# A quenched invariance principle for certain ballistic random walks in i.i.d. environments

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**Abstract.** We prove that every random walk in i.i.d. environment in dimension greater than or equal to 2 that has an almost sure positive speed in a certain direction, an annealed invariance principle and some mild integrability condition for regeneration times also satisfies a quenched invariance principle. The argument is based on intersection estimates and a theorem of Bolthausen and Sznitman.

## 1. INTRODUCTION

Let $d \geq 1$. A Random Walk in Random Environment (RWRE) on $\mathbb{Z}^d$ is defined as follows. Let $\mathcal{M}^d$ denote the space of all probability measures on $\mathcal{E}_d = \{\pm e_i\}_{i=1}^d$ and let $\Omega = \left(\mathcal{M}^d\right)^{\mathbb{Z}^d}$. An *environment* is a point $\omega = \{\omega(x,e)\}_{x\in\mathbb{Z}^d,\, e\in\mathcal{E}_d} \in \Omega$. Let $P$ be a probability measure on $\Omega$. For the purposes of this paper, we assume that $P$ is an i.i.d. measure, i.e.

$$P = Q^{\mathbb{Z}^d}$$

for some distribution $Q$ on $\mathcal{M}^d$ and that $Q$ is *uniformly elliptic*, i.e. there exists a $\kappa > 0$ such that for every $e \in \mathcal{E}_d$,

$$Q(\{\omega(0,\cdot) : \omega(0,e) < \kappa\}) = 0.$$

For an environment $\omega \in \Omega$, the *Random Walk* on $\omega$ is a time-homogenous Markov chain with transition kernel

$$P_\omega\left(X_{n+1} = x + e \,|\, X_n = x\right) = \omega(x,e).$$

The **quenched law** $P_\omega^x$ is defined to be the law on $\left(\mathbb{Z}^d\right)^{\mathbb{N}}$ induced by the transition kernel $P_\omega$ and $P_\omega^x(X_0 = x) = 1$. With some abuse of notation, we write $P_\omega$ also for $P_\omega^0$. We let $\mathcal{P}^x = P \otimes P_\omega^x$ be the joint law of the environment and the walk, and the **annealed** law is defined to be its marginal

$$\mathbb{P}^x(\cdot) = \int_\Omega P_\omega^x(\cdot)\,dP(\omega).$$

We use $\mathbb{E}^x$ to denote expectations with respect to $\mathbb{P}^x$. We consistently omit the superscript $x$ if $x = 0$.

We say that the RWRE $\{X(n)\}_{n\geq 0}$ satisfies the law of large numbers with deterministic speed $v$ if $X_n/n \to v$, $\mathbb{P}$-a.s. For $x \geq 0$, let $[x]$ denote the largest integer less than or equal to $x$. We say that

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the RWRE $\{X(n)\}_{n\geq 0}$ satisfies the *annealed* invariance principle with deterministic, positive definite covariance matrix $\sigma_{\mathbb{P}}^2$ if the linear interpolations of the processes

$$B^n(t) = \frac{X([nt]) - [nvt]}{\sqrt{n}}\,, t \geq 0 \tag{1.1}$$

converge in distribution (with respect to the supremum topology on the space of continuous function on $[0,1]$) as $n \to \infty$, under the measure $\mathbb{P}$, to a Brownian motion of covariance $\sigma_{\mathbb{P}}^2$. We say the process $\{X(n)\}_{n\geq 0}$ satisfies the *quenched* invariance principle with variance $\sigma_{\mathbb{P}}^2$ if for $P$-a.e. $\omega$, the above convergence holds under the measure $P_\omega^0$. Our focus in this paper are conditions ensuring that when an annealed invariance principle holds, so does a quenched one.

To state our results, we need to recall the regeneration structure for random walk in i.i.d. environment, developed by Sznitman and Zerner in [SZ99]. We say that $t$ is a regeneration time (in direction $e_1$) for $\{X(\cdot)\}$ if

$$\langle X(s), e_1\rangle < \langle X(t), e_1\rangle \text{ whenever } s < t$$

and

$$\langle X(s), e_1\rangle \geq \langle X(t), e_1\rangle \text{ whenever } s > t\,.$$

When $\omega$ is distributed according to an i.i.d. $P$ such that the process $\{\langle X(n), e_1\rangle\}_{n\geq 0}$ is $\mathbb{P}$-almost surely transient to $+\infty$, it holds by [SZ99] that, $\mathbb{P}$-almost surely, there exist infinitely many regeneration times for $\{X(\cdot)\}$. Let

$$t^{(1)} < t^{(2)} < \dots\,,$$

be all of the regeneration times for $\{X(\cdot)\}$. Then, the sequence $\{(t^{(k+1)} - t^{(k)}), (X(t^{(k+1)}) - X(t^{(k)}))\}_{k\geq 1}$ is an i.i.d. sequence under $\mathbb{P}$. Further, if $\lim_{n\to\infty} n^{-1}\langle X(n), e_1\rangle > 0$, $\mathbb{P}$-a.s., then we get, see [SZ99], that

$$\mathbb{E}(t^{(2)} - t^{(1)}) < \infty. \tag{1.2}$$

The main result of this paper is the following:

**Theorem 1.1** *Let $d \geq 4$ and let $Q$ be a uniformly elliptic distribution on $\mathcal{M}^d$. Set $P = Q^{\mathbb{Z}^d}$. Assume that the random walk $\{X(n)\}_{n\geq 0}$ satisfies the law of large numbers with a positive speed in the direction $e_1$, that is*

$$\lim_{n\to\infty} \frac{X(n)}{n} = v\,, \mathbb{P} - a.s \quad \textit{with } v \textit{ deterministic such that } \langle v, e_1\rangle > 0\,. \tag{1.3}$$

*Assume further that the process $\{X(n)\}_{n\geq 0}$ satisfies an annealed invariance principle with variance $\sigma_{\mathbb{P}}^2$.*

*Assume that there exists an $\epsilon > 0$ such that $\mathbb{E}(t^{(1)})^\epsilon < \infty$ and, with some $r \geq 2$,*

$$\mathbb{E}[(t^{(2)} - t^{(1)})^r] < \infty\,. \tag{1.4}$$

*If $d = 4$, assume further that (1.4) holds with $r > 8$. Then, the process $\{X(\cdot)\}$ satisfies a quenched invariance principle with variance $\sigma_{\mathbb{P}}^2$.*

(The condition $r \geq 2$ for $d \geq 5$ can be weakened to $r > 1 + 4/(d+4)$ by choosing in (3.7) below $r' = r$ and modifying appropriatly the value of $K_d$ in Proposition 3.1 and Corollary 3.2.) We suspect, in line with Sznitman's conjecture concerning condition $T'$, see [Szn02], that (1.4) holds for $d \geq 2$ and all $r > 0$ as soon as (1.3) holds.

A version of Theorem 1.1 for $d = 2, 3$ is presented in Section 4. For $d = 1$, the conclusion of Theorem 1.1 does not hold, and a quenched invariance principle, or even a CLT, requires a different centering [Zei04, Gol07, Pet08]. (This phenomenom is typical of dimension $d = 1$, as demonstrated in [RAS06] in the context of the totally asymmetric, non-nearest neighbor, RWRE.) Thus, some restriction on the dimension is needed.

Our proof of Theorem 1.1 is based on a criterion from [BS02], which uses two independent RWRE's in the same environment $\omega$. This approach seems limited, in principle, to $d \geq 3$ (for technical reasons, we restrict attention to $d \geq 4$ in the main body of the paper), regardless of how good tail estimates on regeneration times hold. An alternative approach to quenched CLT's, based on martingale methods but still using the existence of regeneration times with good tails, was developed by Rassoul-Agha and Seppäläinen in [RAS05], [RAS07a], and some further ongoing work of these authors. While their approach has the potential of reducing the critical dimension to $d = 2$, at the time this paper was written, it had not been succesful in obtaining statements like in Theorem 1.1 without additional structural assumptions on the RWRE. [1]

Since we will consider both the case of two independent RWRE's in different environments and the case of two RWRE's evolving in the same environment, we introduce some notation. For $\omega_i \in \Omega$, we let $\{X_i(n)\}_{n\geq 0}$ denote the path of the RWRE in environment $\omega_i$, with law $P^0_{\omega_i}$. We write $P_{\omega_1,\omega_2}$ for the law $P^0_{\omega_1} \times P^0_{\omega_2}$ on the pair $(\{X_1(\cdot), X_2(\cdot)\})$. In particular,

$$E_{P\times P}[P_{\omega_1,\omega_2}(\{X_1(\cdot)\} \in A_1, \{X_2(\cdot)\} \in A_2)] = \mathbb{P}(\{X_1(\cdot)\} \in A_1) \cdot \mathbb{P}(\{X_2(\cdot)\} \in A_2)$$

represents the annealed probability that two walks $\{X_i(\cdot)\}$, $i = 1, 2$, in independent environments belong to sets $A_i$, while

$$E_P[P_{\omega,\omega}(\{X_1(\cdot)\} \in A_1, \{X_2(\cdot)\} \in A_2)] = \int P_\omega(\{X_1(\cdot)\} \in A_1) \cdot P_\omega(\{X_2(\cdot)\} \in A_2) dP(\omega)$$

is the annealed probability for the two walks in the *same* environment.

We use throughout the notation

$$t_i^{(1)} < t_i^{(2)} < \dots\,, \quad i = 1, 2$$

for the sequence of regeneration times of the process $\{X_i(\cdot)\}$. Note that whenever $P$ satisfies the assumptions in Theorem 1.1, the estimate (1.4) holds for $(t_i^{(2)} - t_i^{(1)})$, as well.

**Notation** Throughout, $C$ denotes a constant whose value may change from line to line, and that may depend on $d$ and $\kappa$ only. Constants that may depend on additional parameters will carry this dependence in the notation. Thus, if $F$ is a fixed function then $C_F$ denotes a constant that may change from line to line, but that depends on $F, d$ and $\kappa$ only. For $p \geq 1$, $\|\cdot\|_p$ denotes the $L^p$ norm on $\mathbb{R}^d$ or $\mathbb{Z}^d$, while $\|\cdot\|$ denotes the supremum norm on these spaces.

## 2. An intersection estimate and proof of the quenched CLT

As mentioned in the introduction, the proof of the quenched CLT involves considering a pair of RWRE's $(X_1(\cdot), X_2(\cdot))$ in the same environment. The main technical tool needed is the following proposition, whose proof will be provided in Section 3. Let $H_K := \{x \in \mathbb{Z}^d : \langle x, e_1\rangle > K\}$.

**Proposition 2.1** *We continue under the assumptions of Theorem* 1.1. *Let*

$$W_K := \{\{X_1(\cdot)\} \cap \{X_2(\cdot)\} \cap H_K \neq \emptyset\}\,.$$

*Then*

$$E_P[P_{\omega,\omega}(W_K)] < CK^{-\kappa_d} \tag{2.1}$$

[1] After the first version of this paper was completed and posted, Rassoul-Agha and Seppäläinen posted a preprint [RAS07b] in which they prove a statement similar to Theorem 1.1, for all dimensions $d \geq 2$, under somewhat stronger assumptions on moments of regeneration times. While their approach differs significantly from ours, and is somewhat more complicated, we learnt from their work an extra ingredient that allowed us to extend our approach and prove Theorem 1.1 in all dimensions $d \geq 2$. For the convenience of the reader, we sketch the argument in Section 4 below.

*where* $\kappa_d = \kappa_d(\epsilon, r) > 0$ *for* $d \geq 4$.

We can now bring the

**Proof of Theorem** 1.1 (assuming Proposition 2.1). For $i = 1, 2$, define $B_i^n(t) = n^{-1/2}(X_i([nt]) - [nvt]))$, where the processes $\{X_i\}$ are RWRE's in the same environment $\omega$, whose law is $P$. We introduce the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$, and the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued variable

$$\beta_i^n(\cdot) = \text{the polygonal interpolation of } \tfrac{k}{n} \to B_i^n(\tfrac{k}{n}),\ k \geq 0\ . \tag{2.2}$$

It will also be useful to consider the analogously defined space $C([0, T], \mathbb{R}^d)$, of continuous $\mathbb{R}^d$-valued functions on $[0, T]$, for $T > 0$, which we endow with the distance

$$d_T(v, v') = \sup_{s \leq T} |v(s) - v'(s)| \wedge 1\,. \tag{2.3}$$

With some abuse of notation, we continue to write $\mathbb{P}$ for the law of the pair $(\beta_1^n, \beta_2^n)$. By Lemma 4.1 of [BS02], the claim will follow once we show that for all $T > 0$, for all bounded Lipschitz functions $F$ on $C([0, T], \mathbb{R}^d)$ and $b \in (1, 2]$:

$$\sum_m \left(E_P[E_\omega(F(\beta_1([b^m]))E_\omega(F(\beta_2([b^m]))] - \mathbb{E}[F(\beta_1([b^m]))]\mathbb{E}[F(\beta_2([b^m])]\right) < \infty\,. \tag{2.4}$$

When proving (2.4), we may and will assume that $F$ is bounded by 1 with Lipschitz constant 1.

Fix constants $1/2 > \theta > \theta'$. Write $N = [b^m]$. Let

$$s_i^m := \min\{t > N^\theta/2 :\ X_i(t) \in H_{N^{\theta'}},\, t \text{ is a regeneration time for } X_i(\cdot)\}\,, i = 1, 2.$$

Define the events

$$A_i^m := \{s_i^m \leq N^\theta\}\,, i = 1, 2,$$

and

$$C_m := \{\{X_1(n + s_1^m)\}_{n \geq 0} \cap \{X_2(n + s_2^m)\}_{n \geq 0} = \emptyset\}\,, B_m := A_1^m \cap A_2^m \cap C_m\,.$$

Finally, write $\mathcal{F}_i := \sigma(X_i(t), t \geq 0)$ and

$$\mathcal{F}_i^\Omega := \sigma\{\omega_z : \text{there exists a } t \text{ such that } X_i(t) = z\} \vee \mathcal{F}_i, i = 1, 2.$$

Note that, for $i = 1, 2$,

$$\begin{aligned}
\mathbb{P}\left((A_i^m)^c\right) &\leq \mathbb{P}(\max_{j=1}^{N^\theta}[t_i^{(j+1)} - t_i^{(j)}] \geq N^\theta/4) + \mathbb{P}(t_i^{(1)} > N^\theta/4) + \mathbb{P}(X_i(N^\theta/2) \notin H_{N^{\theta'}}) \\
&\leq \frac{4^r N^\theta \mathbb{E}[(t_i^{(2)} - t_i^{(1)})^r]}{N^{r\theta}} + \frac{4^\epsilon \mathbb{E}\left([t_i^{(1)}]^\epsilon\right)}{N^{\theta\epsilon}} + \mathbb{P}\left(\sum_{j=1}^{N^{\theta'}} (t_i^{(j+1)} - t_i^{(j)}) > \frac{N^\theta}{4}\right) \\
&\leq N^{-\delta'} + 4N^{\theta' - \theta}\mathbb{E}\left[t_i^{(2)} - t_i^{(1)}\right] \leq 2N^{-\delta'}\,,
\end{aligned} \tag{2.5}$$

with $\delta' = \delta'(\epsilon, \theta) > 0$ independent of $N$. Using the last estimate and Proposition 2.1, one concludes that

$$\sum_m E_P[P_{\omega,\omega}(B_m^c)] < \infty. \tag{2.6}$$

Now,

$$|\mathbb{E}[F(\beta_1^{[b^m]})F(\beta_2^{[b^m]})] - \mathbb{E}[\mathbf{1}_{B_m} F(\beta_1^{[b^m]})F(\beta_2^{[b^m]})]| \leq \mathbb{P}(B_m^c)\,. \tag{2.7}$$

Let the process $\bar{\beta}_i^{[b^m]}(\cdot)$ be defined exactly as the process $\beta_i^{[b^m]}(\cdot)$, except that one replaces $X_i(\cdot)$ by $X_i(\cdot + s_i^m)$. On the event $A_i^m$, we have by construction that

$$\sup_t \left|\beta_i^{[b^m]}(t) - \bar{\beta}_i^{[b^m]}(t)\right| \leq 2N^{\theta - 1/2}\,,$$

and therefore, on the event $A_1^m \cap A_2^m$,

$$\left|[F(\beta_1^{[b^m]})F(\beta_2^{[b^m]})] - [F(\bar\beta_1^{[b^m]})F(\bar\beta_2^{[b^m]})]\right| \le CN^{\theta-1/2}\,, \tag{2.8}$$

for some constant $C$ (we used here that $F$ is Lipschitz (with constant 1) and bounded by 1).

On the other hand, writing $\omega'$ for an independent copy of $\omega$ with the same distribution $P$,

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{B_m}F(\bar\beta_1^{[b^m]})F(\bar\beta_2^{[b^m]})] = E_P(E_\omega[\mathbf{1}_{B_m}F(\bar\beta_1^{[b^m]})F(\bar\beta_2^{[b^m]}])\\
=\;& \mathbb{E}\left(\mathbf{1}_{A_m^1}F(\bar\beta_1^{[b^m]})E_\omega[\mathbf{1}_{A_m^2\cap C_m}F(\bar\beta_2^{[b^m]}]) \mid \mathcal{F}_1^\Omega]\right)\\
=\;& E_P\left(E_\omega\left[\mathbf{1}_{A_m^1}F(\bar\beta_1^{[b^m]})E_\omega[\mathbf{1}_{A_m^2\cap C_m}F(\bar\beta_2^{[b^m]}]) \mid \mathcal{F}_1^\Omega]\right]\right)\\
=\;& E_P\left(E_\omega\left[\mathbf{1}_{A_m^1}F(\bar\beta_1^{[b^m]})E_{\omega'}[\mathbf{1}_{A_m^2\cap C_m}F(\bar\beta_2^{[b^m]}]) \mid \mathcal{F}_1^\Omega]\right]\right)\\
=\;& E_P\left(E_{\omega,\omega'}\left[\mathbf{1}_{A_m^1}F(\bar\beta_1^{[b^m]})\mathbf{1}_{A_m^2\cap C_m}F(\bar\beta_2^{[b^m]}])\right]\right)\\
=\;& E_P\left(E_{\omega,\omega'}\left[\mathbf{1}_{B_m}F(\bar\beta_1^{[b^m]})F(\bar\beta_2^{[b^m]}])\right]\right)\,.
\end{aligned} \tag{2.9}$$

The third equality follows from the fact that we multiply by the indicator of the event of non-intersection. Since

$$\left|E_P\left(E_{\omega,\omega'}\left[\mathbf{1}_{B_m}F(\beta_1^{[b^m]})F(\beta_2^{[b^m]}])\right]\right) - E_P\left(E_{\omega,\omega'}\left[F(\beta_1^{[b^m]})F(\beta_2^{[b^m]}])\right]\right)\right| \le \mathbb{P}(B_m^c)\,,$$

and

$$E_P\left(E_{\omega,\omega'}\left[F(\beta_1^{[b^m]})F(\beta_2^{[b^m]}])\right]\right) = \mathbb{E}\left[F(\beta_1^{[b^m]})\right]\mathbb{E}\left[F(\beta_2^{[b^m]}])\right]\,,$$

we conclude from the last two displays, (2.9) and (2.8) that

$$\left|\mathbb{E}[F(\beta_1^{[b^m]})F(\beta_2^{[b^m]})] - \mathbb{E}\left[F(\beta_1^{[b^m]})\right]\mathbb{E}\left[F(\beta_2^{[b^m]}])\right]\right| \le 2\mathbb{P}(B_m^c) + 2CN^{\theta-1/2}\,.$$

Together with (2.6), we conclude that (2.4) holds, and complete the proof of Theorem 1.1. □

## 3. Intersection structure

In this section we prove Proposition 2.1, that is we establish estimates on the probability that two independent walks in the same environment intersect each other in the half space $H_K = \{x \in \mathbb{Z}^d : \langle x, e_1\rangle > K\}$. It is much easier to obtain such estimates for walks in different environments, and the result for different environments will be useful for the case of walks in the same environment.

### 3.1 The conditional random walk.

Under the assumptions of Theorem 1.1, the process $\{\langle X(\cdot), e_1\rangle\}$ is $\mathbb{P}$-a.s. transient to $+\infty$. Let

$$D := \{\forall_{n\ge 0}, \langle X(n), e_1\rangle \ge \langle X(0), e_1\rangle\}\,.$$

By e.g. [SZ99], we have that

$$\mathbb{P}(D) > 0\,. \tag{3.1}$$

### 3.2 Intersection of paths in independent environments.

In this subsection, we let $\omega^{(1)}$ and $\omega^{(2)}$ be independent environments, each distributed according to $P$. Let $\{Y_1(n)\}$ and $\{Y_2(n)\}$ be random walks in the environments (respectively) $\omega^{(1)}$ and $\omega^{(2)}$, with starting points $U_i = Y_i(0)$. In other words, $\{Y_1(n)\}$ and $\{Y_2(n)\}$ are independent samples taken from the annealed measures $\mathbb{P}^{U_i}(\cdot)$. For $i = 1, 2$ set

$$D_i^{U_i} = \{\langle Y_i(n), e_1\rangle \geq \langle U_i, e_1\rangle \text{ for } n \geq 0\}, \quad i = 1, 2.$$

For brevity, we drop $U_i$ from the notation and use $\mathbf{P}$ for $\mathbb{P}^{U_1} \times \mathbb{P}^{U_2}$ and $\mathbf{P}^D$ for $\mathbb{P}^{U_1}(\cdot|D_1^{U_1}) \times \mathbb{P}^{U_2}(\cdot|D_2^{U_2})$.

First we prove some basic estimates. While the estimates are similar for $d = 4$ and $d \geq 5$, we will need to prove them separately for the two cases.

*3.2.1 Basic estimates for $d \geq 5$.*

**Proposition 3.1** *($d \geq 5$) With notation as above and assumptions as in Theorem* 1.1*,*

$$\mathbf{P}^D\left(\{Y_1(\cdot)\} \cap \{Y_2(\cdot)\} \neq \emptyset\right) < C\|U_1 - U_2\|^{-K_d}$$

*where $K_d = \frac{d-4}{4+d}$*

The proof is very similar to the proof of Lemma 5.1 of [Ber06], except that here we need a quantitative estimate that is not needed in [Ber06].

*Proof of Proposition* 3.1. We first note that the (annealed) law of $\{Y_i(\cdot) - U_i\}$ does not depend on $i$, and is identical to the law of $\{X(\cdot)\}$. We also note that on the event $D_i^{U_i}$, $t_i^{(1)} = 0$.

For $z \in \mathbb{Z}^d$, let

$$F_i(z) = \mathbf{P}^D(\exists_k Y_i(k) = z)$$

and let

$$F_i^{(R)}(z) = F(z) \cdot \mathbf{1}_{\|z-U_i\|>R}.$$

We are interested in $\|F_i\|_2$ and in $\|F_i^{(R)}\|_2$, noting that none of the two depends on $i$ or $U_i$. We have that

$$F_i(z) = \sum_{n=1}^{\infty} G_i(z,n) \quad \text{and} \quad F_i^{(R)}(z) = \sum_{n=1}^{\infty} G_i^{(R)}(z,n) \tag{3.2}$$

where

$$G_i(z,n) = \mathbf{P}^D(\exists_{t_i^{(n)} \leq k < t_i^{(n+1)}} Y_i(k) = z).$$

and

$$G_i^{(R)}(z,n) = \mathbf{P}^D(\exists_{t_i^{(n)} \leq k < t_i^{(n+1)}} Y_i(k) = z) \cdot \mathbf{1}_{\|z-U_i\|>R}.$$

are the occupation functions of $\{Y_i(\cdot)\}$.

By the triangle inequality,

$$\|F_i\|_2 \leq \sum_{n=1}^{\infty} \|G_i(\cdot,n)\|_2 \tag{3.3}$$

and

$$\|F_i^{(R)}\|_2 \leq \sum_{n=1}^{\infty} \|G_i^{(R)}(\cdot,n)\|_2. \tag{3.4}$$

Thus we want to bound the norm of $G_i(\cdot,n)$ and $G_i^{(R)}(\cdot,n)$. We start with $G_i(\cdot,n)$. Thanks to the i.i.d. structure of the regeneration slabs (see [SZ99]),

$$G_i(\cdot,n) = Q_i^n \star J\,,$$

where $Q_i^n$ is the distribution function of $Y_i(t_i^{(n)})$ under $\mathbb{P}(\cdot|D_i^{U_i})$,

$$J(z) = \mathbf{P}^D(\exists_{0=t_i^{(1)} \leq k < t_i^{(2)}} Y_i(k) - U_i = z),$$

and $\star$ denotes (discrete) convolution. Positive speed ($\langle v, e_1\rangle > 0$) tells us that

$$\Gamma := \|J\|_1 \le \mathbb{E}(t^{(2)} - t^{(1)}|D) < \infty$$

and thus

$$\|G_i(\cdot, n)\|_2 \le \Gamma\|Q_i^n\|_2$$

Under the law $\mathbf{P}^D$, $Q_i^n$ is the law of a sum of integrable i.i.d. random vectors $\Delta Y_i^k = Y_i(t_i^{k+1}) - Y_i(t_i^k)$, that due to the uniform ellipticity condition are non-degenerate. By the same computation as in [Ber06, Proof of claim 5.2], we get

$$\|Q_i^n\|_2 \le Cn^{-d/4},$$

and thus

$$\|G_i^{(R)}(\cdot, n)\|_2 \le \|G_i(\cdot, n)\|_2 \le Cn^{-d/4}. \tag{3.5}$$

(We note in passing that these estimates can also be obtained from a local limit theorem applied to a truncated version of the variables $\Delta Y_i^k$.) It follows from the last two displays and (3.3) that for $d \ge 5$,

$$\|F_i\|_2 < C \tag{3.6}$$

For $F_i^{(R)}$ we have a fairly primitive bound: by Markov's inequality and the fact that the walk is a nearest neighbor walk, for any $r' > 1$,

$$\begin{aligned} &\|G_i^{(R)}(\cdot, n)\|_2 \le \|G_i^{(R)}(\cdot, n)\|_1 \le \mathbb{E}(\mathbf{1}_{t_i^{(n+1)} > R}(t_i^{(n+1)} - t_i^{(n)})|D) \qquad (3.7)\\ \le\ & \mathbb{E}\left[\, \mathbf{1}_{t_i^{(n)} > \frac{R}{2}}(t_i^{(n+1)} - t_i^{(n)})\,|D\right] + \mathbb{E}\left[\, \mathbf{1}_{t_i^{(n+1)} - t_i^{(n)} > \frac{R}{2}}(t_i^{(n+1)} - t_i^{(n)})\,|D\right] \\ \le\ & \frac{2n\mathbb{E}\left[t_i^{(n+1)} - t_i^{(n)}\right]}{R} + C\mathbb{E}\left(\frac{(t_i^{(n+1)} - t_i^{(n)})^{r'}}{R^{(r'-1)}}\right) \le C\frac{n\mathbb{E}((t^{(2)} - t^{(1)})^2|D)}{R}, \end{aligned}$$

where the choice $r' = 2$ was made in deriving the last inequality. Together with (3.5), we get, with $K = \left[R^{4/(d+4)}\right]$,

$$\|F_i^{(R)}\|_2 \le C\left[\sum_{n=1}^{K} \frac{n}{R} + \sum_{n=K+1}^{\infty} n^{-d/4}\right] \le C\left[K^2/R + K^{1-d/4}\right] \le CR^{(4-d)/(d+4)}. \tag{3.8}$$

Let $R := \|U_2 - U_1\|/2$. An application of the Cauchy-Schwarz inequality yields

$$\mathbf{P}^D\left(\{Y_1(\cdot)\} \cap \{Y_2(\cdot)\} \ne \emptyset\right) \le \|F_1^{(R)}\|_2^2 + 2\|F_1^{(R)}\|_2\|F_1\|_2 = O\left(R^{(4-d)/(d+4)}\right)$$

for $d \ge 5$. □

Now assume that the two walks do intersect. How far from the starting points could this happen? From (3.8) we immediately get the following corollary.

**Corollary 3.2** *($d \ge 5$) Fix $R, Y_1(\cdot)$ and $Y_2(\cdot)$ as before. Let $A_i$ be the event that $Y_1(\cdot)$ and $Y_2(\cdot)$ intersect, but the intersection point closest to $U_i = Y_i(0)$ is at distance $\ge R$ from $Y_i(0)$. Then*

$$\mathbf{P}^D(A_1 \cap A_2) < CR^{(4-d)/(d+4)}. \tag{3.9}$$

*3.2.2 Basic estimates for $d = 4$.*

We will now see how to derive the same estimates for dimension $4$ in the presence of bounds on higher moments of the regeneration times. The crucial observation is contained in the following lemma.

**Lemma 3.3** *Let $d \geq 3$ and let $v_i$ be i.i.d., $\mathbb{Z}^d$-valued random variables satisfying, for some $r \in [2, d-1]$,*

$$\langle v_1, e_1 \rangle \geq 1 \text{ a.s.}, \quad \text{and } E\|v_1\|^r < \infty\,. \tag{3.10}$$

*Assume that, for some $\delta > 0$,*

$$P(\langle v_1, e_1 \rangle = 1) > \delta\,, \tag{3.11}$$

*and*

$$P(v_1 = z | \langle v_1, e_1 \rangle = 1) > \delta, \text{ for all } z \in \mathbb{Z}^d \text{ with } \|z - e_1\|_2 = 1 \text{ and } \langle z, e_1 \rangle = 1\,. \tag{3.12}$$

*Then, with $W_n = \sum_{i=1}^n v_i$, there exists a constant $c > 0$ such that for any $z \in \mathbb{Z}^d$,*

$$P(\exists_i : W_i = z) \leq c|\langle z, e_1 \rangle|^{-r(d-1)/(r+d-1)}\,, \tag{3.13}$$

*and, for all integer $K$,*

$$\sum_{z:\langle z, e_1 \rangle = K} P(\exists_i : W_i = z) \leq 1\,. \tag{3.14}$$

*Proof.* We set $T_K = \min\{n : \langle W_n, e_1 \rangle \geq K\}$. We note first that because of (3.11), for some constant $c_1 = c_1(\delta) > 0$ and all $t > 1$,

$$P(A_t) \leq e^{-c_1 t}\,. \tag{3.15}$$

where

$$A_t = \{\#\{i \leq t : \langle v_i, e_1 \rangle = 1\} < c_1 t\}\,.$$

Set $\bar{v} = E v_1$ and $v = E\langle v_1, e_1 \rangle$. Then, for any $\alpha \leq 1$, we get from (3.10) and the Marcinkiewicz-Zygmund inequality (see e.g. [Sh84, Pg. 469] or, for Burkholder's generalization, [St93, Pg. 341]) that for some $c_2 = c_2(r, v, \alpha)$, and all $K > 0$,

$$P(T_K < K^\alpha / 2v) \leq c_2 K^{-r(2-\alpha)/2}\,. \tag{3.16}$$

Let $\mathcal{F}_n := \sigma(\langle W_i, e_1 \rangle, i \leq n)$ denote the filtration generated by the $e_1$-projection of the random walk $\{W_n\}$. Denote by $W_n^\perp$ the projection of $W_n$ on the hyperplane perpendicular to $e_1$. Conditioned on the filtration $\mathcal{F}_n$, $\{W_n^\perp\}$ is a random walk with independent (not identically distributed) increments, and the assumption (3.12) together with standard estimates shows that, for some constant $c_3 = c_3(\delta, d)$,

$$\sup_{y \in \mathbb{Z}^{d-1}} \mathbf{1}_{A_t^c} P(W_t = y | \mathcal{F}_t) \leq c_3 t^{-(d-1)/2}\,,\ a.s. \tag{3.17}$$

Therefore, writing $z_1 = \langle z, e_1 \rangle$, we get for any $\alpha \leq 1$,

$$\begin{aligned}
&P(\exists_i : W_i = z) \leq P(T_{z_1} < z_1^\alpha / 2v) + P(W_i = z \text{ for some } i \geq z_1^\alpha / 2v) \\
\leq\ & c_2 z_1^{-r(2-\alpha)/2} + \sum_{i=z_1^\alpha/2v}^{z_1} P(W_i = z) \leq c_2 z_1^{-r(2-\alpha)/2} + \sum_{i=z_1^\alpha/2v}^{z_1} E(P(W_i = z | \mathcal{F}_i)) \\
\leq\ & c_2 z_1^{-r(2-\alpha)/2} + \sum_{i=z_1^\alpha/2v}^{z_1} P(A_i) + \sum_{i=z_1^\alpha/2v}^{z_1} E(\mathbf{1}_{T_{z_1}=i} \sup_{y \in \mathbb{Z}^{d-1}} \mathbf{1}_{A_i^c} P(W_i^\perp = y | \mathcal{F}_i)) \\
\leq\ & c_2 z_1^{-r(2-\alpha)/2} + z_1 e^{-c_1 z_1^\alpha / 2v} + c_3 (z_1/2v)^{-\alpha(d-1)/2} \sum_{i=z_1^\alpha/2v}^{z_1} P(T_{z_1} = i)\,,
\end{aligned} \tag{3.18}$$

where the second inequality uses (3.16), and the fifth uses (3.15) and (3.17). The estimate (3.18) yields (3.13) by choosing $\alpha = 2r/(r + d - 1) \leq 1$.

To see (3.14), note that the sum of probabilities is exactly the expected number of visits to $\{z : \langle z, e_1 \rangle = K\}$, which is bounded by 1. □

We are now ready to state and prove the following analogue of Proposition 3.1.

**Proposition 3.4** *($d = 4$) With notation as in Proposition 3.1, $d = 4$ and $r$ in (1.4) satisfying $r > 8$, we have*

$$\mathbf{P}^D\left(\{Y_1(\cdot)\} \cap \{Y_2(\cdot)\} \neq \emptyset\right) < C\|U_1 - U_2\|^{-K_4}$$

*where $K_4 > 0$.*

*Proof.* Fix $\nu > 0$ and write $U = |U_1 - U_2|$. Let $\{v_i\}_{i\geq 1}$ denote an i.i.d. sequence of random variables, with $v_1$ distributed like $Y_1(t^{(2)}) - Y_1(t^{(1)})$ under $\mathbf{P}^D$. This sequence clearly satisfies the assumptions of Lemma 3.3, with $\delta = \kappa^2\mathbb{P}(D)$.

Let $T := \mathbb{E}(t^{(2)} - t^{(1)})$. By our assumption on the tails of regeneration times, for $\nu \in (0,1)$ with $\nu r > 1$,

$$\mathbf{P}^D\left(\exists_{i\geq \frac{U}{8T}} : t_1^{(i+1)} - t_1^{(i)} > i^\nu\right) \leq \sum_{i=U/8T}^{\infty} \frac{C}{i^{\nu r}} \leq CU^{1-\nu r}\,. \tag{3.19}$$

By Doob's maximal inequality, and our assumption on the tails of regeneration times,

$$\begin{aligned}
\mathbf{P}^D\left(\exists_{i\geq \frac{U}{8T}} : t_1^{(i)} > 2Ti\right) &\leq \mathbf{P}^D\left(\exists_{i\geq \frac{U}{8T}} : (t_1^{(i)} - \mathbb{E}t_1^{(i)}) > Ti\right) \\
&\leq \sum_{j=0}^{\infty} \mathbf{P}^D\left(\exists_{i\in[\frac{2^jU}{8T}, \frac{2^{j+1}U}{8T})} : (t_1^{(i)} - \mathbb{E}t_1^{(i)}) > Ti\right) \\
&\leq \sum_{j=0}^{\infty} \mathbf{P}^D\left(\exists_{i\leq \frac{2^{j+1}U}{8T}} : (t_1^{(i)} - \mathbb{E}t_1^{(i)}) > 2^jU/8\right) \\
&\leq C\sum_{j=0}^{\infty} \frac{1}{(2^jU)^{r/2}} \leq \frac{C}{U^{r/2}}\,.
\end{aligned} \tag{3.20}$$

For integer $k$ and $i = 1, 2$, let $s_{k,i} = \max\{n : \langle Y_i(t_i^{(n)}), e_1\rangle \leq k\}$. Let

$$\mathcal{A}_{i,U,\nu} := \cap_{k\geq U/8T}\{t_i^{(s_{k,i}+1)} - t_i^{(s_{k,i})} \leq (2Tk)^\nu\}\,.$$

Combining (3.20) and (3.19), we get

$$\mathbf{P}^D\left((\mathcal{A}_{i,U,\nu})^c\right) \leq C[U^{1-\nu r} + U^{-r/2}]\,. \tag{3.21}$$

For an integer $K$, set $\mathcal{C}_K = \{z \in \mathbb{Z}^d : \langle z, e_1\rangle = K\}$. Note that on the event $\mathcal{A}_{1,U,\nu} \cap \mathcal{A}_{2,U,\nu}$, if the paths $Y_1(\cdot)$ and $Y_2(\cdot)$ intersect at a point $z \in \mathcal{C}_K$, then there exist integers $\alpha, \beta$ such that $|Y_1(t_1^{(\alpha)}) - Y_2(t_2^{(\beta)})| \leq 2(2TK)^\nu$. Therefore, with $W_n = \sum_{i=1}^n v_i$, we get from (3.20) and (3.21) that, with $r_0 = r \wedge 3$,

$$\begin{aligned}
&\mathbf{P}^D\left(\{Y_1(\cdot)\} \cap \{Y_2(\cdot)\} \neq \emptyset\right) \\
\leq\ & 2\mathbf{P}^D\left(t_1^{(U/8T)} \geq U/2\right) + 2\mathbf{P}^D\left((\mathcal{A}_{i,U,\nu})^c\right) \\
&+ \sum_{K>U/8T} \sum_{z\in\mathcal{C}_K} \sum_{z':|z-z'|<2(2TK)^\nu} \mathbf{P}(\exists i : W_i = z)\mathbf{P}(\exists j : W_j = z') \\
\leq\ & C\left[U^{-\epsilon} + U^{1-\nu r} + U^{-r/2} + U^{\left[1-\frac{3r_0}{r_0+3}+4\nu\right]}\right]\,,
\end{aligned} \tag{3.22}$$

as long as $1 - 3r_0/(r_0+3) + 4\nu < 0$, where Lemma 3.3 and (3.21) were used in the last inequality. With $r > 8$ (and hence $r_0 = 3$), one can chose $\nu > 1/r$ such that all exponents of $U$ in the last expression are negative, yielding the conclusion. $\square$

Equivalently to Corollary 3.2, the following is an immediate consequence of the last line of (3.22)

**Corollary 3.5** *With notation as in Corollary* 3.2, $d = 4$ *and* $r$ *in (*1.4*) satisfying* $r > 8$*, we have*

$$\mathbf{P}^D(A_1 \cap A_2) < CR^{-K_4'}.$$

*with* $K_4' = K_4'(r) > 0$.

*3.2.3 Main estimate for random walks in independent environments.*

Let $R > 0$ and let $T_i^Y(R) := \min\{n : Y_i(n) \in H_R\}$.

**Proposition 3.6** *(*$d \geq 4$*) Let* $Y_1(\cdot)$ *and* $Y_2(\cdot)$ *be random walks in independent environments satisfying the assumptions of Theorem* 1.1*, with starting points* $U_1, U_2$ *satisfying* $\langle U_1, e_1\rangle = \langle U_2, e_1\rangle = 0$*. Let*

$$A(R) := \tag{3.23}$$
$$\{\forall_{n<T_1^Y(R)}\langle Y_1(n), e_1\rangle \geq 0\} \cap \{\forall_{m<T_2^Y(R)}\langle Y_2(m), e_1\rangle \geq 0\} \cap \{\forall_{n<T_1^Y(R),m<T_2^Y(R)} Y_1(n) \neq Y_2(m)\}.$$

*Then,*

1. *There exists* $\rho > 0$ *such that for every choice of* $R$ *and* $U_1, U_2$ *as above,*
$$\mathbf{P}\left(A(R)\right) > \rho. \tag{3.24}$$
2. *Let* $\hat{B}_i(n)$ *be the event that* $Y_i(\cdot)$ *has a regeneration time at* $T_i^Y(n)$*, and let*
$$B_i(R) := \bigcup_{n=R/2}^{R} \hat{B}_i(n). \tag{3.25}$$
*Then*
$$\mathbf{P}\left(\{\{Y_1(n)\}_{n=1}^\infty \cap \{Y_2(m)\}_{m=1}^\infty \neq \emptyset\} \cap A(R) \cap B_1(R) \cap B_2(R)\right) < CR^{-\beta_d} \tag{3.26}$$
*with* $\beta_d = \beta_d(r, \epsilon) > 0$ *for* $d \geq 4$.

*Proof.* To see (3.24), note first that due to uniform ellipticity, we may and will assume that $|U_1 - U_2| > C$ for a fixed arbitrary large $C$. Since $\zeta := \mathbb{P}(D_1 \cap D_2) > 0$ does not depend on the value of $C$, the claim then follows from Propositions 3.1 and 3.4 by choosing $C$ large enough such that $\mathbf{P}^D\left(A(R)^c\right) < \zeta/2$.

To see (3.26), note the event $A(R) \cap B_1(R) \cap B_2(R)$ implies the event $D_1^{U_1} \cap D_2^{U_2}$, and further if $\{Y_1(n)\}_{n=1}^\infty \cap \{Y_1(m)\}_{m=1}^\infty \neq \emptyset$ then for $i = 1, 2$ the closest intersection point to $U_i$ is at distance greater than or equal to $R/2$ from $U_i$. Therefore (3.26) follows from Corollary 3.2 and Corollary 3.5. □

### 3.3 Intersection of paths in the same environment.

In this subsection we take $\{X_1(n)\}$ and $\{X_2(n)\}$ to be random walks in the same environment $\omega$, with $X_i(0) = U_i$, $i = 1, 2$, and $\omega$ distributed according to $P$. As in subsection 3.2, we also consider $\{Y_1(n)\}$ and $\{Y_2(n)\}$, two independent random walks evolving in independent environments, each distributed according to $P$. We continue to use $\mathbb{P}^{U_1,U_2}$ (or, for brevity, $\mathbb{P}$) for the annealed law of the pair $(X_1(\cdot), X_2(\cdot))$, and $\mathbf{P}$ for the annealed law of the pair $(Y_1(\cdot), Y_2(\cdot))$. Note that $\mathbf{P} \neq \mathbb{P}$. Our next proposition is a standard statement, based on coupling, that will allow us to use some of the results from Section 3.2, even when the walks evolve in the same environment and we consider the law $\mathbb{P}$.

In what follows, a stopping time $T$ with respect to the filtration determined by a path $X$ will be denoted $T(X)$.

**Proposition 3.7** *With notation as above, let* $T_i(\cdot)$, $i = 1, 2$ *be stopping times such that* $T_i(X_i)$, $i = 1, 2$ *are* $\mathbb{P}$*-almost surely finite. Assume* $X_1(0) = Y_1(0)$ *and* $X_2(0) = Y_2(0)$*. Set*

$$I_X := \left\{\{X_1(n)\}_{n=0}^{T_1(X_1)} \bigcap \{X_2(n)\}_{n=0}^{T_2(X_2)} = \emptyset\right\}$$

*and*

$$I_Y := \left\{ \{Y_1(n)\}_{n=0}^{T_1(Y_1)} \bigcap \{Y_2(n)\}_{n=0}^{T_2(Y_2)} = \emptyset \right\} .$$

*Then, for any nearest neighbor deterministic paths* $\{\lambda_i(n)\}_{n\geq 0}$, $i = 1, 2$,

$$\begin{aligned}&\mathbf{P}\left(Y_i(n) = \lambda_i(n), 0 \leq n \leq T_i(Y_i), i = 1, 2; I_Y\right) \\ &= \mathbb{P}\left(X_i(n) = \lambda_i(n), 0 \leq n \leq T_i(X_i), i = 1, 2; I_X\right) .\end{aligned} \tag{3.27}$$

*Proof.* For every pair of non-intersecting paths $\{\lambda_i(n)\}_{n\geq 0}$, define three i.i.d. environments $\omega^{(1)}$, $\omega^{(2)}$ and $\omega^{(3)}$ as follows: Let $\{J(z)\}_{z\in\lambda_1\cup\lambda_2}$ be a collection of i.i.d. variables, of marginal law $Q$. At the same time, let $\{\eta^j(z)\}_{z\in\mathbb{Z}^d}$, $j = 1, 2, 3$ be three independent i.i.d. environments, each $P$-distributed. Then define

$$\omega^{(1)}(z) = \begin{cases} J(z) & \text{if } z \in \lambda^{(1)} \\ \eta^{(1)}(z) & \text{otherwise,} \end{cases}$$

$$\omega^{(2)}(z) = \begin{cases} J(z) & \text{if } z \in \lambda^{(2)} \\ \eta^{(2)}(z) & \text{otherwise,} \end{cases}$$

and

$$\omega^{(3)}(z) = \begin{cases} J(z) & \text{if } z \in \lambda^{(1)} \cup \lambda^{(2)} \\ \eta^{(3)}(z) & \text{otherwise,} \end{cases}$$

and let $Y_1$ evolve in $\omega^{(1)}$, let $Y_2$ evolve in $\omega^{(2)}$ and let $X_1$ and $X_2$ evolve in $\omega^{(3)}$. Then by construction,

$$P_{\omega^{(1)},\omega^{(2)}}\left(Y_i(n) = \lambda_i(n), 0 \leq n \leq T_i(Y_i)\right) = P_{\omega^{(3)}}\left(X_i(n) = \lambda_i(n), 0 \leq n \leq T_i(X_i)\right) .$$

Integrating and then summing we get (3.27). □

An immediate consequence of Proposition 3.7 is that the estimates of Proposition 3.6 carry over to the processes $(X_1(\cdot), X_2(\cdot))$. More precisely, let $R > 0$ be given and set $T_i^X(R) := \min\{n : X_i(n) \in H_R\}$. Define $A(R)$ and $B_i(R)$ as in (3.23) and (3.25), with the process $X_i$ replacing $Y_i$.

**Corollary 3.8** *($d \geq 4$) Let $X_1(\cdot)$ and $X_2(\cdot)$ be random walks in the same environment satisfying the assumptions of Theorem* 1.1*, with starting points $U_1, U_2$ satisfying $\langle U_1, e_1\rangle = \langle U_2, e_1\rangle = 0$. Then,*

1. *There exists $\rho > 0$ such that for every choice of $R$ and $U_1, U_2$ as above,*
$$\mathbb{P}\left(A(R)\right) > \rho. \tag{3.28}$$
2. *With $C < \infty$ and $\beta_d > 0$ as in (*3.26*),*
$$\mathbb{P}\left(\{\{X_1(n)\}_{n=1}^{\infty} \cap \{X_2(m)\}_{m=1}^{\infty} \neq \emptyset\} \cap A(R) \cap B_1(R) \cap B_2(R)\right) < CR^{-\beta_d} . \tag{3.29}$$

With $\beta_d$ as in (3.29) and $\epsilon$ as in the statement of Theorem 1.1, fix $0 < \psi_d$ satisfying

$$\psi_d < \beta_d(1 - \psi_d) \text{ and } (1 + \epsilon)(1 - \psi_d) > 1 . \tag{3.30}$$

For $R$ integer, let

$$\mathcal{K}_k(R) = \{\exists_{(k+0.5)R^{1-\psi_d}<j<(k+1)R^{1-\psi_d}} \text{ s.t. } T_i(j) \text{ is a regeneration time for } X_i(\cdot)\} ,$$

and let

$$C_i(R) := \bigcap_{k=1}^{\lceil 2R^{\psi_d}\rceil} \mathcal{K}_k(R). \tag{3.31}$$

Proposition 2.1 will follow from the following lemma:

**Lemma 3.9** *($d \geq 4$) Under the assumptions of Theorem* 1.1*, there exist constants $C$ and $\gamma_d > 0$ such that for all integer $K$,*

$$\mathbb{P}\left(W_K \cap \ C_1(K) \cap C_2(K)\right) < CK^{-\gamma_d} .$$

*Proof of Lemma* 3.9. Let $w := [K^{1-\psi_d}]$ and for $k = 1, \ldots, [K^{\psi_d}/2]$ define the event

$$S_k = \left\{ \begin{array}{cc} \forall_{T_1(kw) \leq j < T_1((k+1)w)} X_1(j) > kw & \\ & \text{and} \\ \forall_{T_2(kw) \leq j < T_2((k+1)w)} X_2(j) > kw & \\ & \text{and} \\ \{X_1(j)\}_{j=T_1(kw)}^{T_1((k+1)w)-1} \bigcap \{X_2(j)\}_{j=T_2(kw)}^{T_2((k+1)w)-1} = \emptyset & \end{array} \right\} \tag{3.32}$$

By (3.28),

$$\mathbb{P}\left(S_k | S_1^c \cap S_2^c \cap \cdots \cap S_{k-1}^c\right) \geq \rho .$$

Therefore,

$$\mathbb{P}\left(\cup_k S_k\right) \geq 1 - (1-\rho)^{[K^{\psi_d}/2]}. \tag{3.33}$$

Now, by (3.29),

$$\mathbb{P}\left(S_k \cap C_1(K) \cap C_2(K) \cap W_K\right) < Cw^{-\beta_d} = CK^{-\beta_d(1-\psi_d)}.$$

We therefore get that

$$\mathbb{P}\left(\cup_k S_k \cap C_1(K) \cap C_2(K) \cap W_K\right) < CK^{-\beta_d(1-\psi_d)} K^{\psi_d} = CK^{\psi_d - \beta_d(1-\psi_d)}.$$

Combined with (3.33), we get that

$$\mathbb{P}\left(\{X_1(\cdot)\} \cap \{X_2(\cdot)\} \cap H_K \neq \emptyset \ \cap \ C_1(K) \cap C_2(K)\right) < CK^{-\gamma_d}$$

for every choice of $\gamma_d < \beta_d(1-\psi_d) - \psi_d$. □

*Proof of Proposition* 2.1. Note that by the moment conditions on the regeneration times,

$$\mathbb{P}\left(C_i(K)^c\right) \leq CK^{-\epsilon(1-\psi_d)} + CK \cdot K^{-(1+\epsilon)(1-\psi_d)} = CK^{-\epsilon(1-\psi_d)} + CK^{1-(1+\epsilon)(1-\psi_d)}.$$

By the choice of $\psi_d$, see (3.30), it follows that (2.1) holds for

$$\kappa_d < \min\left\{(1+\epsilon)(1-\psi_d) - 1, \gamma_d\right\}.$$

□

## 4. ADDENDUM - $d = 2, 3$

After the first version of this work was completed and circulated, F. Rassoul-Agha and T. Seppäläinen have made significant progress in their approach to the CLT, and posted an article [RAS07b] in which they derive the quenched CLT for all dimensions $d \geq 2$, under a somewhat stronger assumption on the moments of regeneration times than (1.4). (In their work, they consider finite range, but not necessarily nearest neighbor, random walks, and relax the uniform ellipticity condition.) While their approach is quite different from ours, it incorporates a variance reduction step that, when coupled with the techniques of this paper, allows one to extend Theorem 1.1 to all dimensions $d \geq 2$, with a rather short proof. In this addendum, we present the result and sketch the proof.

**Theorem 4.1** *Let $d = 2, 3$. Let $Q$ and $\{X(n)\}$ be as in Theorem* 1.1*, with $\epsilon = r \geq 40$. Then, the conclusions of Theorem* 1.1 *still hold.*

**Remark:** The main contribution to the condition $r \geq 40$ comes from the fact that one needs to transfer estimates on regenerations times in the direction $e_1$ to regeneration times in the direction $v$, see Lemma 4.5 below. If $e_1 = v$, or if one is willing to assume moment bounds directly on the regeneration times in direction $v$, then the same proof works with $\epsilon > 0$ arbitrary and $r > 14$.

*Proof of Theorem* 4.1 *(sketch).* The main idea of the proof is that the condition “no late intersection of independent random walks in the same environment” may be replaced by the condition “intersections of independent random walks in the same environment are rare”.

Recall, c.f. the notation and proof of Theorem 1.1, that we need to derive a polynomially decaying bound on Var$(E_\omega F(\beta^N))$ for $F : C([0,1],\mathbb{R}^d) \to \mathbb{R}$ bounded Lipschitz and $\beta^N$ the polygonal interpolation as in (2.2). In the sequel, we write $F^N(X) := F(\beta^N)$ if $\beta^N$ is the polygonal interpolation of the scaling (as in (2.2)) of the path $\{X_n\}_{n=0,\ldots,N}$.

For any $k$, let $S_k = \min\{n : X_n \in H_k\}$. For two paths $p_1, p_2$ of length $T_1, T_2$ with $p_i(0) = 0$, let $p_1 \circ p_2$ denote the concatenation, i.e.

$$p_1 \circ p_2(t) = \begin{cases} p_1(t)\,, & t \leq T_1 \\ p_1(T_1) + p_2(t - T_1)\,, & t \in (T_1, T_2]\,. \end{cases}$$

Use the notation $X_i^j = \{X_i, \ldots, X_2, \ldots, X_j\}$. Then, we can write, for any $k$,

$$F^N(X_0^N) = F^N(X_0^{S_k \wedge N} \circ [X^N_{S^k \wedge N} - X_{S^k \wedge N}])\,.$$

Now comes the main variance reduction step, which is based on martingale differences. Order the vertices in an $L^1$ ball of radius $N$ centered at 0 in $\mathbb{Z}^d$ in lexicographic order $\ell(\cdot)$. Thus, $z$ is the predecessor of $z'$, denoted $z = p(z')$, if $\ell(z') = \ell(z) + 1$. Note that (because of our choice of lexicographic order), if $z_1 < z_1'$ then $\ell(z) < \ell(z')$.

Let $\delta > 1/r$ be given such that $2\delta < 1$. Define the event

$$W_N := \{\exists i \in [0, N] : t^{(i+1)} - t^{(i)} > N^\delta/3 \text{ or } t^{(i+N^\delta)} - t^{(i)} > N^{3\delta/2}\}\,.$$

By our assumptions, we have that $\mathbb{P}(W_N) \leq C(N^{-\epsilon\delta} + N^{1-\delta r})$, and hence decays polynomially.

$$\text{Var}(E_\omega F(\beta^N)) \leq \text{Var}(E_\omega F(\beta^N)\mathbf{1}_{W_N^c}) + O(N^{-\delta'})\,,$$

for some $\delta' > 0$. In the sequel we write $\bar{F}^N(X) = F^N(X)\mathbf{1}_{W_N^c}$.

Set $\mathcal{G}_z^N := \sigma(\omega_x : \ell(x) \leq \ell(z), \|x\|_1 \leq N)$, and write $\hat{H}_k = \{z : \langle z, e_1\rangle = k\}$. We have the following martingale difference representation:

$$\begin{aligned} E_\omega \bar{F}^N(X) - \mathbb{E}\bar{F}^N(X) &= \sum_{z:|z|_1 \leq N} \left[\mathbb{E}\left(\bar{F}^N(X)|\mathcal{G}_z\right) - \mathbb{E}\left(\bar{F}^N(X)|\mathcal{G}_{p(z)}\right)\right] \\ &=: \sum_{k=-N}^{N} \sum_{z \in \hat{H}_k, |z|_1 \leq N} \Delta_z^N\,. \end{aligned} \tag{4.1}$$

Because it is a martingale differences representation, we have

$$\text{Var}(E_\omega \bar{F}^N(X)) = \sum_{k=-N}^{N} \sum_{z \in \hat{H}_k, |z|_1 \leq N} \mathbb{E}\left(\Delta_z^N\right)^2\,. \tag{4.2}$$

Because of the estimate $\mathbb{E}[(t^{(1)})^\epsilon] < \infty$, the Lipschitz property of $F$, and our previous remarks concerning $W_N$, the contribution of the terms with $k \leq 2N^\delta$ to the sum in (4.2) decays polynomially. To control the terms with $k > 2N^\delta$, for $z \in \hat{H}_k$ let $\tau_z$ denote the largest regeneration time $t^{(i)}$ smaller than $S_{k-N^\delta}$, and write $\tau_z^+$ for the first regeneration time larger than $S_{k+N^\delta}$. Then,

$$F^N(X) = F^N(X_0^{\tau_z} \circ [X_{\tau_z}^{\tau_z^+} - X_{\tau_z}] \circ [X_{\tau_z^+}^N - X_{\tau_z^+}]) \,.$$

Because of the Lipschitz property of $F$, our rescaling, and the fact that we work on the event $W_N^c$, we have the bound

$$|\bar{F}^N(X_0^{\tau_z} \circ [X_{\tau_z}^{\tau_z^+} - X_{\tau_z}] \circ [X_{\tau_z^+}^N - X_{\tau_z^+}]) - \bar{F}^N(X_0^{\tau_z} \circ [X_{\tau_z^+}^N - X_{\tau_z^+}])| \le 4N^{(3\delta-1)/2} \,.$$

One then obtains by standard manipulations

$$\mathbb{E}\left(\Delta_z^N\right)^2 \le CN^{3\delta-1} E[(E_\omega[\mathbf{1}_{X \text{ visits z}}])^2] \,.$$

Let $I_N$ denote the number of intersections, up to time $N$, of two independent copies of $\{X(n)\}_{n\ge 0}$ *in the same environment*. Then,

$$\sum_{z:\|z\|_1 \le N} [E(E_\omega[\mathbf{1}_{X \text{ visits z}}])^2] = E(E_{\omega\times\omega} I_N) \,. \tag{4.3}$$

Combining these estimates, we conclude that

$$\text{Var}(E_\omega \bar{F}^N(X)) \le CN^{3\delta-1} E(E_{\omega\times\omega} I_N) + N^{-\delta'} \,. \tag{4.4}$$

The proof of Theorem 4.1 now follows from the following lemma.

**Lemma 4.2** *Under the assumptions of Theorem* 4.1*, for* $d \ge 2$ *and* $r > 40$*, we have that for* $r' < r/4 - 1/2$ *and any* $\epsilon' \in (0, 1/2 - 4/r' + 2/(r')^2)$,

$$[E(E_{\omega\times\omega} I_N)] \le CN^{1-\epsilon'} \,, \tag{4.5}$$

*where* $C$ *depends only on* $\epsilon'$.

Indeed, equipped with Lemma 4.2, we deduce from (4.4) that

$$\text{Var}(E_\omega F^N(X)) \le N^{-\delta'} + CN^{1-\epsilon'} N^{3\delta-1} \,.$$

Thus, whenever $\delta > 1/r$ is chosen such that $3\delta < \epsilon'$, (which is possible as soon as $r > 3/\epsilon'$, which in turn is possible for some $\epsilon' < 1/2 - 4/r' + 2/(r')^2$ if $r \ge 40$), $\text{Var}(E_\omega F^N(X)) \le CN^{-\delta}$, for some $\delta > 0$. As mentioned above, this is enough to conclude. □

Before proving Lemma 4.2, we need the following estimate:

**Lemma 4.3** *Let* $S_n$ *be an i.i.d. random walk on* $\mathbb{R}$ *with* $ES_1 = 0$ *and* $E|S_1|^r < \infty$ *for* $r > 3$*. Let* $U_n$ *be a sequence of events such that, for some constant* $a_3 > 3/2$*, and all* $n$ *large,*

$$P(U_n) \ge 1 - \frac{1}{n^{a_3}} \,. \tag{4.6}$$

*In addition we assume that* $\{U_k\}_{k<n}$ *is independent of* $\{S_k - S_n\}_{k\ge n}$ *for every* $n$.

*Let* $a_1 \in (0,1)$ *and* $a_2 > 0$ *be given. Suppose further that for any* $n$ *finite,*

$$P(\text{for all } t \le n,\, S_t \ge \lfloor t^{\frac{a_1}{2}} \rfloor \text{ and } U_t \text{ occurs}) > 0 \,.$$

*Then, there exists a constant* $C = C(a_1, a_2, a_3) > 0$ *such that for any* $T$,

$$P(\text{for all } t \le T,\, S_t \ge \lfloor t^{\frac{a_1}{2}} \rfloor \text{ and } U_t \text{ occurs}) \ge \frac{C}{T^{1/2+a_2}} \,. \tag{4.7}$$

*Proof.* Fix constants $\bar\epsilon > 0$, $\alpha \in (0,1)$ and $\beta \in (1,2)$ (eventually, we will take $\alpha \to 1, \beta \to 2$ and $\bar\epsilon \to \infty$). Throughout the proof, $C$ denote constants that may change from line to line but may depend only on these parameters. Define $b_i = \lfloor i^{\alpha\bar\epsilon} \rfloor$ and $c_i = \lceil i^{\bar\epsilon+1} \rceil$. Consider the sequence of stopping times $\tau_0 = 0$ and

$$\tau_{i+1} = \min\{n > \tau_i : S_n - S_{\tau_i} > c_{i+1} - c_i \text{ or } S_n - S_{\tau_i} < b_{i+1} - c_i\}.$$

Declare an index $i$ good if $S_{\tau_i} - S_{\tau_{i-1}} = c_i - c_{i-1}$. Note that if the indices $i = 1, \ldots, K$ are all good, then $S_n \geq b_{i-1}$ for all $n \in (\tau_{i-1}, \tau_i]$, $i = 1, \ldots, K$.

Let the overshoot $O_i$ of $\{S_n\}$ at time $\tau_i$ be defined as $S_{\tau_i} - S_{\tau_{i-1}} - (c_i - c_{i-1})$ if $i$ is good and $S_{\tau_i} - S_{\tau_{i-1}} - (b_i - c_{i-1})$ if $i$ is not good. By standard arguments (see e.g. [RAS07b, Lemma 3.1]), $E(|O_i|^{r-1}) < \infty$. By considering the martingale $S_n$, we then get

$$P(i \text{ is good}) \sim (1 - \frac{1+\bar\epsilon}{i}), \tag{4.8}$$

as $i \to \infty$. By considering the martingale $S_n^2 - nES_1^2$, we get

$$E(\tau_{i+1} - \tau_i) = \Omega(i^{1+2\bar\epsilon}), \tag{4.9}$$

as $i \to \infty$. In particular,

$$P(\tau_{i+1} - \tau_i > i^{2+2\bar\epsilon+\delta}) \leq \frac{C}{i^{1+\delta}}, \tag{4.10}$$

while, from our assumption on the moments of $S_1$ and Doob's inequality,

$$P(\tau_{i+1} - \tau_i \leq i^{\bar\epsilon\beta}) \leq \frac{C}{i^{r\bar\epsilon(2-\beta)/2}}. \tag{4.11}$$

We assume in the sequel that $r\bar\epsilon(2-\beta)/2 > 2$ and that $a_3(\bar\epsilon\beta + 1) > 5 + 3\bar\epsilon + \delta$ (both these are possible by choosing any $\beta < 2$ so that $a_3\beta > 3$, and then taking $\bar\epsilon$ large). We say that $i+1$ is *very good* if it is good and in addition, $\tau_{i+1} - \tau_i \in [i^{\bar\epsilon\beta}, i^{2+2\bar\epsilon+\delta}]$. By (4.8), (4.10) and (4.11), we get

$$P(i \text{ is very good}) \sim (1 - \frac{1+\bar\epsilon}{i}). \tag{4.12}$$

Declare an index $i$ *excellent* if $i$ is very good and in addition, $U_n$ occurs for all $n \in [\tau_{i-1}, \tau_i)$.

On the event that the first $K$ $i$'s are very good, we have that $\tau_{K-1} \geq CK^{\bar\epsilon\beta+1}$ and $\tau_K \leq CK^{3+2\bar\epsilon+\delta} =: T_K$, and $S_n \geq K^{\bar\epsilon\alpha}$ for $n \in [\tau_{K-1}, \tau_K]$. Letting $\mathcal{M}_K$ denote the event that the first $K-1$ $i$'s are excellent, and $K$ is very good, we then have, for every $n$,

$$P(U_n^c \mathbf{1}_{n \in [\tau_{K-1}, \tau_K)} | \mathcal{M}_K) \leq K^{-a_3(\bar\epsilon\beta+1)} / P(\mathcal{M}_K).$$

We now show inductively that $P(\mathcal{M}_K) \geq C/K^{1+\bar\epsilon}$. Indeed, under the above hypotheses, we get

$$P(U_n^c \text{ for some } n \in [\tau_{K-1}, \tau_K) | \mathcal{M}_K) \leq K^{1+\bar\epsilon+3+2\bar\epsilon+\delta-a_3(\bar\epsilon\beta+1)} = K^{4+\delta-a_3+\bar\epsilon(3-a_3\beta)}$$

and thus, with our choice of constants and (4.8), we conclude that under the above hypothesis,

$$P(\mathcal{M}_{K+1} | \mathcal{M}_K) \sim (1 - \frac{1+\bar\epsilon}{K+1}). \tag{4.13}$$

We thus get inductively that the hypothesis propagates and in particular we get

$$P(i \text{ is excellent for } i \leq K) \geq \frac{C}{K^{1+\bar\epsilon}}. \tag{4.14}$$

Further, if the first $K$ $i$'s are excellent (an event with probability bounded below by $C/K^{1+\bar{\epsilon}}$), we have that $\tau_K \le T_K$. Note that if $t = CK^{\bar{\epsilon}\beta+1}$ then on the above event we have that by time $t$, at least $Ct^{1/(3+2\bar{\epsilon}+\delta)}$ of the $\tau_i$'s are smaller than $t$, and hence $S_t \ge Ct^{\bar{\epsilon}\alpha/(3+2\bar{\epsilon}+\delta)}$. We thus conclude that, for all $T$ large,

$$P(\text{for all } t \le T,\, S_t \ge t^{\bar{\epsilon}\alpha/(3+2\bar{\epsilon}+\delta)},\text{ and } U_t \text{ occurs}) \ge \frac{C}{T^{(1+\bar{\epsilon})/(1+\bar{\epsilon}\beta)}}\,.$$

Taking $\bar{\epsilon}$ large and $\beta$ close to 2 (such that still $r\bar{\epsilon}(2-\beta) > 2$), and $\alpha$ close to 1, completes the proof. $\square$

*Proof of Lemma* 4.2 *(sketch).* Let

$$v = \lim_{n\to\infty} \frac{X_n}{n} \ne 0$$

be the limiting direction of the random walk, and let $u$ be a unit vector which is orthogonal to $v$.

In what follows we will switch from the regenerations in direction $e_1$ that we used until now, and instead use regenerations in the direction $v$, whose definition, given below, is slightly more general than the definition of regenerations in the direction $e_1$ given in Section 1.

**Definition 4.4** *We say that $t$ is a regeneration time for $\{X_n\}_{n=1}^{\infty}$ in direction $v$ if*

- $\langle X_s, v\rangle \le \langle X_{t-1}, v\rangle$ *for every* $s < t-1$.
- $\langle X_t, v\rangle > \langle X_{t-1}, v\rangle$.
- $\langle X_s, v\rangle \ge \langle X_t, v\rangle$ *for every* $s > t$.

We denote by $t^{v,(n)}$ the succesive regeneration times of the RWRE $X_n$ in direction $v$ (when dealing with two RWRE's $X_i(n)$, we will use the notation $t_i^{v,(n)}$). The sequence $t^{v,(n+1)} - t^{v,(n)}$, $n \ge 1$, is still i.i.d., and with $D^v$ defined in the obvious way, the law of $t^{v,(2)} - t^{v,(1)}$ is identical to the law of $t^{v,(1)}$ conditioned on the event $D^v$. The following lemma, of maybe independent interest, shows that, up to a fixed factor, the regeneration time $t^{v,(1)}$ (and hence, also $t^{v,(2)} - t^{v,(1)}$) inherits moment bounds from $t^{(1)}$.

**Lemma 4.5** *Assume* $r > 10$ *and* $\mathbb{E}((t^{(1)})^r) < \infty$. *Then* $\mathbb{E}(\langle X_{t^{v,(1)}}, v\rangle)^{2r'} < \infty$ *and* $\mathbb{E}((t^{v,(1)})^{r'}) < \infty$ *with* $r' < r/4 - 1/2$.

*Proof.* On the event $(D^v)^c$, define $\tau_0 = \min\{n > 0 : \langle X_n, v\rangle \le 0\}$ and set $M = \max\{\langle X_n, v\rangle : n \in [0, \tau_0]\}$. By [Szn02, Lemma 1.2], $\langle X_{t^{v,(1)}}, v\rangle$ is (under the annealed law) stochastically dominated by the sum of a geometric number of independent copies of $M+1$. Hence, if $\mathbb{E}[M^p|(D^v)^c] < \infty$ for some $p$, then $\mathbb{E}|\langle X_{t^{v,(1)}}, v\rangle|^p < \infty$.

Fix a constant $\chi < 1/2(\mathbb{E}t^{(1)} \wedge \mathbb{E}(t^{(2)} - t^{(1)}))$ small enough so that $(2 + 2\|v\|_2)\chi < \|v\|_2^2$. Now fix some (large) number $x$. On the event $M > x$, either

- $t^{(\chi x)} \ge x$

or

- $t^{(k+1)} - t^{(k)} \ge \chi k \quad$ for some $k > \chi x$

or

- $\{|t^{(k)} - \mathbb{E}t^{(k)}| > \chi k\}$ or $\{\|X_{t^{(k)}} - \mathbb{E}X_{t^{(k)}}\| > \chi k\}$ for some $k > \chi x$.

(Indeed, on the event $M > x$ with $x$ large, the RWRE has to satisfy that at some large time $t > x$, $\langle X_t, v\rangle$ is close to 0 instead of close to $\|v\|_2^2 t$.)

Due to the moment bounds on $t^{(1)}$ and $t^{(2)} - t^{(1)}$, and the chosen value of $\chi$, we have $\mathbb{P}(t^{(\chi x)} \ge x) \le Cx^{-r/2}$. We also have

$$\mathbb{P}(t^{(k+1)} - t^{(k)} \ge \chi k\,, \quad \text{some } k > \chi x) \le C \sum_{k=\chi x}^{\infty} k^{-r} = Cx^{-r+1}\,,$$

and

$$\mathbb{P}(\{|t^{(k)}-\mathbb{E}t^{(k)}|>\chi k\} \text{ or } \{\|X_{t^{(k)}}-\mathbb{E}X_{t^{(k)}}\|>\chi k\}\,,\ \text{some } k>\chi x)\leq C\sum_{k=\chi x}^{\infty} k^{-r/2}=Cx^{-r/2+1}\,.$$

We conclude that $\mathbb{E}M^p \leq C + C\int_1^\infty x^{p-1}x^{-r/2+1}dx < \infty$ if $p < r/2-1$. This proves that

$$\mathbb{E}|\langle X_{t^{v,(1)}}, v\rangle|^p < \infty \quad \text{if } p < r/2-1. \tag{4.15}$$

We can now derive moment bounds on $t^{v,(1)}$ (which imply also moment bounds on $\bar{t}^v := t^{v,(2)} - t^{v,(1)}$). Suppose $\mathbb{E}((t^{v,(1)})^{p'}) = \infty$. For any $\epsilon'' > 0$ we can then find a sequence of integers $x_m \to \infty$ such that $\mathbb{P}(t^{v,(1)} > x_m) \geq C/x_m^{p'+\epsilon''}$. Therefore, using (4.15) and the assumed moment bounds,

$$\begin{aligned} &\mathbb{P}(|t^{v,(x_m)}-\mathbb{E}(t^{v,(x_m)})|>x_m/2)\\ \geq\ &\mathbb{P}(t^{v,(1)}-\mathbb{E}(t^{v,(1)})>x_m)\mathbb{P}(|t^{v,(x_m)}-t^{v,(1)}-(x_m-1)\mathbb{E}(\bar{t}^v)|<\chi x_m)\geq Cx_m^{-(p'+\epsilon'')}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(\left|t^{v,(x_m)}-\mathbb{E}\left[t^{v,(x_m)}\right]\right|>x_m/2\ ;\ |\langle X_{t^{v,x_m}},v\rangle-\mathbb{E}\langle X_{t^{v,x_m}},v\rangle|<\chi x_m\right)&\\ \geq \frac{C}{x_m^{p'+\epsilon''}}-\frac{C}{x_m^{p/2}}\geq\frac{C}{x_m^{p'+\epsilon''}}\,,& \end{aligned} \tag{4.16}$$

if $p' < p/2 < r/4 - 1/2$. On the other hand, the event depicted in (4.16) implies that at some time $t$ larger than $x_m$, the ratio $\langle X_t, v\rangle/t$ is not close to $\|v\|_2^2$, an event whose probability is bounded above (using the regeneration times $t^{(n)}$) by

$$Cx_m^{-r/2}+C\sum_{k=Cx_m}^{\infty}k^{-r/2}\leq Cx_m^{1-r/2}\,.$$

Since $1-r/2 < -p'$, we achieved a contradiction. □

Consider temporarily the walks $X_1$ and $X_2$ as evolving in independent environments. We define the following i.i.d. one dimensional random walk:

$$S_n=\left\langle X_1\left(t_1^{v,(n)}\right)-X_2\left(t_2^{v,(n)}\right),u\right\rangle.$$

Set $r' < r/4 - 1/2$. For $\kappa$ and $\eta$ to be determined below, we define the events

$$B_n=\left\{t_i^{v,(n)}-t_i^{v,(n-1)}<n^\eta, i=1,2\right\}\,,$$

$$C_n=\left\{\left|\langle X_i(t_i^{v,(n)})-E(X_i(t_i^{v,(n)})),v\rangle\right|<n^\kappa, i=1,2\right\}\,,$$

$$D_n=\left\{\max_{k\in[n-n^\kappa,n]}|\langle X_i(t_i^{v,(k)})-X_i(t_i^{v,(n)}),u\rangle|<n^\eta, i=1,2\right\}\,,$$

and $U_n = B_n\cap C_n\cap D_n$. By our assumptions, Lemma 4.5, and standard random walk estimates, $P(B_n^c)\leq n^{-\eta r'}$, $P(C_n^c)\leq n^{-r'(2\kappa-1)}$ and $P(D_n^c)\leq n^{-r'(2\eta-\kappa)/2}$. With $r' > 15/2$, choose $\kappa > 1/2, \eta < 1/2$ such that $r'\eta > 3/2$, $r'(2\kappa-1) > 3/2$ and $r'(2\eta-\kappa) > 3$, to deduce that $P(U_n^c) \leq n^{-a_3}$ for some $a_3 > 3/2$. (This is possible with $\eta$ close to $1/2$ and $\kappa$ close to $2(\eta+1)/5$.)

Fix $\eta' \in (0, 1/2-\eta)$ and define the event

$$A(T)=\{\text{for all } n\leq T, S_n\geq \lfloor n^{\frac{1}{2}-\eta'}\rfloor\}\,.$$

Note that there exists $k_0$ such that on the event $A(T)\cap_{n=1}^T U_n$, $X_1[k_0,T/2]\cap X_2[k_0,T/2]=\emptyset$.

From Lemma 4.3 we have that $\mathbb{P}^D(A(T) \cap_{n=1}^T U_n) \geq C/T^{1/2+a_2}$ , for some constant $a_2 > 0$. By ellipticity, this implies

$$\mathbb{P}^D(X_1[1, t_1^{(T)}] \cap X_2[1, t_2^{(T)}] = \emptyset) \geq C/T^{1/2+a_2} \,. \tag{4.17}$$

uniformly over the starting points. (This estimate, which was derived initially for walks in independent environments, obviously holds for walks in the same environment, i.e. under $\mathbb{P}$, too, because it involves a non-intersection event.)

Fix $T$, and let

$$G(T) = \sum_{i,j} \mathbf{1}_{X_1(i)=X_2(j)} \mathbf{1}_{\langle X_1(i),v\rangle \in [T-0.5,T+0.5)}.$$

We want to bound the sum of

$$F(T) = E\left[E_{\omega\times\omega}\left(G(T)\right)\right].$$

**Claim 4.6**

$$\sum_{t=1}^{N} E\left[P_{\omega\times\omega}\left(G(t) \neq 0\right)\right] \leq CN^{1/2+a_2+1/r'} \,. \tag{4.18}$$

*Proof.* We define variables $\{\psi_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=0}^\infty$ inductively as follows:

$$\psi_1 := \max\{t_1^{v,(1)}, t_2^{v,(1)}\}\,, \quad \theta_0 = 0$$

and then, for every $n \geq 1$,

$$\theta_n := \min\{k > \psi_n : G(k) \neq 0\}\,, \quad \psi_{n+1} := \max\{\tau_1(\theta_n), \tau_2(\theta_n)\}\,,$$

with

$$\tau_i(k) := \min\{\langle X_{t_i^{v,(m)}}, v\rangle : \langle X_{t_i^{v,(m)}}, v\rangle > k+1\}.$$

We define $h_n = \psi_n - \theta_{n-1}$ and $j_n = \theta_n - \psi_n$. By (4.17), for every $k$

$$\mathbb{P}(j_n > k | j_1, \ldots, j_{n-1}, h_1, \ldots, h_n) \geq C/k^{1/2+a_2} \,. \tag{4.19}$$

Let

$$K := \min\left\{n : \sum_{i=1}^{n} j_i > N\right\}.$$

Let $Y_i^{(N)} = \max_{k=0}^N [t_i^{v,(k+1)} - t_i^{v,(k)}]$ be the length of the longest of the first $N$ regenerations of $X_i$, $i = 1, 2$, in direction $v$, and set $Y_N = \max(Y_1^{(N)}, Y_2^{(N)})$. Then

$$\sum_{t=1}^{N} \mathbf{1}_{G(t)\neq 0} \leq K \cdot Y_N \,.$$

We see below in (4.21) that $E(Y_N^p) \leq CN^{p/r'}$ for $p < r'$. In addition, by the moment bound (4.19), for any $t$,

$$\mathbf{P}(K > t) = \mathbf{P}\left(\sum_{i=1}^{t} j_i < N\right) \leq \exp\left(-C\frac{t}{N^{\frac{1}{2}+a_2}}\right).$$

From here we get

$$\sum_{t=1}^{N} E\left[P_{\omega\times\omega}\left(G(t) \neq 0\right)\right] \leq CN^{1/2+a_2+1/r'+\epsilon''} \tag{4.20}$$

for every $\epsilon'' > 0$. The fact that $a_2$ was an arbitrary positive number allows the removal of $\epsilon''$ from (4.20).
$\square$

In addition, $G(t)$ is bounded by the product of the length of the $\{X_1\}$ regeneration containing $t$ and that of the $\{X_2\}$ regeneration containing $t$. So for all $t < N$,

$$G(t) \leq Y_1^{(N)} \cdot Y_2^{(N)},$$

and therefore, for any $p < r'/2$,

$$E\left[G(t)^p\right] \leq \sqrt{E((Y_1^{(N)})^{2p})E((Y_2^{(N)})^{2p})} \leq CN^{2p/r'},$$

where in the last inequality we used the estimate

$$E((Y_i^{(N)})^{2p}) \leq A^{2p} + 2pN \int_A^\infty y^{2p-1} P(\tau_i^{(2)} - \tau_i^{(1)} > y)dy \leq A^{2p} + CNA^{2p-r'}, \tag{4.21}$$

with $A = N^{1/r'}$. Thus, with $1/q = (p-1)/p$,

$$\begin{aligned} E[G(t)] &= E[G(t) \cdot \mathbf{1}_{G(t)\neq 0}] \leq \left(EG(t)^p\right)^{1/p} \left(E\left[P_{\omega\times\omega}\left(G(t) \neq 0\right)\right]\right)^{1/q} \\ &\leq CN^{2/r'} \left(E\left[P_{\omega\times\omega}\left(G(t) \neq 0\right)\right]\right)^{1/q}. \end{aligned} \tag{4.22}$$

Thus,

$$E[E_{\omega,\omega} I_N] \leq \sum_{t=1}^N E[G(t)] \leq CN^{2/r'} N^{1/p} \left( \sum_{t=1}^N E\left[P_{\omega\times\omega}\left(G(t)\neq 0\right)\right]\right)^{1/q} \tag{4.23}$$

Using (4.18), we see that

$$E[E_{\omega,\omega} I_N] \leq CN^{\frac{2}{r'} + \frac{1}{p} + (\frac{p-1}{p})(\frac{1}{2} + \frac{1}{r}' + a_2)}.$$

By choosing $2p < r'$ close to $r'$ and $a_2$ small, we can get in the last exponent any power strictly larger than $4/r' + 1/2 - 2/(r')^2$.

□

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