. 1. The signals $g(t)$ (solid) and $g(t-5\cdot 2^{-7})$ (dotted), sampled at 2^7 equally spaced points.

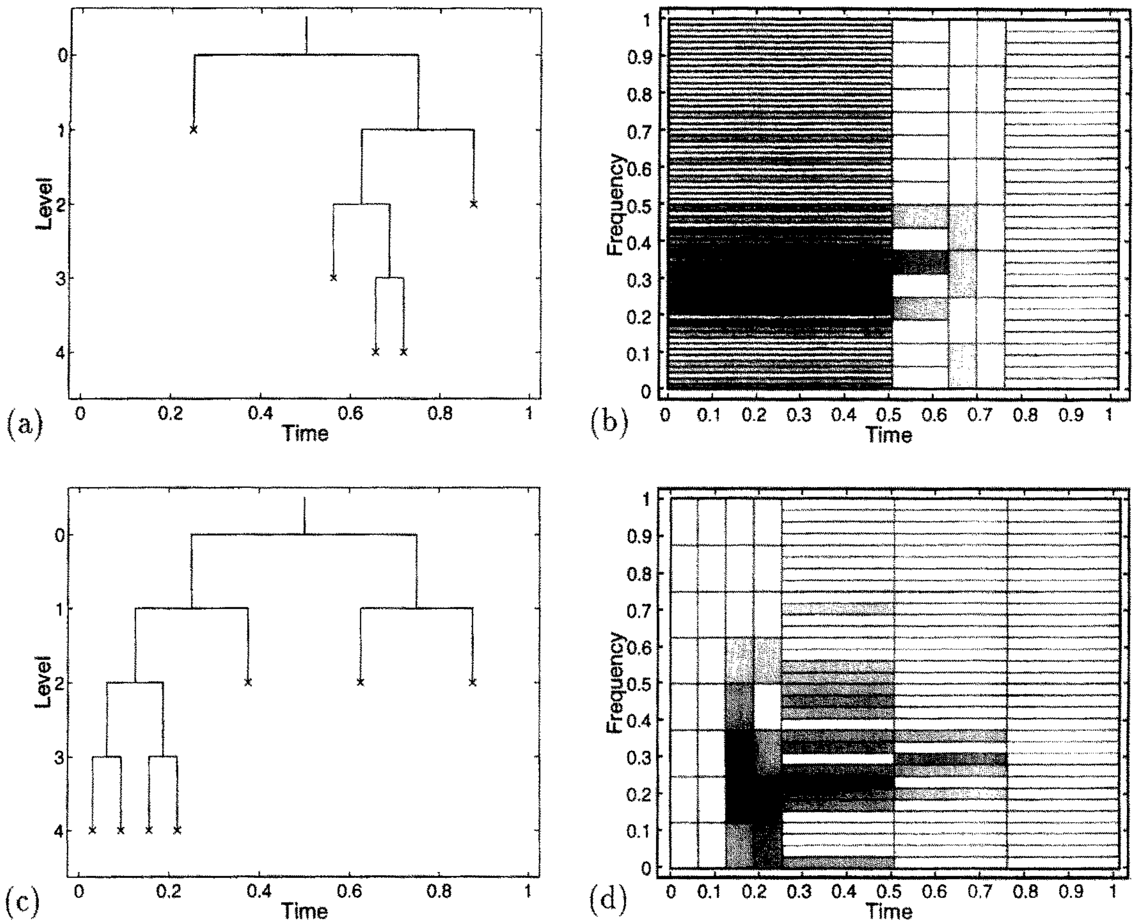

Fig. 2. Local cosine decomposition (LCD): (a) the best expansion tree of $g(t)$; (b) the time-frequency representation of $g(t)$ in its best basis, entropy = 2.57; (c) the best expansion tree of $g(t-5\cdot 2^{-7})$; (d) the time-frequency representation of $g(t-5\cdot 2^{-7})$ in its best basis, entropy = 2.39.

$2^7=128$ samples. For definiteness, we choose the Shannon entropy as the cost function, defined by [19] $\mathscr{M}(\{x_i\})=-\sum_i \|x_i\|^2 \ln \|x_i\|^2$. Figs. 2 and 3 depict the 'best-basis' expansions under the LCD and the SIAP-LTD algorithms, respectively. A comparison of Figs. 2(b) and 2(d) readily reveals the sensitivity of LCD to temporal shifts while the 'best-basis' SIAP-LTD representation is indeed shift-invariant and characterized by a lower entropy (Fig. 3).

The structure of this paper is as follows. In Section 2, we define 'identity to within a resolution time shift' for functions and bases and elucidate the concept of 'shift-invariance up to a resolution level' for a best-basis decomposition. In Section 3, we construct a library of smooth localized trigonometric bases, which includes all translations of bases within the library. The library is formed into resolution levels in a tree structure, and a folding operator, which facilitates an efficient transform, is introduced. In Section 4, we describe the best-basis search algorithm and suboptimal versions which are based on *locally* adapting the polarity or holding it in specific values at the finest resolution level. It is proved that the resultant best-basis decomposition is indeed shift-invariant. The paper is concluded in Section 5.

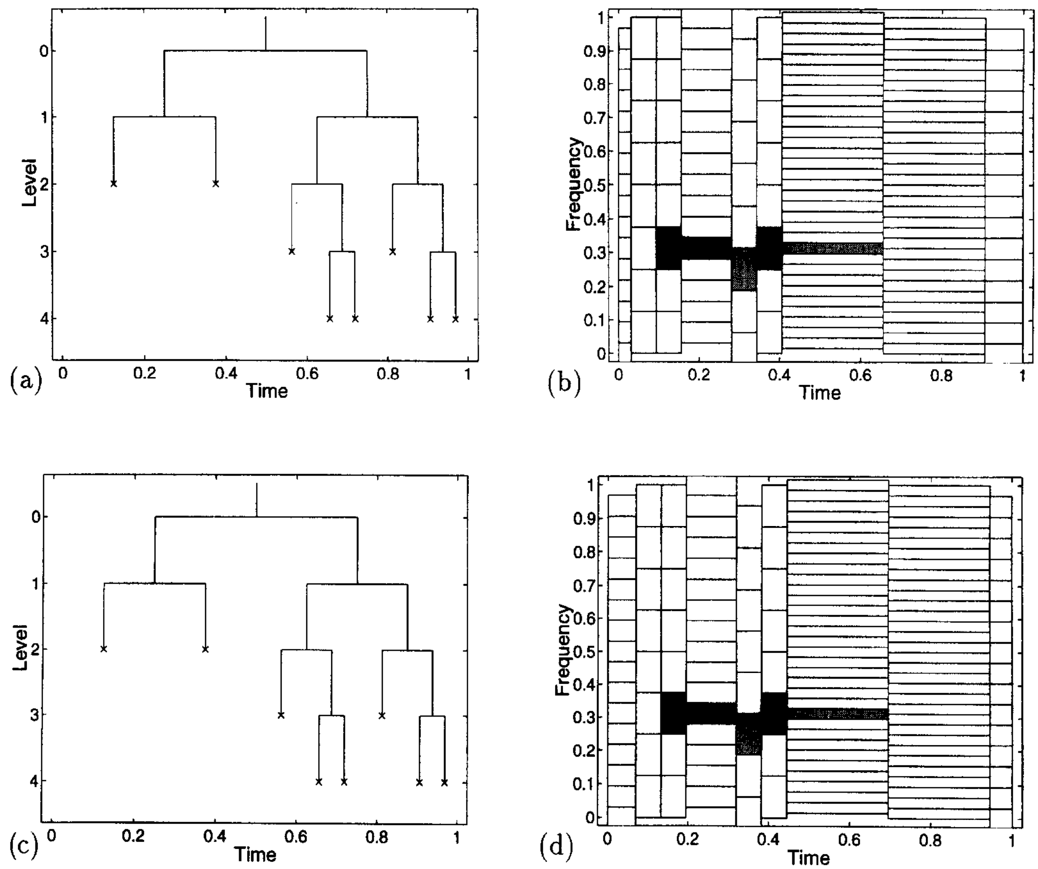

Fig. 3. Shift-invariant adapted-polarity local trigonometric decomposition (SIAP-LTD): (a) the best expansion tree of $g(t)$; (b) the time-frequency representation of $g(t)$ in its best basis, entropy = 1.44; (c) the best expansion tree of $g(t-5\cdot 2^{-7})$; (d) the time-frequency representation of $g(t-5\cdot 2^{-7})$ in its best basis, entropy = 1.44.

2. Shift-invariant best-basis expansions

Best-basis expansions are shift-invariant if for any pair of signals, which are identical to within a time shift, their respective time-frequency representations are identical to within the *same* time shift. One way to achieve shift-invariance is to adjust the time localization of the basis functions, which are chosen for representing an analyzed signal [47,11,36]. That is, when the signal is translated in time by τ, a new best-basis is selected whose elements are also translated by τ compared to the former best basis. Consequently, the expansion coefficients, that are now associated with translated basis functions, stay unchanged and the time-frequency representation is shifted in time by τ. In practice, we may consider discrete-time translations of the form $\tau = 2^{-J}q$, where q takes integer values and J is as large as necessary. For simplicity, we shall restrict ourselves to periodic functions with period 1, and use the ordinary inner product of $L_2[0,1]$. We designate this Hilbert space as $\hat{L}_2[0,1]$ to indicate the periodization. i.e.,

$$g(t+n) = g(t), \quad n \in \mathbb{Z}, \qquad \langle g,g\rangle = \int_0^1 |g(t)|^2\,\mathrm{d}t < \infty$$

for all $g \in \hat{L}_2[0,1]$.

Definition 1. $f, g \in \hat{L}_2[0,1]$ are said to be identical to within a resolution J time-shift ($J > 0$) if there exists $q \in \mathbb{Z}$, $0 \leqslant q < 2^J$, such that $g(t) = f(t - 2^{-J}q)$ for all $t \in [0,1]$.

Let $\mathscr{B}$ denote a library of orthonormal bases in $\hat{L}_2[0,1]$, $\mathscr{M}$ an additive information cost functional and $\mathscr{M}(Bg)$ the information cost of representing $g \in \hat{L}_2[0,1]$ on a basis $B \in \mathscr{B}$. The best basis for g, relative to a library of bases $\mathscr{B}$ and an information cost functional $\mathscr{M}$, is that $B \in \mathscr{B}$ for which $\mathscr{M}(Bg)$ is minimal [19].

Definition 2. Bases B_1, $B_2 \in \mathscr{B}$ are said to be identical to within a resolution J time-shift ($J > 0$) if there exists $q \in \mathbb{Z}$, $0 \leqslant q < 2^J$, such that $\psi(t - 2^{-J}q) \in B_2$ if and only if $\psi(t) \in B_1$.

Definition 3. A best-basis decomposition is said to be shift-invariant up to a resolution level J ($J > 0$) if for any $f, g \in \hat{L}_2[0,1]$ which are identical to within a resolution J time shift, their respective best bases B_f and B_g are identical to within the same time shift.

It is evident that a best-basis decomposition, which is shift-invariant up to a resolution level J, is also shift-invariant up to a lower resolution level, because the translation is on a finer grid. In case of uniformly sampled discrete functions of length $N = 2^J$, an invariance to discrete translation is equivalent to shift invariance up to a resolution level J.

3. Smooth local trigonometric bases

In this section we construct a library of orthonormal bases of $\hat{L}_2[0,1]$ which consist of sines or cosines multiplied by smooth compactly supported functions. The basis functions are localized in time-frequency plane owing to the finite support and smoothness of the window functions.

Let $r = r(t)$ be a function in the class $C^s(\mathbb{R})$ for some $s \geqslant 0$ (class of s-times continuously differentiable functions), satisfying the following conditions:

$$|r(t)|^2 + |r(-t)|^2 = 1 \quad \text{for all } t \in \mathbb{R}, \tag{1}$$

$$r(t) = \begin{cases} 0 & \text{if } t \leqslant -1, \\ 1 & \text{if } t > 1. \end{cases} \tag{2}$$

Then $r((t-\alpha)/\varepsilon)r((\beta - t)/\varepsilon)$ is a window function supported on the interval $[\alpha - \varepsilon, \beta + \varepsilon]$. The function r is called a *rising cutoff function* [59], since it rises from being identically zero to being identically one. The role of $\varepsilon > 0$ is to allow overlap of windows, and thus control the smoothness of the window function [42–44]. An example of a continuously differentiable real-valued rising cutoff function $r_1 \in C^1$ is given by

$$r_1(t) = \begin{cases} 0 & \text{if } t \leqslant -1, \\ \sin\left[\frac{\pi}{4}\left(1 + \sin\frac{\pi}{2}t\right)\right] & \text{if } -1 < t < 1, \\ 1 & \text{if } t \geqslant 1, \end{cases} \tag{3}$$

and depicted in Fig. 4, along with a corresponding window function on $[\alpha, \beta]$ for $\varepsilon < (\beta - \alpha)/2$. By modulating a window function we get a smooth local trigonometric function that is supported on the same interval. Let

$$C_{I,k}^{0,0}(t) = \sqrt{\frac{2}{\beta - \alpha}} \cos\left[\frac{\pi}{\beta - \alpha}\left(k + \frac{1}{2}\right)(t - \alpha)\right], \tag{4a}$$

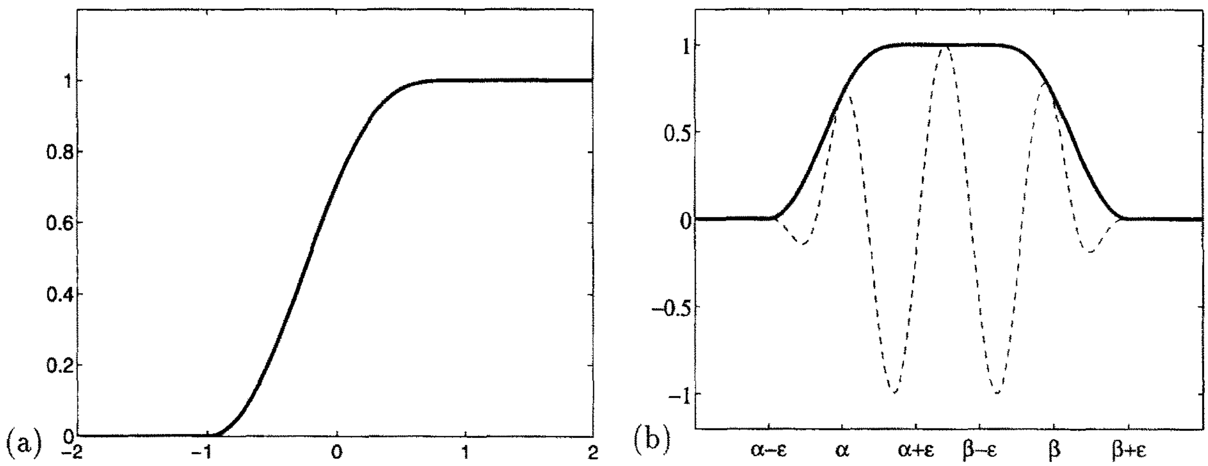

Fig. 4. (a) An example of a rising cutoff function in C^1: (b) The corresponding window function on $[\alpha, \beta]$ for $\varepsilon < (\beta - \alpha)/2$ (solid), and a modulated function (dashed).

$$C_{I,k}^{0,1}(t) = \sqrt{\frac{2}{\beta - \alpha}}\, h_k \cos\left[\frac{\pi}{\beta - \alpha} k(t - \alpha)\right], \tag{4b}$$

$$C_{I,k}^{1,0}(t) = \sqrt{\frac{2}{\beta - \alpha}}\, h_{k+1} \sin\left[\frac{\pi}{\beta - \alpha} (k+1)(t - \alpha)\right], \tag{4c}$$

$$C_{I,k}^{1,1}(t) = \sqrt{\frac{2}{\beta - \alpha}} \sin\left[\frac{\pi}{\beta - \alpha} \left(k + \frac{1}{2}\right)(t - \alpha)\right], \tag{4d}$$

denote sets of modulating trigonometric functions on the interval $I = [\alpha, \beta]$, where

$$h_k = \begin{cases} 1/\sqrt{2} & k = 0, \\ 1 & k \neq 0, \end{cases}$$

are weight factors needed to insure orthonormality [2]. The smooth local trigonometric functions on I are defined by

$$\phi_{I,k}^{\rho_0,\rho_1}(t) = \bar{r}\left(\frac{t - \alpha}{\varepsilon}\right) r\left(\frac{\beta - t}{\varepsilon}\right) C_{I,k}^{\rho_0,\rho_1}(t), \quad k \in \mathbb{Z}_+, \quad \rho_0, \rho_1 \in \{0, 1\}, \tag{5}$$

where $\bar{r}$ denotes the complex conjugate of r. The parities of the function $C_{I,k}^{\rho_0,\rho_1}$ at the end-points α and β are specified by ρ_0 and ρ_1, respectively, according to

$$C_{I,k}^{\rho_0,\rho_1}(\alpha + t) = (-1)^{\rho_0} C_{I,k}^{\rho_0,\rho_1}(\alpha - t), \tag{6a}$$

$$C_{I,k}^{\rho_0,\rho_1}(\beta + t) = -(-1)^{\rho_1} C_{I,k}^{\rho_0,\rho_1}(\beta - t), \tag{6b}$$

$\rho_0, \rho_1 \in \{0, 1\}$. That is, even parity at the left end-point is specified by $\rho_0 = 0$ (respectively, odd parity by $\rho_0 = 1$), whereas even parity at the right end-point is specified by $\rho_1 = 1$ (respectively, odd parity by $\rho_1 = 0$).

Let $\mathscr{I} = \{I_{\ell,n,m}\}$ be a set of intervals of the form

$$I_{\ell,n,m} = [2^{-\ell} n + 2^{-J} m, 2^{-\ell}(n + 1) + 2^{-J} m), \tag{7}$$

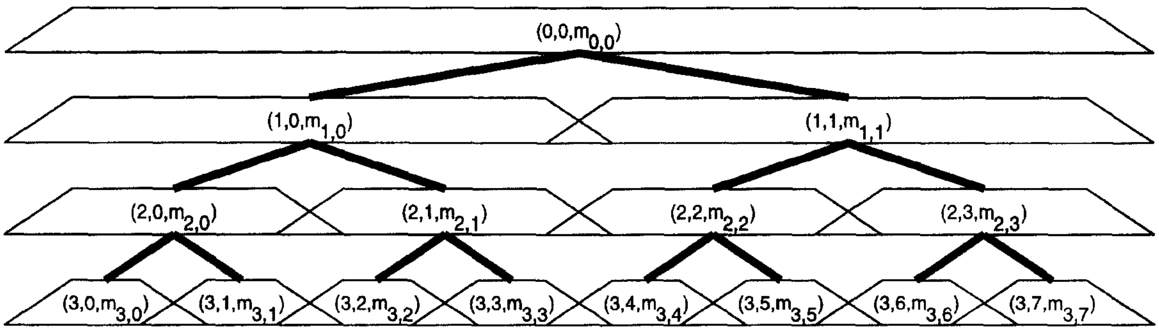

Fig. 5. The smooth local trigonometric bases organized in a binary tree structure. Each node in the tree is indexed by the triplet (ℓ,n,m) and represents a subset of the basis functions.

$0\leqslant\ell\leqslant L\leqslant J$, $0\leqslant n<2^{\ell}$, $0\leqslant m<2^{J-\ell}$, and consider the set of functions defined by

$$\left\{\psi^{\rho_0,\rho_1}_{\ell,n,m,k}(t)\equiv\sum_{q\in\mathbb{Z}}\phi^{\rho_0,\rho_1}_{I_{\ell,n,m},k}(t+q):I_{\ell,n,m}\in\mathscr{I},\ k\in\mathbb{Z},\ \rho_0,\rho_1\in\{0,1\}\right\}. \tag{8}$$

We call ℓ the resolution-level index, n the position index, m the shift index, k the frequency index ($k\in\mathbb{Z}_+$) and $\rho_0,\rho_1\in\{0,1\}$ the polarity indices. As proven in the sequel, the set of functions defined in (8) is a redundant set that spans $\hat{L}_2[0,1]$. Our objective is to construct out of it a library of orthonormal bases, from which the best basis can be efficiently searched for a given signal. The library of bases must include any basis which is a shifted version of another basis within the library. That is, if two bases are identical to within a resolution J time shift, then both either belong or do not belong to the library. This requirement is essential for the shift invariance of the best-basis decomposition.

Pursuing the stated objective, we organize the set of the basis functions in a tree structure that facilitates efficient constructions of orthonormal bases and a fast search for the best basis. The tree configuration is depicted in Fig. 5. Each node in the tree is indexed by the triplet (ℓ,n,m), and represents a subspace with different time-frequency localization characteristics:

$$B^{\rho_0,\rho_1}_{\ell,n,m}=\{\psi^{\rho_0,\rho_1}_{\ell,n,m,k}:k\in\mathbb{Z}_+\}, \tag{9}$$

$$V^{\rho_0,\rho_1}_{\ell,n,m}=\operatorname{clos}_{\hat{L}_2(\mathbb{R})}\{B^{\rho_0,\rho_1}_{\ell,n,m}\}. \tag{10}$$

Lemma 1. *The set $B^{\rho_0,\rho_1}_{\ell,n,m}$ is an orthonormal basis of the subspace $V^{\rho_0,\rho_1}_{\ell,n,m}$.*

The proof is detailed in Appendix A. In some places, to simplify notation where there is no possibility of confusion, we replace the set of indices (ℓ,n,m) by their related interval $I=I_{\ell,n,m}$.

The transform of a given function to an orthonormal basis involves computations of inner products with the basis functions. Here, an efficient computation is attainable by introducing a folding operator F [59] and a periodic folding operator $Q:\hat{L}_2[0,1]\to\hat{L}_2[0,1]$ defined, respectively, by

$$F(\alpha,\rho)g(t)=\begin{cases} r\left(\dfrac{t-\alpha}{\varepsilon}\right)g(t)+(-1)^{\rho}r\left(\dfrac{\alpha-t}{\varepsilon}\right)g(2\alpha-t) & \text{if } \alpha<t<\alpha+\varepsilon,\\ \bar{r}\left(\dfrac{\alpha-t}{\varepsilon}\right)g(t)-(-1)^{\rho}\bar{r}\left(\dfrac{t-\alpha}{\varepsilon}\right)g(2\alpha-t) & \text{if } \alpha-\varepsilon<t<\alpha,\\ g(t) & \text{otherwise,}\end{cases} \tag{11}$$

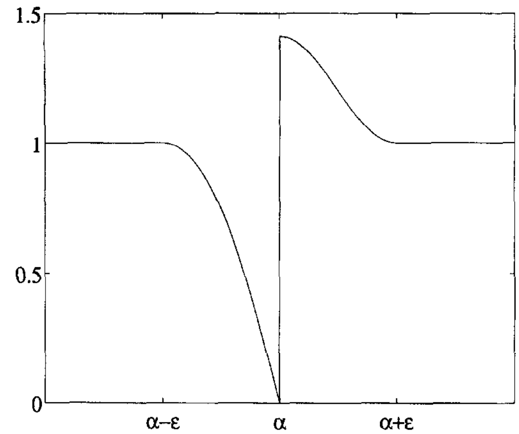

Fig. 6. Action of $F(\alpha, 0)$ on the constant function $g(t) = 1$.

and

$$Q(\alpha,\rho) = \prod_{q\in\mathbb{Z}} F(\alpha + q, \rho). \tag{12}$$

The adjoint of F is given by

$$F^*(\alpha,\rho)g(t) = \begin{cases} \bar{r}\left(\frac{t-\alpha}{\varepsilon}\right) g(t) - (-1)^{\rho} r\left(\frac{\alpha - t}{\varepsilon}\right) g(2\alpha - t) & \text{if } \alpha < t < \alpha + \varepsilon, \\ r\left(\frac{\alpha - t}{\varepsilon}\right) g(t) + (-1)^{\rho} \bar{r}\left(\frac{t-\alpha}{\varepsilon}\right) g(2\alpha - t) & \text{if } \alpha - \varepsilon < t < \alpha, \\ g(t) & \text{otherwise.} \end{cases} \tag{13}$$

Due to identity (1), F is unitary and Q is a unitary isomorphism of $\hat{L}_2[0,1]$. In particular, the unfolding operators F^* and Q^* are the inverses of F and Q, respectively. The *action region* of the folding operator $F(\alpha,\rho)$ is $(\alpha-\varepsilon, \alpha+\varepsilon)$, since outside this region it acts like the identity. The polarity of $F(\alpha,\rho)$ around $t=\alpha$ is odd–even for $\rho=0$ and even–odd for $\rho=1$. That is, if g is smooth, then folding it at α with polarity $\rho=0$, for example, makes the left part, specifically[1] $\mathbf{1}_{(-\infty,\alpha]}F(\alpha,0)g$, a function that is smooth when extended odd to the right, and makes the right part $(\mathbf{1}_{[\alpha,\infty)}F(\alpha,0)g)$ a function that is smooth when extended even to the left. Fig. 6 shows the result of the action of $F(\alpha,0)$ on the constant function $g(t)=1$, using the cutoff function defined by (3).

A pair of unfolding opertors $F^*(\alpha,\rho_0)$ and $F^*(\beta,\rho_1)$ commute whenever $\varepsilon < (\beta-\alpha)/2$. In this case, the actions of $F^*(\alpha,\rho_0)$ and $F^*(\beta,\rho_1)$ on a function g that is supported on an interval $I=[\alpha,\beta]$, simplify to multiplications by $\bar{r}((t-\alpha)/\varepsilon)$ and $r((\beta-t)/\varepsilon)$, respectively:

$$F^*(\alpha,\rho_0)F^*(\beta,\rho_1)g(t) = \bar{r}\left(\frac{t-\alpha}{\varepsilon}\right) r\left(\frac{\beta-t}{\varepsilon}\right) \tilde{g}(t), \tag{14}$$

[1] $\mathbf{1}_I$ denotes the indicator function for the interval I, i.e. the function that is 1 in I and 0 elsewhere.

where $\tilde{g}$ is the extension of g to the outside region of I, with the appropriate parities at the end-points:

$$\tilde{g}=\begin{cases}(-1)^{\rho_0}g(2\alpha-t) & \text{if } t<\alpha,\\ g(t) & \text{if } t\in[\alpha,\beta],\\ -(-1)^{\rho_1}g(2\beta-t) & \text{if } t>\beta.\end{cases} \tag{15}$$

Denote by χ_I the periodic extension of the indicator function for the interval I, i.e., $\chi_{[\alpha,\beta]}\equiv\mathbf{1}_{\{[\alpha+q,\beta+q]:q\in\mathbb{Z}\}}$. Accordingly, the local trigonometric functions defined in Eq. (5), and the basis functions defined in Eq. (8) satisfy

$$\phi_{I,k}^{\rho_0,\rho_1}(t)=F^*(\alpha,\rho_0)F^*(\beta,\rho_1)\mathbf{1}_I C_{I,k}^{\rho_0,\rho_1}(t), \tag{16}$$

$$\psi_{I,k}^{\rho_0,\rho_1}(t)=Q^*(\alpha,\rho_0)Q^*(\beta,\rho_1)\chi_I C_{I,k}^{\rho_0,\rho_1}(t); \tag{17}$$

whence

$$\langle\psi_{I,k}^{\rho_0,\rho_1},g\rangle=\langle\chi_I C_{I,k}^{\rho_0,\rho_1},Q(\alpha,\rho_0)Q(\beta,\rho_1)g\rangle=\langle C_{I,k}^{\rho_0,\rho_1},\mathbf{1}_I F(\alpha,\rho_0)F(\beta,\rho_1)g\rangle. \tag{18}$$

Consequently, we can compute the inner product of a given signal g with a basis function in two conventional stages: At the first stage the signal is preprocessed by folding it. Then, each segment is transformed by a trigonometric basis which has the appropriate parity properties at the end-points. In the discrete case, the trigonometric transform is DCT-II for even–even parity, DCT-IV for even–odd parity, DST-II for odd–odd parity and DST-IV for odd–even parity; all having fast implementation algorithms [9, 41, 48].

The window of the local trigonometric function defined in (5) has an ascending part which is supported on $[\alpha-\varepsilon,\alpha+\varepsilon]$ and a descending part which is supported on $[\beta-\varepsilon,\beta+\varepsilon]$. If the ascending parts of adjacent windows are disjoint, as well as their descending parts, then their associated intervals are called *compatible* [2]. That is, intervals $I'=[\alpha,\beta)$ and $I''=[\beta,\gamma)$ are called compatible if $\alpha+\varepsilon<\beta-\varepsilon<\beta+\varepsilon<\gamma-\varepsilon$.

Lemma 2. If I' and I'' are adjacent compatible intervals and the corresponding subspaces have the same connecting polarity index, then an orthogonal sum of the subspaces corresponds to the union of the intervals, *i.e.*, $V_{I'}^{\rho_0,\rho_1}\oplus V_{I''}^{\rho_1,\rho_2}=V_{I'\cup I''}^{\rho_0,\rho_2}$.

Proof. Let $I'=[\alpha_0,\alpha_1)$ and $I''=[\alpha_1,\alpha_2)$ be adjacent compatible intervals, and let $Q_j=Q(\alpha_j,\rho_j)$, $j=0,1,2$, be the periodic folding operators at the end-points. It is shown in Appendix B that the operators $P_{I'}=Q_0^*Q_1^*\chi_{I'}Q_1Q_0$, $P_{I''}=Q_1^*Q_2^*\chi_{I''}Q_2Q_1$ and $P_{I'\cup I''}=Q_0^*Q_2^*\chi_{I'\cup I''}Q_2Q_0$ are orthogonal projections onto $V_{I'}^{\rho_0,\rho_1}$, $V_{I''}^{\rho_1,\rho_2}$ and $V_{I'\cup I''}^{\rho_0,\rho_2}$ respectively. The same stages of the proof of Lemma 4.3 in [59, p. 113], outlined as well in Appendix B for convenience, yield that $P_{I'}+P_{I''}=P_{I'\cup I''}$ and $P_{I'}P_{I''}=P_{I''}P_{I'}=0$, which completes the proof. □

This lemma implies that a basis on the interval $I'\cup I''$ can be switched with a union of bases on I' and I''. Accordingly, an orthonormal basis of $\hat{L}_2[0,1]$ is constructed by taking the collection of basis functions that correspond to a disjoint compatible cover of $[0,1)$, or any other interval of a unit length. The indexing scheme of such a compatible cover is depicted in Fig. 7, and refers to the following proposition.

Proposition 1. *Let $\{I_j\}_{j=0}^{M}$ be a compatible partition of a unit-length interval by intervals $I_j=[\alpha_j,\alpha_{j+1})$. Let $\{\rho_j\}_{j=0}^{M+1}$ be a collection of $\rho_j\in\{0,1\}$ where $\rho_{M+1}=\rho_0$. Then $\{B_{I_j}^{\rho_j,\rho_{j+1}}:0\leqslant j\leqslant M\}$ forms an orthonormal basis of $\hat{L}_2[0,1]$.*

Proof. Lemmas 1 and 2 imply that $\{B_{I_j}^{\rho_j,\rho_{j+1}}:0\leqslant j\leqslant M\}$ is an orthonormal basis of $V_{[0,1)}^{\rho_0,\rho_0}$, so we shall show that $V_{[0,1)}^{\rho,\rho}\equiv\hat{L}_2[0,1]$ for $\rho\in\{0,1\}$.

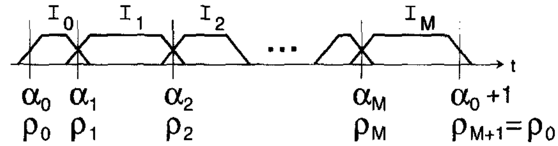

Fig. 7. Indexing scheme of a compatible partition of a unit length interval, employed for generating smooth local trigometric bases.

Clearly, $V^{\rho,\rho}_{[0,1)} \subset \hat{L}_2[0,1]$ since $\{\psi^{\rho,\rho}_{[0,1),k} : k \in \mathbb{Z}_+\} \in \hat{L}_2[0,1]$ is an orthonormal basis of $V^{\rho,\rho}_{[0,1)}$. Now suppose that $g \perp V^{\rho,\rho}_{[0,1)}$, $g \in \hat{L}_2[0,1]$, then

$$\langle \psi^{\rho,\rho}_{[0,1),k}, g\rangle = \langle C^{\rho,\rho}_{[0,1),k}, \mathbf{1}_{[0,1)}F(0,\rho)F(1,\rho)g\rangle = \langle \mathbf{1}_{[0,1)}C^{\rho,\rho}_{[0,1),k}, Q(0,\rho)g\rangle = 0$$

for all $k \in \mathbb{Z}_+$. Hence, g is identically zero, because $\{\mathbf{1}_{[0,1)}C^{\rho,\rho}_{[0,1),k} : k \in \mathbb{Z}_+\}$ is an orthonormal basis of $L_2[0,1]$ [2], and Q is a unitary isomorphism of $\hat{L}_2[0,1]$. □

Recall that the set of basis functions defined in (8) is structured in a tree whose nodes are associated with the intervals $I_{\ell,n,m}$. We can build out of this set a library of orthonormal bases by taking subsets which correspond to a compatible partition of a unit-length interval. The polarity indices of the basis functions are practicably specified by a single integer P $(0 \leqslant P < 2^{2^L})$. Let $p(j)$ denote the polarity index at $t = \alpha_0 + j2^{-L}$, and let $P = [p(2^L - 1), \ldots, p(1), p(0)]_2$ be the binary representation of P. Then the polarity indices of the basis functions on an interval $I = [\alpha, \beta)$, that belong to the disjoint cover of $[\alpha_0, \alpha_0 + 1)$, are given by

$$\rho(\alpha) = p[2^L(\alpha - \alpha_0)], \tag{19a}$$

$$\rho(\beta) = p[2^L(\beta - \alpha_0)]. \tag{19b}$$

Notice that the length of an interval at the resolution level ℓ $(\ell \leqslant L)$ is a multiple of 2^{-L}. Thus, $(\alpha - \alpha_0)$ and $(\beta - \alpha_0)$ are also multiples of 2^{-L}, whenever $[\alpha, \beta)$ belongs to the partition of $[\alpha_0, \alpha_0 + 1)$. Derivable from Proposition 1, we have the following.

Proposition 2. *Let $E = \{(\ell, n, m)\}$ denote a collection of indices $0 \leqslant \ell \leqslant L$, $0 \leqslant n < 2^\ell$ and $0 \leqslant m < 2^{J-\ell}$ satisfying*

(i) *The segments $\{I_{\ell,n,m} : (\ell,n,m) \in E\}$ are a disjoint compatible cover of $[\alpha_0, \alpha_0 + 1)$, for some $0 \leqslant \alpha_0 < 1$.*
(ii) *Nodes (ℓ, n_1, m_1), $(\ell, n_2, m_2) \in E$ at the same resolution level have identical shift index $(m_1 = m_2)$.*
Then for any polarity $0 \leqslant P < 2^{2^L}$ we have an orthonormal basis of $\hat{L}_2[0,1]$, given by

$$\{B^{\rho(\alpha),\rho(\beta)}_{[\alpha,\beta)} : [\alpha,\beta) = I_{\ell,n,m},\ (\ell,n,m) \in E\},$$

and the set of all (E,P) as specified above generates a library of orthonormal bases.

Condition (ii) precludes a relative shift between nodes within the same resolution level. This condition is actually unnecessary for the construction of a library of orthonormal bases. However, such a supplementary constraint limits the size of the library and thus controls the computational complexity of the best-basis search algorithm, while still retaining shift invariance.

4. The best basis selection

Let $\mathscr{B}$ represent the library of orthonormal bases of Proposition 2. Denote by $\mathscr{M}$ an additive information cost functional, and by $\mathscr{M}(Bg)$ the information cost of representing g on a basis B. The best basis for $g \in \hat{L}_2[0,1]$ in $\mathscr{B}$ relative to $\mathscr{M}$ is defined as that $B \in \mathscr{B}$ for which $\mathscr{M}(Bg)$ is minimal [19]. In this section we introduce an efficient search algorithm for the best basis, that relies on the tree structure of $\mathscr{B}$.

Denote by $A_{\ell,n,m}^{p_0,p_1}$ the best basis for g restricted to the subspace $V_{\ell,n,m}^{p_0,p_1}$. Since $B_{0,0,m}^{p,p}$ spans $\hat{L}_2[0,1]$ for any shift index m ($0 \leqslant m < 2^J$) and polarity index $p \in \{0,1\}$ (refer to Proposition 1), the best basis for g is $A_{0,0,m}^{p,p}$ combined with the best shift and polarity indices. These parameters, namely m and p, are determined recursively together with the best basis.

Let $m_0 = m$ and $p_0(0) = p$ designate, respectively, the shift and polarity at the coarsest resolution level ($\ell = 0$). Suppose that at the resolution level ℓ we have found m_ℓ, $\{p_\ell(i) \,|\, 0 \leqslant i < 2^\ell\}$ and $A_{\ell,n,m_\ell}^{p_\ell(n),p_\ell(n+1)}$ for all $0 \leqslant n < 2^\ell$, where we set $p_\ell(2^\ell + i) = p_\ell(i)$ owing to the periodicity of $\hat{L}_2[0,1]$. Then we will choose $m_{\ell-1}$, $\{p_{\ell-1}(i) \,|\, 0 \leqslant i < 2^{\ell-1}\}$ and $A_{\ell-1,n,m_{\ell-1}}^{p_{\ell-1}(n),p_{\ell-1}(n+1)}$ for $0 \leqslant n < 2^{\ell-1}$ so as to minimize the information cost.

It is shown in the sequel that shift invariance is acquired by merely considering two optional values of $m_{\ell-1}$: m_ℓ and $m_\ell + 2^{J-\ell}$. These two options correspond to either no relative shift or $2^{-\ell}$ shift between the resolution levels $\ell-1$ and ℓ. For each of the two choices we find the polarity indices and the best basis for g at the resolution level $\ell - 1$. Then we compare the information costs and select that value of $m_{\ell-1}$ which yields a cheaper representation.

Fig. 8(a) depicts the relation between intervals at the resolution levels $\ell-1$ and ℓ, for the case where $m_{\ell-1} = m_\ell$. The interval $I_{\ell,2n+1,m_\ell}$ can be joined up with its *left* adjoining interval $I_{\ell,2n,m_\ell}$ into a parent interval $I_{\ell-1,n,m_\ell}$. Accordingly, the polarity indices at the resolution level $\ell-1$ are simply $p_{\ell-1}(n) = p_\ell(2n)$, and by employing Lemma 2 we have

$$A_{\ell-1,n,m_\ell}^{p_{\ell-1}(n),p_{\ell-1}(n+1)} = \begin{cases} B_{\ell-1,n,m_\ell}^{p_{\ell-1}(n),p_{\ell-1}(n+1)} & \text{if } \mathscr{M}'_B \leqslant \mathscr{M}'_A, \\ A_{\ell,2n,m_\ell}^{p_\ell(2n),p_\ell(2n+1)} \cup A_{\ell,2n+1,m_\ell}^{p_\ell(2n+1),p_\ell(2n+2)} & \text{otherwise}, \end{cases} \tag{20}$$

where $\mathscr{M}'_A = \mathscr{M}(A_{\ell,2n,m_\ell}^{p_\ell(2n),p_\ell(2n+1)}g) + \mathscr{M}(A_{\ell,2n+1,m_\ell}^{p_\ell(2n+1),p_\ell(2n+2)}g)$ is the information cost of the children and $\mathscr{M}'_B = \mathscr{M}(B_{\ell-1,n,m_\ell}^{p_{\ell-1}(n),p_{\ell-1}(n+1)}g)$ is the information cost of the parent. In this case, the information cost of g when

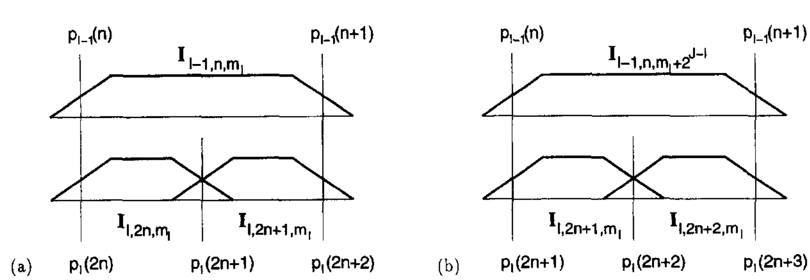

Fig. 8. Joining up adjacent intervals at the resolution level ℓ into a parent interval at a coarser resolution level: (a) the levels have the same shift index; (b) the intervals at the level $\ell-1$ are translated with respect to the intervals at the level ℓ.

expanded at the resolution level $\ell - 1$ is given by

$$\mathscr{M}'_{\ell-1} = \sum_{n=0}^{2^{\ell-1}-1} \mathscr{M}(A^{p_\ell(2n),p_\ell(2n+2)}_{\ell-1,n,m_\ell} g). \tag{21}$$

For the other alternative of $m_{\ell-1}$ ($m_{\ell-1} = m_\ell + 2^{J-\ell}$), the relation between the intervals at the resolution levels $\ell - 1$ and ℓ is depicted in Fig. 8(b). Now, the interval $I_{\ell,2n+1,m_\ell}$ can be joined up with its *right* adjoining interval $I_{\ell,2n+2,m_\ell}$ into a parent interval $I_{\ell-1,n,m_\ell+2^{J-\ell}}$. The polarity indices at the resolution level $\ell - 1$ are given by $p_{\ell-1}(n) = p_\ell(2n+1)$, and consequently

$$A^{p_{\ell-1}(n),p_{\ell-1}(n+1)}_{\ell-1,n,m_\ell+2^{J-\ell}} = \begin{cases} B^{p_{\ell-1}(n),p_{\ell-1}(n+1)}_{\ell-1,n,m_\ell+2^{J-\ell}} & \text{if } \mathscr{M}''_B \leqslant \mathscr{M}''_A, \\ A^{p_\ell(2n+1),p_\ell(2n+2)}_{\ell,2n+1,m_\ell} \cup A^{p_\ell(2n+2),p_\ell(2n+3)}_{\ell,2n+2,m_\ell} & \text{otherwise,} \end{cases} \tag{22}$$

where $\mathscr{M}''_A = \mathscr{M}(A^{p_\ell(2n+1),p_\ell(2n+2)}_{\ell,2n+1,m_\ell} g) + \mathscr{M}(A^{p_\ell(2n+2),p_\ell(2n+3)}_{\ell,2n+2,m_\ell} g)$ is the information cost of the children and $\mathscr{M}''_B = \mathscr{M}(B^{p_{\ell-1}(n),p_{\ell-1}(n+1)}_{\ell-1,n,m_\ell+2^{J-\ell}} g)$ is the information cost of the parent. In this case, the information cost of g when expanded at the resolution level $\ell - 1$ is given by

$$\mathscr{M}''_{\ell-1} = \sum_{n=0}^{2^{\ell-1}-1} \mathscr{M}(A^{p_\ell(2n+1),p_\ell(2n+3)}_{\ell-1,n,m_\ell+2^{J-\ell}} g). \tag{23}$$

The value of $m_{\ell-1}$ is thus determined according to the lower information cost, i.e.,

$$m_{\ell-1} = \begin{cases} m_\ell & \text{if } \mathscr{M}'_{\ell-1} \leqslant \mathscr{M}''_{\ell-1}, \\ m_\ell + 2^{J-\ell} & \text{otherwise.} \end{cases} \tag{24}$$

The corresponding best basis and polarity indices at the resolution level $\ell - 1$ are retained for the next stage of the procedure, which is carried out up to the level $\ell = 0$. The algorithm is initiated at the level $\ell = L$ ($L \leqslant J$), specified by the shortest intervals that are required for segmentation. At this level, we estimate the shift index m_L and polarity indices $\{p_L(n),\ 0 \leqslant n < 2^L\}$, and impose

$$A^{p_L(n),p_L(n+1)}_{L,n,m_L} = B^{p_L(n),p_L(n+1)}_{L,n,m_L}, \quad 0 \leqslant n < 2^L. \tag{25}$$

To simplify notation, the set of polarity indices at the resolution level L is organized into a single integer P_L ($0 \leqslant P_L < 2^{2^L}$), using its binary representation $P_L = [p_L(2^L-1), \ldots, p_L(1), p_L(0)]_2$. The optimal shift and polarity at the finest resolution level are given by

$$(m_L, P_L) = \arg \min_{\substack{0 \leqslant m < 2^{J-L} \\ 0 \leqslant P < 2^{2^L}}} \left\{ \sum_{n=0}^{2^L-1} \mathscr{M}(B^{p(n),p(n+1)}_{L,n,m} g) \right\}. \tag{26}$$

Proposition 3. The best basis expansion stemming from the previously described recursive algorithm is shift-invariant up to a resolution level J.

Proof. Let $f, g \in \hat{L}_2[0,1]$ be identical to within a resolution J time shift. Then there exists an integer $0 \leqslant q < 2^J$ such that $g(t) = f(t - q2^{-J})$. Denote the best bases for f and g by A_f and A_g, respectively. It is shown in Appendix C that $B^{\rho_0,\rho_1}_{[\alpha,\beta)} \subset A_f$ implies $B^{\rho_0,\rho_1}_{[\alpha+q2^{-J},\beta+q2^{-J})} \subset A_g$ for all $[\alpha,\beta) \in \mathscr{I}$ and $\rho_0, \rho_1 \in \{0,1\}$.Consequently, if $\psi(t)$ is a basis function in A_f, then $\psi(t-q2^{-J})$ is a basis function in A_g. Thus, A_f and A_g are identical to within a $q2^{-J}$ time shift. □

In practice, the minimization process of the information cost at the finest resolution level L need not to be performed by a sequential consideration of 2^{2^L} polarity values. Normally, the influence of the polarity indices on the information cost is less significant than the influence of the shift index. Furthermore, an ill-adapted polarity bit (a single polarity index specified at a certain end-point) is possibly eliminated at a coarser level by merging intervals on its both sides. Hence, to maintain a manageable computational complexity, we settle for suboptimal polarity indices which are locally adapted to the signal. Instead of pursuing a global minimum, as advised in (26), we estimate for each $0 \leqslant m < 2^{J-L}$ the locally adapted polarity indices, and choose that $m = m_L$ which leads to the lowest information cost.

For an additive information-cost functional, the orthogonal decomposition (Lemma 2) implies that any polarity bit affects only the costs of its two adjoining segments. In particular, the value of the nth polarity-bit $p_L(n)$, is completely subject to the values of his adjacent polarity bits, namely $p_L(n-1)$ and $p_L(n+1)$. Denote by $\pi_m(n)$ the optimal value of the nth polarity bit for a shift m. On the supposition that $\pi_m(n-1)$ and $\pi_m(n+1)$ correspond to the minimal local information cost about the nth end-point, we have

$$\pi_m(n) = \begin{cases} 0 & \text{if } C_{m,n}(0) \leqslant C_{m,n}(1), \\ 1 & \text{otherwise,} \end{cases} \tag{27}$$

where

$$C_{m,n}(\rho) = \min_{\rho_0,\rho_1 \in \{0,1\}} \left\{ \mathcal{M}(B_{L,n,m}^{\rho_0,\rho} g) + \mathcal{M}(B_{L,n+1,m}^{\rho,\rho_1} g) \right\}, \quad \rho \in \{0,1\}, \tag{28}$$

designates the local information cost about the nth end-point for a shift m. If the assumption is true for all polarity indices and for all shifts, then the optimal shift and polarity at the finest resolution level are given by

$$m_L = \arg \min_{0 \leqslant m < 2^{J-L}} \left\{ \sum_{n=0}^{2^L-1} \mathcal{M}(B_{L,n,m}^{\pi_m(n),\pi_m(n+1)} g) \right\}, \tag{29}$$

$$p_L(n) = \pi_{m_L}(n), \quad 0 \leqslant n < 2^L. \tag{30}$$

Clearly, the optimal shift and polarity, obtainable by (26), minimize the global information cost but not necessarily the local costs about each end-point. Hence, the shift and locally adapted polarity, computed by (29) and (30), are suboptimal and may result in a higher information cost. However, the representation is still shift-invariant due to the consistency in their computation. The following steps summarize the execution of SIAP-LTD:

Step 0. Specify an information cost funcional $\mathcal{M}$ and maximum depth of decomposition L.

Step 1. Use Eq. (18) and the trigonometric transforms DCT-II, DCT-IV, DST-II and DST-IV to expand g into the subsets $B_{L,n,m}^{\rho_0,\rho_1}$ for $0 \leqslant n < 2^L$, $0 \leqslant m < 2^{J-L}$ and $\rho_0, \rho_1 \in \{0,1\}$.

Step 2. Estimate the shift and polarity indices at the finest resolution level using Eqs. (29) and (30), and impose Eq. (25).

Step 3. For $\ell = L, \ldots, 1$:

1. Expand g into the subsets $B_{\ell-1,n,m_\ell}^{p_{\ell-1}(n),p_{\ell-1}(n+1)}$ and $B_{\ell-1,n,m_\ell+2^{J-\ell}}^{p_{\ell-1}(n),p_{\ell-1}(n+1)}$ for $0 \leqslant n < 2^{\ell-1}$.
2. Let $m_{\ell-1} = m_\ell$ and compute the information cost of g at the resolution level $\ell-1$ by Eq. (21).
3. Let $m_{\ell-1} = m_\ell + 2^{J-\ell}$ and compute the information cost of g at the resolution level $\ell-1$ by Eq. (23).
4. Determine the value of $m_{\ell-1}$ according to (24) and keep the corresponding $p_{\ell-1}(n)$ and $A_{\ell-1,n,m_{\ell-1}}^{p_{\ell-1}(n),p_{\ell-1}(n+1)}$ for $0 \leqslant n < 2^{\ell-1}$.

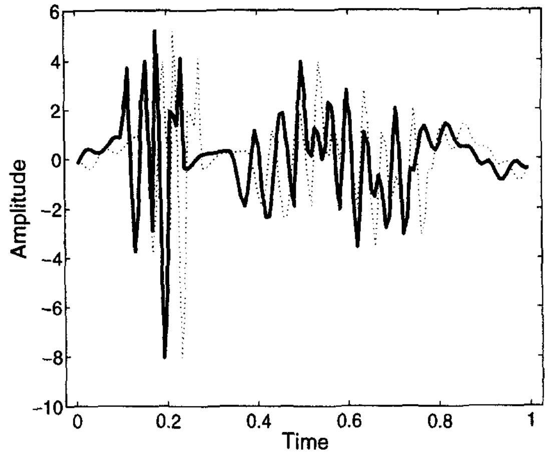

Fig. 9. The signals $f(t)$ (solid) and $f(t-5\cdot 2^{-7})$ (dotted), sampled at 2^7 equally spaced points.

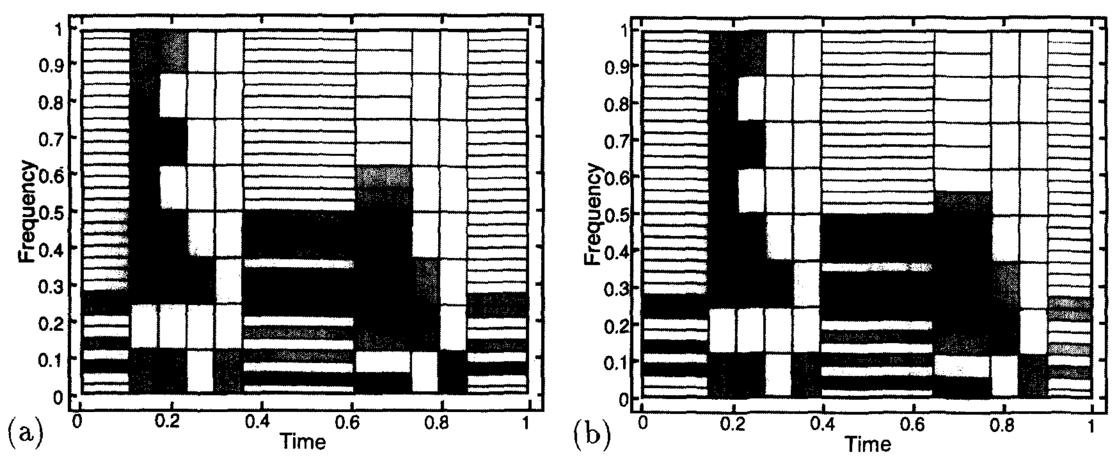

Fig. 10. Shift-invariant local cosine decomposition (SI-LCD): (a) the time-frequency representation of $f(t)$ in its best basis, entropy $= 3.01$; (b) the time-frequency representation of $f(t-5\cdot 2^{-7})$ in its bsest basis. entropy $= 3.01$.

The computational complexity of executing SIAP-LTD is $O[N(L+2^{J-L+1})\log_2 N]$, where N denotes the length of the signal. More specifically, Steps 1 and 2 take, respectively, $O(2^{J-L+1}N\log_2 N)$ and $O(2^{J-L+2}N)$ operations, and Step 3 requires twice as much operations as the conventional LCD [19], i.e., $O(NL\log_2 N)$ operations. The complexity of SIAP-LTD is thus comparable to that of LCD with the benefits of shift invariance and a higher quality (lower 'information cost') 'best basis'.

The LCD may be viewed as a degenerate form of SIAP-LTD characterized by a polarity $P_L = 0$ and shift $m_0 = 0$. In this case, no relative shift between resolution levels is allowed for (m_ℓ is non-adaptively set to zero for all $0\leqslant\ell\leqslant L$), and the resultant representation is shift-variant. The SIAP-LTD provides two degrees of freedom that generate independently shift invariance and adaptive-polarity foldings. While the relative shifts between resolution levels are required to obtain shift invariance, the adaptation of the polarity indices at the finest resolution level is mainly intended to reduce the information cost and thus improve the time-frequency representation. This improvement is notable for signals that have dominant frequencies within each segment, such as the signal which is depicted in Fig. 1, or for signals that possess definite parity properties at the

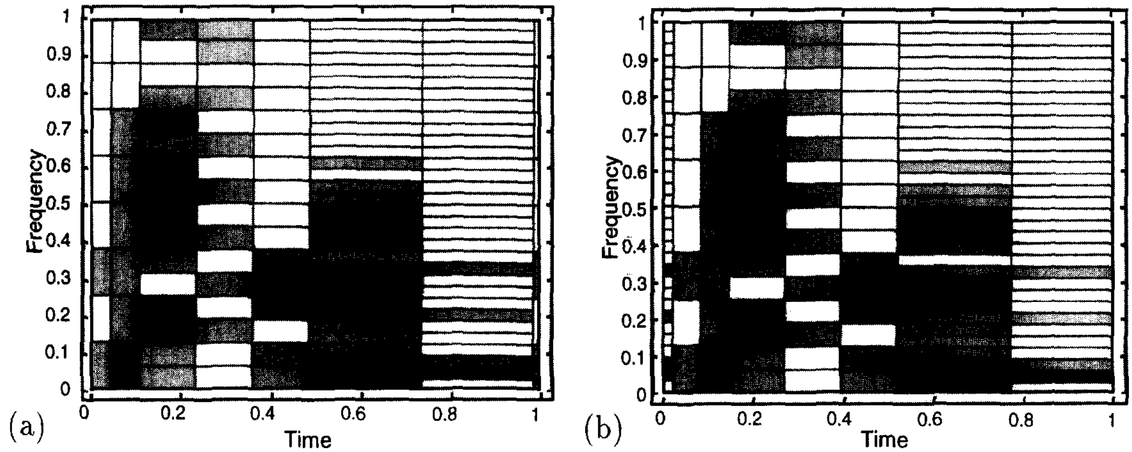

Fig. 11. Shift-invariant local cosine decomposition (SI-LCD): (a) the time-frequency representation of $f(t)$ in its best basis, entropy = 3.07; (b) the time-frequency representation of $f(t-5\cdot 2^{-7})$ in its bsest basis. entropy = 3.07.

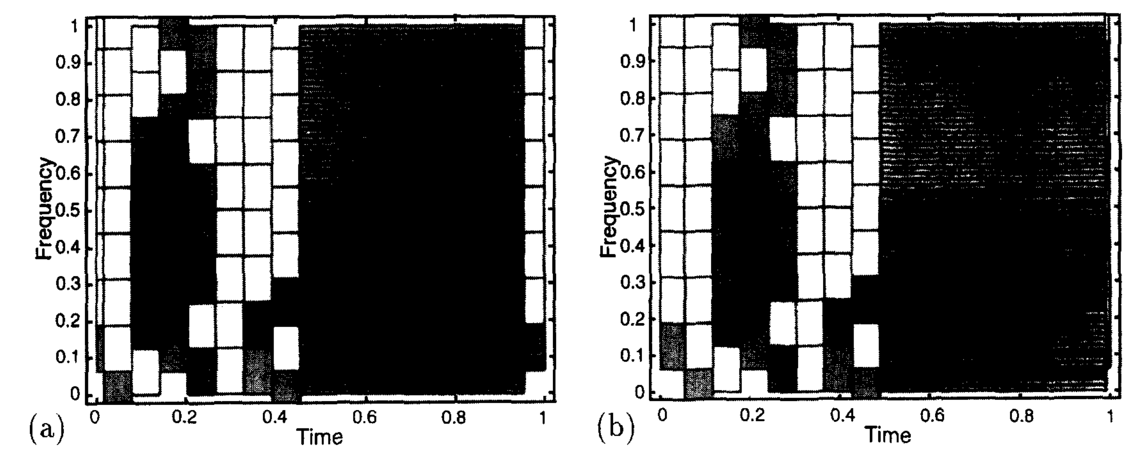

Fig. 12. Shift-invariant adapted-polarity local trigonometric decomposition (SIAP-LTD): (a) the time-frequency representation of $f(t)$ in its best basis, entropy = 2.86; (b) the time-frequency representation of $f(t-5\cdot 2^{-7})$ in its bsest basis, entropy = 2.86.

end-points of the segments. Otherwise, the polarity can be forced to a value whose bits are identical ($P_L = 0$ or $P_L = 2^{2^L} - 1$), without suppressing the shift invariance. By (5), if the polarity bits are restricted to zeros (respectively ones), then the library of bases consists of smooth local cosines (respectively sines). Accordingly, we call the best-basis search algorithms *shift-invariant local cosine decomposition* (SI-LCD) when P_L is forced to zero, and *shift-invariant local sine decomposition* (SI-LSD) when P_L is forced to $2^{2^L} - 1$.

As an example, we refer to the signals $f(t)$ and $f(t-5\cdot 2^{-7})$ depicted in Fig. 9. The time-frequency representations attained by SI-LCD (Fig. 10), SI-LSD (Fig. 11) and SIAP-LTD (Fig. 12) are all shift-invariant and have similar information costs. Whereas that obtained by LCD (Fig. 13) yields variations in the energy spread and leads to a higher shift-dependent information cost.

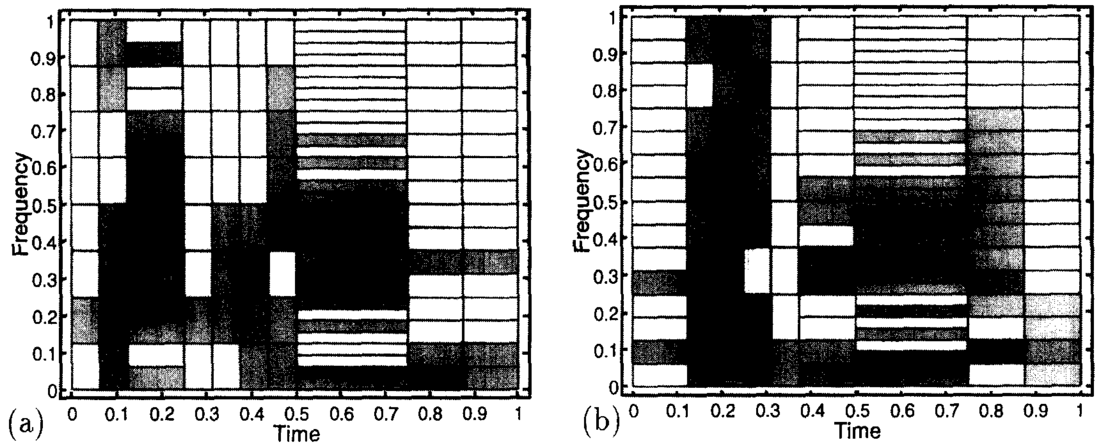

Fig. 13. Local cosine decomposition (LCD): (a) the time-frequency representation of $f(t)$ in its best basis, entropy = 3.12; (b) the time-frequency representation of $f(t - 5 \cdot 2^{-7})$ in its best basis. entropy = 3.27.

5. Conclusion

Shift-invariant 'best-basis' expansions necessitate an extended library of bases that includes all the shifted versions of bases within the library. Due to the enlarged size of the library, it is extremely important to organize the library in a structure that facilitates a fast search for the best basis. If a multiresolution analysis is profitable, then such a structure could be a binary tree whose nodes and levels represent subspaces and resolutions, respectively. The shift invariance is acquired by considering a relative shift between expansions in distinct resolution levels, which is determined in accordance with the minimization process of the information cost. Thus, the best-basis representation is not only shift-invariant, but is also characterized by a lower information cost.

In this paper, an extended library of smooth local trigonometric bases is defined, and an appropriate fast 'best-basis' search algorithm, named *shift-invariant adapted-polarity local trigonometric decomposition* (SIAP-LTD), is introduced. When compared with the local cosine decomposition (LCD) [19], SIAP-LTD is advantageous in three respects. First, it leads to a best-basis expansion that is shift-invariant. Second, the resulting representation is characterized by a lower information cost. Third, the polarity of the folding operator is adapted to the parity properties of the segmented signal at the end-points. We showed that a *locally* adapted polarity yields a manageable computational complexity which is comparable to that of the LCD.

A similar procedure is available for other types of bases, most notably wavelets and wavelet-packet bases [47, 11, 12, 24, 36]. In these cases, the adaptive relative shift is equivalent to an adaptive down-sampling when expanding each parent node. By choosing between the odd and even samples after the low-pass and high-pass filters, we obtain an orthonormal best-basis representation which is shift-invariant and of a higher quality (lower information cost). In addition, the shift-invariant nature of the information cost renders this quantity a characteristic of the signal, so it can be used as a measure of the relative efficiency of various libraries (i.e., various scaling function selections) with respect to a given cost function.

It is worth mentioning that while a fixed action-region was used for the folding operator (a fixed ε in (11)), it is possible to dilate it in coarser resolution levels, as long as the segments of the signal are compatible. That is, in each resolution level, if a parent node has been chosen for the best expansion then the radii of the action regions at its end-points are maximized, subject to the compatibility restriction. Such a *variable folding* operator may lead to better time-frequency localization properties of basis functions, compared to fixed folding [19] and multiple folding [26].

Appendix A. Proof of Lemma 1

We need to show that

$$\langle \psi_{I,i}^{\rho_0,\rho_1}, \psi_{I,j}^{\rho_0,\rho_1} \rangle = \delta_{i,j} \tag{A.1}$$

for all $I \in \mathscr{I}$, $i,j \in \mathbb{Z}_+$ and $\rho_0, \rho_1 \in \{0,1\}$, where $\delta_{i,j}$ denotes the Kronecker delta.

This can be proved by a straightforward computation of the inner product and using the properties of the rising cutoff function. Here a simpler proof, which is based on Eq. (18), is provided. Define

$$\Lambda_{i,j} = \langle \psi_{I,i}^{\rho_0,\rho_1}, \psi_{I,j}^{\rho_0,\rho_1} \rangle\,; \tag{A.2}$$

then by (18),

$$\Lambda_{i,j} = \langle F^*(\alpha,\rho_0)F^*(\beta,\rho_1)\chi_I C_{I,i}^{\rho_0,\rho_1}, F^*(\alpha,\rho_0)F^*(\beta,\rho_1)\chi_I C_{I,j}^{\rho_0,\rho_1} \rangle, \tag{A.3}$$

where α and β are the end-points of the interval I. Since F, the folding operator, is unitary, it follows that

$$\Lambda_{i,j} = \langle C_{I,i}^{\rho_0,\rho_1}, \chi_I C_{I,j}^{\rho_0,\rho_1} \rangle. \tag{A.4}$$

Whence $\Lambda_{i,j} = \delta_{i,j}$ because the set $\{\chi_I C_{I,k}^{\rho_0,\rho_1} \,:\, k \in \mathbb{Z}_+\}$ is an orthonormal basis for $L_2(I)$.

Appendix B. Proof of Lemma 2

Let $I' = [\alpha_0, \alpha_1)$ and $I'' = [\alpha_1, \alpha_2)$ be adjacent compatible intervals, and let $Q_j = Q(\alpha_j, \rho_j)$, $j = 0,1,2$, be the periodic folding operators at the end-points. First we show that the operator $P_{I'} = Q_0^* Q_1^* \chi_{I'} Q_1 Q_0$ is an orthogonal projection onto $V_{I'}^{\rho_0,\rho_1}$.

By Lemma 1, the set $B_{I'}^{\rho_0,\rho_1} = \{\psi_{I',k}^{\rho_0,\rho_1} \,:\, k \in \mathbb{Z}_+\}$ is an orthonormal basis of $V_{I'}^{\rho_0,\rho_1}$. Eq. (16) and definition of $\psi_{I',k}^{\rho_0,\rho_1}$ imply that

$$\psi_{I',k}^{\rho_0,\rho_1}(t) = Q_0^* Q_1^* \chi_{I'} C_{I',k}^{\rho_0,\rho_1}(t)\,, \tag{B.1}$$

whence

$$P_{I'} \psi_{I',k}^{\rho_0,\rho_1} = Q_0^* Q_1^* \chi_{I'} Q_1 Q_0 Q_0^* Q_1^* \chi_{I'} C_{I',k}^{\rho_0,\rho_1} = Q_0^* Q_1^* \chi_{I'} C_{I',k}^{\rho_0,\rho_1} = \psi_{I',k}^{\rho_0,\rho_1} \tag{B.2}$$

for Q_0 and Q_1 are unitary. Now let $w \in \hat{L}_2[0,1]$ be in $(V_{I'}^{\rho_0,\rho_1})^\perp$. Then

$$\langle w, \psi_{I',k}^{\rho_0,\rho_1} \rangle = \langle w, Q_0^* Q_1^* \chi_{I'} C_{I',k}^{\rho_0,\rho_1} \rangle = \langle Q_1 Q_0 w, \mathbf{1}_{I'} C_{I',k}^{\rho_0,\rho_1} \rangle = 0 \quad \text{for all } k \in \mathbb{Z}_+\,. \tag{B.3}$$

Thus, $Q_1 Q_0 w$ is identically zero on I', since $\{\mathbf{1}_{I'} C_{I',k}^{\rho_0,\rho_1} \,:\, k \in \mathbb{Z}_+\}$ is an orthonormal basis for $L_2(I')$, and so

$$P_{I'} w = Q_0^* Q_1^* \chi_{I'} Q_1 Q_0 w = 0\,. \tag{B.4}$$

Consequently,

$$P_{I'}(v + w) = v \quad \text{for all } v \in V_{I'}^{\rho_0,\rho_1} \text{ and } w \perp V_{I'}^{\rho_0,\rho_1}\,. \tag{B.5}$$

In the same manner, $P_{I''} = Q_1^* Q_2^* \chi_{I''} Q_2 Q_1$ and $P_{I' \cup I''} = Q_0^* Q_2^* \chi_{I' \cup I''} Q_2 Q_0$ are orthogonal projections onto $V_{I''}^{\rho_1,\rho_2}$ and $V_{I' \cup I''}^{\rho_0,\rho_2}$, respectively.

Owing to the compatibility of the intervals, the set of operators $\{Q_0 Q_1 Q_2\}$ form a commuting family. Additionally, Q_0 commutes with $\chi_{I''}$, Q_2 commutes with $\chi_{I'}$, and Q_1 commutes with $[\chi_{I'} + \chi_{I''}] = \chi_{I' \cup I''}$.

Following the same stages of the proof of Lemma 4.3 in [59, p. 113], we have

$$\begin{aligned} P_{I'} + P_{I''} &= Q_0^* Q_1^* \chi_{I'} Q_1 Q_0 + Q_1^* Q_2^* \chi_{I''} Q_2 Q_1 = Q_1^* \left[Q_0^* \chi_{I'} Q_0 + Q_2^* \chi_{I''} Q_2 \right] Q_1 \\ &= Q_0^* Q_2^* Q_1^* \left[\chi_{I'} + \chi_{I''} \right] Q_1 Q_2 Q_0 = Q_0^* Q_2^* \chi_{I' \cup I''} Q_2 Q_0 = P_{I' \cup I''} \end{aligned} \tag{B.6}$$

and

$$P_{I'} P_{I''} = Q_0^* Q_1^* \chi_{I'} Q_1 Q_0 Q_1^* Q_2^* \chi_{I''} Q_2 Q_1 = Q_0^* Q_1^* Q_2^* \chi_{I'} \chi_{I''} Q_0 Q_2 Q_1 = 0 . \tag{B.7}$$

Hence $V_{I'}^{\rho_0,\rho_1} \oplus V_{I''}^{\rho_1,\rho_2} = V_{I' \cup I''}^{\rho_0,\rho_2}$.

Appendix C. Proof of Proposition 3

Let $f, g \in \hat{L}_2[0,1]$ be identical to within a resolution J time shift, and let A_f and A_g denote their respective best bases. Then there exists an integer $0 \leqslant q < 2^J$ such that

$$g(t) = f(t - q2^{-J}) . \tag{C.1}$$

We show by induction that

$$B_{[\alpha,\beta)}^{\rho_0,\rho_1} \subset A_f \tag{C.2}$$

implies

$$B_{[\alpha+q2^{-J},\beta+q2^{-J})}^{\rho_0,\rho_1} \subset A_g \tag{C.3}$$

for all $I = [\alpha, \beta) \in \mathscr{I}$ and $\rho_0, \rho_1 \in \{0, 1\}$. Or equivalently,

$$B_{\ell,n,m}^{\rho_0,\rho_1} \subset A_f \tag{C.4}$$

implies

$$B_{\ell,\tilde{n},\tilde{m}}^{\rho_0,\rho_1} \subset A_g, \qquad \tilde{n} \equiv n + (m+q) \operatorname{div} 2^{J-\ell}, \quad \tilde{m} \equiv (m+q) \bmod 2^{J-\ell} \tag{C.5}$$

for all $0 \leqslant \ell \leqslant L$, $0 \leqslant n < 2^\ell$, $0 \leqslant m < 2^{J-\ell}$ and $\rho_0, \rho_1 \in \{0,1\}$.

First we validate the claim for the finest resolution level $\ell = L$. Suppose that $B_{L,n_0,m_L}^{\rho_0,\rho_1} \subset A_f$. Then the information cost for representing f at the finest resolution level is minimized for shift m_L and polarity P_L where $p_L(n_0) = \rho_0$, $p_L(n_0+1) = \rho_1$. That is,

$$(m_L, P_L) = \arg \min_{\substack{0 \leqslant m < 2^{J-L} \\ 0 \leqslant P < 2^{2^L}}} \left\{ \sum_{n=0}^{2^L-1} \mathscr{M}(B_{L,n,m}^{p(n),\,p(n+1)} f) \right\} , \tag{C.6}$$

$$P_L = [p_L(2^L-1), \ldots, p_L(1), p_L(0)]_2 , \quad p_L(2^L) = p_L(0) , \tag{C.7}$$

$$p_L(n_0) = \rho_0 , \quad p_L(n_0+1) = \rho_1 . \tag{C.8}$$

It stems from (C.1) and definition of $\psi_{\ell,n,m,k}^{\rho_2,\rho_3}$ that

$$\langle f, \psi_{\ell,n,m,k}^{\rho_2,\rho_3} \rangle = \langle g, \psi_{\ell,\tilde{n},\tilde{m},k}^{\rho_2,\rho_3} \rangle , \qquad \tilde{n} = n + (m+q) \operatorname{div} 2^{J-\ell}, \ \tilde{m} = (m+q) \bmod 2^{J-\ell} \tag{C.9}$$

and, accordingly,

$$\mathscr{M}(B_{\ell,n,m}^{\rho_2,\rho_3} f) = \mathscr{M}(B_{\ell,\tilde{n},\tilde{m}}^{\rho_2,\rho_3} g) \tag{C.10}$$

for all $0 \leqslant \ell \leqslant L$, $0 \leqslant n < 2^{\ell}$, $0 \leqslant m < 2^{J-\ell}$, $k \in \mathbb{Z}_+$ and $\rho_2, \rho_3 \in \{0,1\}$. Hence, the information cost for representing g at the finest resolution level is minimized for shift $\tilde{m}_L = (m_L + q) \bmod 2^{J-L}$ and polarity $\tilde{P}_L$ where $\tilde{p}_L(\tilde{n}) = p_L(n)$ and $\tilde{n} = n + (m_L + q)$ div 2^{J-L}. That is,

$$(\tilde{m}_L, \tilde{P}_L) = \arg \min_{\substack{0 \leqslant m < 2^{J-L} \\ 0 \leqslant P < 2^{2^L}}} \left\{ \sum_{n=0}^{2^L - 1} \mathscr{M}(B_{L,n,m}^{p(n),\, p(n+1)} g) \right\}, \tag{C.11}$$

$$\tilde{P}_L = [\tilde{p}_L(2^L - 1), \ldots, \tilde{p}_L(1), \tilde{p}_L(0)]_2, \quad \tilde{p}_L(2^L) = \tilde{p}_L(0), \tag{C.12}$$

$$\tilde{p}_L(\tilde{n}) = p_L(n). \tag{C.13}$$

Consequently,

$$B_{L,\tilde{n}_0,\tilde{m}_L}^{\rho_0,\rho_1} = B_{L,\tilde{n}_0,\tilde{m}_L}^{\tilde{p}_L(\tilde{n}_0),\, \tilde{p}_L(\tilde{n}_0+1)} \subset A_g. \tag{C.14}$$

Now, suppose that the claim is true for all levels finer than l ($l < L$), and assume that (C.4) exists for $\ell = l$. Then by (20) and (22)

$$\mathscr{M}(B_{l,n,m_l}^{\rho_0,\rho_1} f) \leqslant \mathscr{M}(A_{l+1,2n+\gamma,m_{l+1}}^{\rho_0,\rho_2} f) + \mathscr{M}(A_{l+1,2n+1+\gamma,m_{l+1}}^{\rho_2,\rho_1} f), \tag{C.15}$$

where $m_{l+1} \equiv m_l - \gamma \cdot 2^{J-l-1}$, $\rho_2 \equiv p_{l+1}(2n+1+\gamma)$ and $\gamma \in \{0,1\}$ such that $m_{l+1} \in [0, 2^{J-l-1})$. Notice that γ is an indication of a relative shift between the resolution levels l and $l+1$.

The inductive hypothesis together with Eq. (C.10) lead to the identities

$$\mathscr{M}(A_{l+1,2n+\xi+\gamma,m_{l+1}}^{\rho_0,\rho_2} f) = \mathscr{M}(A_{l+1,2\tilde{n}+\xi+\tilde{\gamma},\tilde{m}_{l+1}}^{\rho_0,\rho_2} g), \quad \xi \in \{0,1\}, \tag{C.16}$$

$$\mathscr{M}(A_{l+1,2n+\xi+\gamma,m_{l+1}}^{\rho_2,\rho_1} f) = \mathscr{M}(A_{l+1,2\tilde{n}+\xi+\tilde{\gamma},\tilde{m}_{l+1}}^{\rho_2,\rho_1} g), \quad \xi \in \{0,1\}, \tag{C.17}$$

where $\tilde{m}_{l+1} \equiv \tilde{m}_l - \tilde{\gamma} \cdot 2^{J-l-1}$ and $\tilde{\gamma} \in \{0,1\}$ such that $\tilde{m}_{l+1} \in [0, 2^{J-l-1})$. Consequently, using again (C.10) we have

$$\mathscr{M}(B_{l,\tilde{n},\tilde{m}_l}^{\rho_0,\rho_1} g) \leqslant \mathscr{M}(A_{l+1,2\tilde{n}+\tilde{\gamma},\tilde{m}_{l+1}}^{\rho_0,\rho_2} g) + \mathscr{M}(A_{l+1,2\tilde{n}+1+\tilde{\gamma},\tilde{m}_{l+1}}^{\rho_2,\rho_1} g), \tag{C.18}$$

$$\tilde{p}_l(\tilde{n}) = p_l(n), \quad \tilde{n} = n + (m+q) \text{ div } 2^{J-l}. \tag{C.19}$$

So by (20) and (22) we conclude that

$$B_{l,\tilde{n},\tilde{m}_l}^{\rho_0,\rho_1} = B_{l,\tilde{n},\tilde{m}_l}^{\tilde{p}_l(\tilde{n}),\, \tilde{p}_l(\tilde{n}+1)} \subset A_g, \tag{C.20}$$

proving as well the validity of the claim for $\ell = l$. Thus, A_f and A_g are identical to within a $q2^{-J}$ time shift.

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