Large Exceedances for Multidimensional Lévy Processes

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Abstract Three results on hitting a rare set by the increments of an $\mathbb{R}^d$ valued random process with stationary independent increments are presented: the first time that it occurs, the duration of such a segment and the typical trajectory during the segment.

1 Introduction

Large exceedances in Markov processes are of theoretical and applied relevance, especially in the context of biomolecular (DNA and protein) data, for assessing statistical significance of a sequence segment composition [KA90, KDK90]. In the context of sequential decision procedures, the false alarm rate in detection of change points by the commonly used CUSUM method corresponds to the location of the first segment with cumulative log-likelihood score exceeding the decision threshold, cf. [Sie85]. Another example pertains to one–server light traffic queues where the event

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of an unusually long waiting time for completion of service is characterized by segments of high exceedance, cf. [Ana88, Igl72].

It is helpful to describe the one dimensional problem first. Let $\{X_i\}$ be i.i.d. real ($\mathbb{R}$ valued) random variables of *negative* mean and law μ, and let $\{S_n, n \geq 0\}$ be the partial sum process induced by $\{X_i\}$. Consider the rare segments $\{m$ to $n\}$ in which $S_n - S_m > y$ for large values of y. Of special interest is the position and duration of the first such segment, and the empirical distribution of the increments X_i during these large exceedance segments. Formally, let the position of the first exceedance above level y be

$$T(y) = \inf\{n : \text{ for some m} \leq n, \ \ S_n - S_m > y\} \ ,$$

and determine the duration of this exceedance as

$$L(y) = T(y) - \max\{m : \ S_{T(y)} - S_m > y\} = T(y) - \tau(y) \ .$$

Dembo and Karlin [DK91b] established the a.s. convergence $L(y)/y \to 1/\int xe^{\lambda^* x} d\mu$ as $y \to \infty$, provided X_i are bounded and λ^* is the unique positive solution of $\int e^{\lambda^* x} d\mu = 1$. They further ascertained the empirical measure of X_i during these large exceedances which converges a.s. (and in the weak topology) to the Gibbs law μ^* where $\mu^*(B) = \int_B e^{\lambda^* x} d\mu$ for any measurable set B. It is proved by Iglehart [Igl72] that $T(y)/e^{\lambda^* y}$ converges in distribution, as $y \to \infty$, to an exponential law. In [DK91a, KD92] these results are extended to describe the behavior of large exceedances for increments governed by an irreducible finite state Markov chain.

In vector scoring of sequences, successive positions are vectors $X_i \in \mathbb{R}^d$ with components corresponding to different attributes. For example, for protein sequences, the components could be charge, hydrophobicity, and steric measurements of the amino acids. High quality segments correspond to indices $\tau(y)$ and $T(y)$ of the sequence such that $S_{T(y)} - S_{\tau(y)}$ first attains a high multivariate score corresponding to the rare set yA (y large). Such segments reflect desirable vector scoring arrays (e.g. for DNA segments having simultaneous high purine content and high DNA stability); in the queuing context, such segments correspond to large waiting times in queues with correlated customer behavior patterns; for the sequential detection problem they relate to simultaneous tests among three or more alternatives using pairwise likelihood ratios. The methods of [DK91a, DK91b] fail in high dimensions ($d > 1$), as soon as the set A is not a union of finitely many half-spaces. A more amenable approach is via large deviations analysis. Preliminary results are presented in [DZ93, Section 5.5], based on Mogulskii's [Mog76] large deviations characterization of the sample path of random walks in $\mathbb{R}^d$. Utilizing results of Freidlin and Wentzell [FW84] (see also [Puk92, deA93]) we analyze here the continuous time version, namely large exceedances of $\mathbb{R}^d$

valued Lévy processes X_t with increments satisfying Cramér's condition (i.e., $E[e^{\langle\lambda,X_1\rangle}]$ is finite for all $\lambda\in\mathbb{R}^d$. Hereafter, $\langle\lambda,x\rangle$ denotes the inner product of $\lambda,x\in\mathbb{R}^d$). For example, in the queuing context, these exceedances give information about the biases of the arrival process and service times during busy periods in which large overflow occurs (see Example 2 below).

In contrast with [DZ93, Section 5.5] where the special case of Brownian motion is sketched, here a more involved proof is needed due to the discontinuities (jumps) of the process X_t at random times (in particular, see the proof of Lemma 6 below). We also obtain here stronger results regarding the behavior of $T(y)$ (see Theorem 3 below).

2 Statement of main results

Let $\{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$ valued random process of stationary independent increments (infinitely divisible process) with initial value $X_0=0$, and logarithmic moment generating function $\Lambda(\lambda)=\log E[e^{\langle\lambda,X_1\rangle}]$, assumed to be finite for all $\lambda\in\mathbb{R}^d$. Specifically, for such processes [JS80, II.4.19],

$$\Lambda(\lambda)=\langle\lambda,b\rangle+\frac{1}{2}\langle\Sigma\lambda,\lambda\rangle+\int_{\mathbb{R}^d\setminus\{0\}}\left(e^{\langle\lambda,x\rangle}-1-\langle\lambda,x\rangle\right)\nu(dx)\ , \tag{1}$$

where $b=E[X_1]\neq 0$ is the drift vector of the process X_t, Σ is a symmetric nonnegative definite $d\times d$ matrix (which corresponds to the covariance of the Gaussian part of X_1), and ν is a Borel σ-finite measure on $\mathbb{R}^d$ for which the latter integral is finite for all $\lambda\in\mathbb{R}^d$. For our later needs we recall the Fenchel–Legendre transform of $\Lambda(\lambda)$ defined by

$$\Lambda^*(x)=\sup_{\lambda\in\mathbb{R}^d}\{\langle\lambda,x\rangle-\Lambda(\lambda)\}\ .$$

The domain of definition of $\Lambda^*(\cdot)$ designated $\mathcal{D}_{\Lambda^*}$ consists of all x for which $\Lambda^*(x)$ is finite. It is also useful to introduce

$$V(x,t)=\sup_{\lambda\in\mathbb{R}^d}\{\langle\lambda,x\rangle-t\Lambda(\lambda)\}=t\,\Lambda^*\left(\frac{x}{t}\right)\quad,\quad x\in\mathbb{R}^d,\ t>0, \tag{2}$$

with $V(x,0)=\infty$ for $x\neq 0$ and $V(0,0)=0$, and for every set $E\subset\mathbb{R}^d$ to define the *quasi-potential*

$$V_E=\inf_{x\in E,t\geq 0}V(x,t)\ . \tag{3}$$

It is convenient to replace y by $1/\epsilon$ and consider the rescaled process $Y_t^\epsilon=\epsilon X_{t/\epsilon}$. The increments $Y_t^\epsilon-Y_s^\epsilon$ are of mean $(t-s)b$ and variance $\mathbf{O}(t-s)$. Our aim is to estimate the probability of the rare events $\{Y_t^\epsilon-Y_s^\epsilon\in A\}$ for small ϵ. For this objective, we require that $\overline{A}$ (the closure of A) is disjoint from the half ray $\{\tau b\}_{\tau\geq 0}$. The set A can be unbounded.

To formalize the results, we define the following random times

$$T_\epsilon = \inf\{t : \ \exists s \in [0,t) \text{ such that } \ Y_t^\epsilon - Y_s^\epsilon \in A\}$$

$$\tau_\epsilon = \sup\{s \in [0, T_\epsilon) : \ Y_{T_\epsilon}^\epsilon - Y_s^\epsilon \in A\} \tag{4}$$

$$L_\epsilon = T_\epsilon - \tau_\epsilon$$

Under appropriate conditions on A, the main results of this paper are of the following form. There exist positive finite constants V^*, L^* and a suitable point x^* in A such that

$$\lim_{\epsilon \to 0} \epsilon \log T_\epsilon = V^* \ \text{ in probability ,} \tag{5}$$

$$\lim_{\epsilon \to 0} L_\epsilon = L^* \ \text{ in probability,} \tag{6}$$

and

$$\lim_{\epsilon \to 0} \sup_{0 \le s \le L^*} |U_s^\epsilon - u_s^*| = 0 \ \text{ in probability,} \tag{7}$$

where $U_s^\epsilon = Y_{\tau_\epsilon + s}^\epsilon - Y_{\tau_\epsilon}^\epsilon$ for $s \ge 0$, and u^* is the straight line $u_s^* = (s/L^*)x^*$, for $0 \le s \le L^*$.

The interpretation of (5) is that since the hitting probability of the segment $X_t - X_s$ to the set $\frac{1}{\epsilon}A$ is exponentially small, of the order $e^{-(V^*+o(1))/\epsilon}$, the waiting time for the first such segment is of order $e^{(V^*+o(1))/\epsilon}$ with probability tending to one. The limits (6) and (7) assert that the duration of such a segment is of order $1/\epsilon$, and its trajectory behaves as a deterministic straight line $u_s^* = (s/L^*)x^*$.

A sufficient condition for (5)–(7) to hold is stated next.

Theorem 1 *Let A be a closed, convex set of non-empty interior A^o, such that $A^o \cap \{\rho z : \rho > 0, z \in \mathcal{D}_{\Lambda^*}\}$ is non-empty. If for $\delta > 0$ small enough, A excludes the cone*

$$K_\delta = \{x : \langle x, b\rangle \ge (1-\delta)|x||b|\} \ ,$$

then the limit relation (5) holds with $V^* = V_A$ defined in (3). If further $\Lambda^*(\cdot)$ is finite everywhere then there exist unique $x^* \in A$ and $t^* > 0$ such that $V(x^*, t^*) = V_A$, and the limit relations (6)–(7) hold with $L^* = t^*$ and x^* defined above.

Remark: With the exception of the above theorem, the set A is not assumed to be convex. In particular, in Theorem 5 we present weaker conditions on the set A which suffice for (5)–(7) to hold.

Example 1: Consider the measure $\nu \equiv 0$ in (1), i.e. X_t is a linear transformation of the standard

Brownian motion. Ignoring possible degeneracies, we take $\Sigma = I$. Here, $\Lambda^*(x) = |x-b|^2/2$ is finite everywhere, and $V_{\{x\}} = V(x, |x|/|b|) = |x||b| - \langle x, b\rangle$. Therefore, if A is a closed, convex set of non-empty interior which, for $\delta > 0$ small enough, excludes the cone K_δ, then by Theorem 1 the limit relations (5)–(7) hold with $V^* = \inf_{x\in A}\{|x||b| - \langle x, b\rangle\}$, x^* the unique point of A for which $V^* = V_{\{x\}}$, and $L^* = |x^*|/|b|$. In the particular case of $A = \cap_{i=1}^d \{x : x_i \geq a_i\}$, corresponding to the simultaneous exceedances in all d coordinates it is easy to check that $x^* = (a_1, a_2, \ldots, a_d)$ as soon as $a_i \geq b_i|x^*|/|b|$ for $i = 1, \ldots, d$.

Example 2: Let the arrival process into a service station, denoted N_t, be a compound Poisson, non-negative integer valued random process, with finite moment generating function (i.e. $b = 0$, $\Sigma = 0$ and the measure ν in (1) is supported on the positive integers). Suppose the service times are exponentially distributed with parameter $\mu > E[N_1]$, and that the service station allows an infinite queue. The number of customers waiting for service at time t is $\sup_{s\leq t}\{(N_t - W_t) - (N_s - W_s)\}$, where W_t is a Poisson(μ) process. Note that $X_t = (X_t^1, X_t^2) = (N_t - W_t, N_t)$ is an $\mathbb{R}^2$-valued Lévy process. Let $A = \{(x_1, x_2) : x_1 \geq 1\}$. It is straight forward to check that (P–1)-(P–3) of Theorem 5 hold, with (5)-(7) following. The second component of U_s^ϵ corresponds to the (scaled) arrival process during a busy period in which the number of customers exceeds the high level $1/\epsilon$. The information implied in (6) and (7) may help in overflow prevention.

The key to the proof of the limit relations (5)–(7) depends on the following conditions (fixed time estimates), whose scope of validity is discussed in Section 4.

Condition (C–1) *There exist $L^* \in (0,\infty)$, $x^* \in A$ and $V^* \in (0,\infty)$ such that:*

$$\textit{for all}\, T > L^* \qquad \lim_{\epsilon\to 0} \epsilon \log \mathrm{P}(T_\epsilon \leq T) = -V^* \quad . \tag{8}$$

$$\textit{for all}\, \delta > 0,\ T > L^* \qquad \limsup_{\epsilon\to 0} \epsilon \log \mathrm{P}(|L_\epsilon - L^*| \geq \delta \textit{ and } T_\epsilon \leq T) < -V^* \quad , \tag{9}$$

and

$$\limsup_{\epsilon\to 0} \epsilon \log \mathrm{P}(\sup_{0\leq s\leq L^*} |U_s^\epsilon - u_s^*| \geq \delta \textit{ and } T_\epsilon \leq T) < -V^* \ , \tag{10}$$

where $u_s^* = (s/L^*)x^*$.

Condition (C–2) $\displaystyle \lim_{\eta\to 0}\lim_{C\to\infty}\limsup_{\epsilon\to 0}\sup_{t\geq C} \epsilon \log \mathrm{P}(Y_t^\epsilon \in A^\eta) < -2V^*$ *, where* $A^\eta = \{x : \inf_{y\in A}|y - x| \leq \eta\}$.

The estimates of (C–1) state the most likely way for a "one–shot" hit of A by the increments process $Y_t^\epsilon - Y_s^\epsilon$ during a *finite* time interval, as well as an assessment of the probability of such an event (at least on an exponential scale). Condition (C–2) provides for the confinement of L_ϵ to a bounded time interval, by virtue of which the problem can be decoupled to a sequence of independent "one–shot" attempts at hitting A. In detailing these steps (Lemmas 1–4 below) we achieve the following two theorems.

Theorem 2 *Assume that both (C–1) and (C–2) apply. Then, the limit relations (5)–(7) hold.*

Theorem 3 *Assume that both (C–1) and (C–2) apply. For $n_\epsilon \to \infty$ such that $\epsilon \log n_\epsilon \to 0$, let $p_\epsilon = P(T_\epsilon \le n_\epsilon)$. Then, $n_\epsilon^{-1} p_\epsilon T_\epsilon$ converges in distribution to an Exponential (1) random variable. If also*

$$x \in A \Rightarrow \{\gamma x : \gamma \ge 1\} \subset A \ , \tag{11}$$

then the limit relation (5) holds almost surely.

3 Proofs of Theorems 2 and 3

The main difficulty in proving Theorem 2 is that it involves events on an infinite time horizon; this precludes using directly the fixed time estimates of (C–1). The proof proceeds by reducing the infinite time horizon to finite time intervals which are loosely coupled and applying the estimates of (C–1) on the latter intervals. The first step to this end is the following upper bound on T_ϵ.

Lemma 1 *For any $\delta > 0$,*

$$\lim_{\epsilon \to 0} \mathrm{P}(T_\epsilon > e^{(V^*+\delta)/\epsilon}) = 0 \, .$$

Proof: Split the time interval $[0, e^{(V^*+\delta)/\epsilon}]$ into disjoint intervals of equal length $\Delta = (L^*+1)$ each. Let N_ϵ be the (integer part of the) number of such intervals. Observe that

$$\mathrm{P}(T_\epsilon > e^{(V^*+\delta)/\epsilon}) \le \mathrm{P}(Y^\epsilon_{k\Delta+t} - Y^\epsilon_{k\Delta+s} \notin A \qquad 0 \le s \le t \le \Delta \, , \, k = 0, \ldots, N_\epsilon - 1) \, .$$

The above events are independent for different values of k as they correspond to disjoint segments of $Y^\epsilon_\cdot$. Moreover, by the stationarity of the increments of $Y^\epsilon_\cdot$, they are of equal probability. Hence,

$$\mathrm{P}(T_\epsilon > e^{(V^*+\delta)/\epsilon}) \le [1 - \mathrm{P}(T_\epsilon \le \Delta)]^{N_\epsilon} \ ,$$

while

$$N_\epsilon \ge c e^{(V^*+\delta)/\epsilon} \ ,$$

for some $0 < c < \infty$ (independent of ϵ). Since for all $\epsilon > 0$ small enough (8) implies

$$\mathrm{P}(T_\epsilon \le L^* + 1) \ge e^{-(V^*+\delta/2)/\epsilon} \ ,$$

it follows that for all $\epsilon > 0$ small enough,

$$\mathrm{P}(T_\epsilon > e^{(V^*+\delta)/\epsilon}) \leq (1 - e^{-(V^*+\delta/2)/\epsilon})^{ce^{(V^*+\delta)/\epsilon}} \leq \exp(-ce^{\delta/2\epsilon}) \to_{\epsilon\to 0} 0\,. \tag{12}$$

■

Lemma 1 is not enough yet, since the upper bounds on T_ϵ are unbounded (as $\epsilon \to 0$). To continue we need the following short time estimate which allows for discretizing $Y^\epsilon_\cdot$.

Lemma 2 *For any* $\eta > 0$,

$$\limsup_{\epsilon\to 0} \epsilon \log \mathrm{P}(\sup_{0\leq t\leq \epsilon} |Y^\epsilon_t| > \eta) = -\infty \tag{13}$$

Proof: Note that

$$\{ \sup_{0\leq t\leq\epsilon} |Y^\epsilon_t| > \eta\} \subseteq \{ \sup_{0\leq \tau\leq 1} |Z_\tau| > \frac{\eta}{\epsilon} - |b|\}\ ,$$

where $Z_\tau = X_\tau - \tau b$ is a Martingale. Bounding the latter event by the union of $2d$ one–dimensional events involving thresholding the coordinates of Z_τ, it suffices to show that for every $\lambda \in \mathbb{R}^d$

$$\lim_{\epsilon\to 0} \epsilon \log \mathrm{P}(\sup_{0\leq\tau\leq 1} \{\langle\lambda, Z_\tau\rangle\} \geq \frac{1}{\epsilon}) = -\infty\ . \tag{14}$$

To this end, fix λ and note that for every θ, $M_\tau = e^{\theta\langle\lambda, Z_\tau\rangle}$ is a positive sub-martingale. Hence, by Doob's maximal inequality, for every $\theta \geq 0$,

$$\begin{aligned}
\mathrm{P}(\sup_{0\leq\tau\leq 1} \{\langle\lambda, Z_\tau\rangle\} \geq \frac{1}{\epsilon}) &= \mathrm{P}(\sup_{0\leq\tau\leq 1} M_\tau \geq e^{\theta/\epsilon}) \\
\leq e^{-\theta/\epsilon} E[M_1] &= e^{-\theta/\epsilon + \Lambda(\theta\lambda) - \theta\langle\lambda, b\rangle}\ .
\end{aligned}$$

Since $\Lambda(\cdot)$ is finite everywhere, (14) follows by letting first $\epsilon \to 0$ and then $\theta \to \infty$. ■

The following lemma provides for the confinement to the increments within finite time lags.

Lemma 3 *There exists a constant* $C < \infty$ *such that*

$$\lim_{\epsilon\to 0} \mathrm{P}(L_\epsilon \geq C) = 0\,.$$

Proof: Choose η, δ small enough and C large enough for (C–2) to yield

$$\limsup_{\epsilon\to 0} \epsilon \log K^2_\epsilon \sup_{t\geq C} \mathrm{P}(Y^\epsilon_t \in A^\eta) < 0\ , \tag{15}$$

where $K_\epsilon = \lfloor \epsilon^{-1} e^{(V^*+\delta)/\epsilon} \rfloor + 1$. Now cover the time interval $[0, e^{(V^*+\delta)/\epsilon}]$ by K_ϵ non-overlapping sub-intervals of size ϵ each, and let $\overline{Y}_t^\epsilon$ be the piecewise constant process obtained by considering $Y^\epsilon_{\epsilon\lfloor t/\epsilon \rfloor}$. Note that the event $\{L_\epsilon \geq C\}$ is contained in the union

$$\{T_\epsilon > e^{(V^*+\delta)/\epsilon}\} \bigcup \{ \sup_{t \leq e^{(V^*+\delta)/\epsilon}} |Y_t^\epsilon - \overline{Y}_t^\epsilon| > \eta/2\} \bigcup \{\overline{T}_\epsilon(C,\eta) \leq e^{(V^*+\delta)/\epsilon}\} ,$$

where

$$\overline{T}_\epsilon(C,\eta) = \inf\{t : \ \exists s \in [0, t-C) \text{ such that } \ \overline{Y}_t^\epsilon - \overline{Y}_s^\epsilon \in A^\eta\} .$$

Consequently, by the union of events bound and the stationarity of increments of $Y^\epsilon_\cdot$

$$\mathrm{P}(L_\epsilon \geq C) \leq \mathrm{P}(T_\epsilon > e^{(V^*+\delta)/\epsilon}) + K_\epsilon \mathrm{P}(\sup_{0 \leq t \leq \epsilon} |Y_t^\epsilon| > \eta/2) + K_\epsilon^2 \sup_{t \geq C} \mathrm{P}(Y_t^\epsilon \in A^\eta) .$$

Using (12), (13), and (15), one has that for some constant c_1 and all $\epsilon > 0$ small enough,

$$\mathrm{P}(L_\epsilon \geq C) \leq e^{-ce^{\delta/2\epsilon}} + e^{-c_1/\epsilon} \to_{\epsilon \to 0} 0 . \tag{16}$$

■

Lemma 4 *Let C be the constant from Lemma 3 and for each fixed integer n define the decoupled random times*

$$T_{\epsilon,n} = \inf\{t : \ Y_t^\epsilon - Y_s^\epsilon \in A \ \textit{for some } s \textit{ where } t > s \geq 2nC\lfloor t/(2nC) \rfloor\} .$$

Then,

$$\lim_{n \to \infty} \lim_{\epsilon \to 0} \mathrm{P}(T_{\epsilon,n} \neq T_\epsilon) = 0 \ .$$

Proof: Divide $[C, \infty)$ into the disjoint intervals $I_\ell = [(2\ell-1)C, (2\ell+1)C)$, $\ell = 1, \ldots$ Define the events

$$J_\ell = \{Y_t^\epsilon - Y_\tau^\epsilon \in A \ \text{ for some } \ \tau \leq t \ , t, \tau \in I_\ell\} ,$$

and the stopping time

$$N = \inf\{\ell \geq 1 : J_\ell \text{ occurs } \} .$$

By the stationarity and independence of the increments of $Y^\epsilon_\cdot$, the events J_ℓ are independent and equally probable. Let $p = \mathrm{P}(J_\ell)$. Then, $\mathrm{P}(N = \ell) = p(1-p)^{\ell-1}$ for $\ell \in \mathbb{Z}_+$. Hence,

$$\begin{aligned} \mathrm{P}(\{T_\epsilon < T_{\epsilon,n}\} \cap \{L_\epsilon < C\}) \ &\leq \ \mathrm{P}(\bigcup_{k=1}^\infty \{N = kn\}) \\ &= \ \sum_{k=1}^\infty p(1-p)^{kn-1} = \frac{p(1-p)^{n-1}}{1-(1-p)^n} \leq \frac{1}{n} \end{aligned}$$

Since by definition $T_\epsilon \le T_{\epsilon,n}$, the proof is completed by applying Lemma 3. ∎

Returning to the proof of the theorem, it is enough to consider the rare events of interest with respect to the decoupled times for n large enough. This procedure results with a sequence of i.i.d. random variables corresponding to disjoint segments of $Y^\epsilon_\cdot$ of length $2nC$ each. The fixed time estimates of (C–1) can then be applied. In particular, with $N_\epsilon = \lfloor (2nC)^{-1} e^{(V^*-\delta)/\epsilon} \rfloor + 1$ denoting the number of such segments in $[0, e^{(V^*-\delta)/\epsilon}]$, the following lower bound on $T_{\epsilon,n}$ is obtained

$$\begin{aligned} \mathrm{P}(T_{\epsilon,n} < e^{(V^*-\delta)/\epsilon}) &\le \sum_{k=0}^{N_\epsilon - 1} \mathrm{P}(\left\lfloor \frac{T_{\epsilon,n}}{2nC} \right\rfloor = k) \\ &\le N_\epsilon \mathrm{P}(T_{\epsilon,n} < 2nC) \le N_\epsilon \mathrm{P}(T_\epsilon < 2nC) \\ &\le (\frac{e^{(V^*-\delta)/\epsilon}}{2nC} + 1) \mathrm{P}(T_\epsilon \le 2nC) . \end{aligned}$$

Therefore, with n large enough for $2nC > L^*$, the estimate (8) implies that

$$\lim_{\epsilon \to 0} \mathrm{P}(T_{\epsilon,n} < e^{(V^*-\delta)/\epsilon}) \le \lim_{\epsilon \to 0} \frac{e^{(V^*-\delta)/\epsilon}}{2nC} e^{-(V^*-\delta/2)/\epsilon} = 0 .$$

Hence, for all $\delta > 0$, by Lemma 4

$$\lim_{\epsilon \to 0} \mathrm{P}(T_\epsilon < e^{(V^*-\delta)/\epsilon}) = \lim_{n \to \infty} \lim_{\epsilon \to 0} \mathrm{P}(T_{\epsilon,n} < e^{(V^*-\delta)/\epsilon}) = 0 ,$$

and (5) results in view of the upper bound of Lemma 1.

Define now

$$\tau_{\epsilon,n} = \sup\{s : \ s \in [0, T_{\epsilon,n}) , \quad Y^\epsilon_{T_{\epsilon,n}} - Y^\epsilon_s \in A\} .$$

Clearly, $T_{\epsilon,n} \ge T_\epsilon$ and if $T_{\epsilon,n} = T_\epsilon$ then also $\tau_{\epsilon,n} = \tau_\epsilon$. Moreover, for all n and all ϵ, the distribution of $\{Y^\epsilon_{\tau_{\epsilon,n}+s} : 0 \le s \le T_{\epsilon,n} - \tau_{\epsilon,n}\}$ is the same as the conditional distribution of $\{Y^\epsilon_{\tau_\epsilon+s} : 0 \le s \le T_\epsilon - \tau_\epsilon\}$ given $T_\epsilon \le 2nC$. The estimates of (C–1) imply that for all $\delta > 0$ and any n large enough

$$\lim_{\epsilon \to 0} \mathrm{P}(|L_\epsilon - L^*| \ge \delta \ | \ T_\epsilon \le 2nC) = 0 \ ,$$

and

$$\lim_{\epsilon \to 0} \mathrm{P}(\sup_{0 \le s \le L^*} |U^\epsilon_s - u^*_s| \ge \delta \ | \ T_\epsilon \le 2nC) = 0 .$$

When combined with Lemma 4, the limit relations (6) and (7) are confirmed. ∎

Proof of Theorem 3: Let T_{ϵ,n_ϵ} be defined as in Lemma 4, but with n_ϵ instead of $2nC$. By the same argument as in this lemma, $P(T_{\epsilon,n_\epsilon} \ne T_\epsilon) \to 0$ as $\epsilon \to 0$. Fix $y > 0$ and let $m_\epsilon = \lfloor y/p_\epsilon \rfloor$ and

$y_\epsilon = p_\epsilon m_\epsilon$. The event $\{n_\epsilon^{-1} p_\epsilon T_{\epsilon,n_\epsilon} > y_\epsilon\}$ is merely the intersection of m_ϵ independent events each of which occurs with probability $(1 - p_\epsilon)$. Consequently,

$$P(n_\epsilon^{-1} p_\epsilon T_{\epsilon,n_\epsilon} > y_\epsilon) = (1 - p_\epsilon)^{m_\epsilon} .$$

Since $\epsilon \log n_\epsilon \to 0$ it follows from (5) that $p_\epsilon \to 0$ and $y_\epsilon \to y$. Therefore, $(1 - p_\epsilon)^{m_\epsilon} \to e^{-y}$ and the Exponential limit law of $n_\epsilon^{-1} p_\epsilon T_\epsilon$ follows.

Our assumption (11) implies that the stopping times T_ϵ/ϵ are monotonically non-increasing in ϵ (sample-wise). Consequently, the almost sure convergence in (5) follows as soon as for every fixed $\delta > 0$, and every $\gamma > 0$ arbitrarily small

$$\limsup_{n\to\infty} |\epsilon_n \log T_{\epsilon_n} - V^*| \le \delta \ \text{ almost surely}, \tag{17}$$

where $\epsilon_n = (1 - \gamma)^n$. By (12), for some $c_2 < \infty$,

$$\sum_{n=1}^{\infty} \mathrm{P}(T_{\epsilon_n} > e^{(V^*+\delta)/\epsilon_n}) \le c_2 + \sum_{n=1}^{\infty} e^{-ce^{\delta/(2(1-\gamma)^n)}} < \infty . \tag{18}$$

Let $\overline{C} = \max(C, (L^* + 1)/2)$ where C is the constant from Lemma 3. Let $k_\epsilon = \lfloor (2\overline{C})^{-1} e^{(V^*-\delta)/\epsilon} \rfloor + 1$ and note that the event $\{T_\epsilon < e^{(V^*-\delta)/\epsilon} \cap L_\epsilon \le C\}$ is contained in $\cup_{i=0}^{2k_\epsilon - 1} \mathcal{A}_{\overline{C}i}$, where

$$\mathcal{A}_u = \{\, Y_{u+t}^\epsilon - Y_{u+s}^\epsilon \in A \ \text{ for some } \ 2\overline{C} \ge t > s \ge 0\} .$$

By the stationarity of increments of $Y_\cdot^\epsilon$, one has that $P(\mathcal{A}_u) = P(\mathcal{A}_0) = P(T_\epsilon \le 2\overline{C})$. Therefore,

$$P(T_\epsilon < e^{(V^*-\delta)/\epsilon}) \le P(L_\epsilon \ge C) + 2k_\epsilon P(T_\epsilon \le 2\overline{C}) .$$

For all $\epsilon > 0$ small enough (8) implies that

$$\mathrm{P}(T_\epsilon \le 2\overline{C}) \le e^{-(V^*-\delta/2)/\epsilon} . \tag{19}$$

Combining (16) and (19) it follows that

$$\sum_{n=1}^{\infty} \mathrm{P}(T_{\epsilon_n} < e^{(V^*-\delta)/\epsilon_n}) < \infty . \tag{20}$$

Applying the Borel-Cantelli lemma, (17) follows from (18) and (20). ∎

4 Large deviations and the set A

We turn now to use the Large Deviations Principle (LDP) associated with sample path of $Y^\epsilon_\cdot$ in order to establish (C–1) and (C–2) as soon as the set A satisfy certain explicit geometrical conditions. To this end, let $D([0,t])$ be the space of functions continuous from the right and having left–hand limits, equipped with the uniform (sup norm) topology. The laws μ_ϵ of the processes Y^ϵ_s, $s \in [0,t]$, are supported on this metric space, and satisfy there the LDP with the following rate function

$$I_t(\phi) = \begin{cases} \int_0^t \Lambda^*(\dot{\phi}_s)\, ds, & \text{if } \phi \in \mathcal{AC}_t, \phi_0 = 0 \\ \\ \infty & \text{otherwise} \end{cases}$$

where $\mathcal{AC}_t$ is the space of absolutely continuous functions $\phi : [0,t] \to \mathbb{R}^d$. In the present context the LDP is summarized in the following theorem.

Theorem 4 *(a) For any $t < \infty$ and any $\alpha < \infty$, $\Psi_I(\alpha) = \{\phi : I_t(\phi) \leq \alpha\}$ is a compact set with respect to the sup norm topology.*
(b) For any measurable set of functions $\Gamma \subseteq D([0,t])$,

$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq - \inf_{\phi \in \overline{\Gamma}} I_t(\phi) \;, \tag{21}$$

and

$$\liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \geq - \inf_{\phi \in \Gamma^o} I_t(\phi) \;. \tag{22}$$

Here measurable is with respect to the σ-algebra generated by the coordinate maps $s \mapsto f(s)$, completed by the common null sets of $\{\mu_\epsilon : \epsilon > 0\}$.

Part (a) above is referred to as $I_t(\cdot)$ being a good rate function. The bounds of (b) are called the large deviations upper and lower bounds. Note that the notation μ_ϵ does not indicate the value of t considered. In our applications this value will be made clear via the definitions of the relevant sets.

Proof of the above theorem can be found in [FW84, Puk92, deA93]. It can also be easily deduced by modifying either the proof of Schilder's theorem in Section 5.2 of [DZ93], or the alternative proofs of Schilder's theorem presented in [DS89, Var84].

The *cost* associated with a termination point $x \in \mathbb{R}^d$ at time $t \in (0,\infty)$ is defined as

$$J(x,t) = \inf_{\{\phi \in \mathcal{AC}_t :\, \phi_0 = 0, \phi_t = x\}} I_t(\phi) \,. \tag{23}$$

Lemma 5 *For all $x \in \mathbb{R}^d, t > 0$,*

$$J(x,t) = I_t(\frac{s}{t}x) = V(x,t) \,, \tag{24}$$

where $V(x,t)$ is defined in (2). Moreover, $V(x,t)$ is a convex, nonnegative, lower semicontinuous function on $\mathbb{R}^d \times [0,\infty)$.

Proof: By its definition, $\Lambda^*(\cdot)$ is a convex function. Hence, for all $t > 0$ and any $\phi \in \mathcal{AC}_t$ with $\phi_0 = 0$, by Jensen's inequality,

$$I_t(\phi) = t\int_0^t \Lambda^*(\dot{\phi}_s)\frac{ds}{t} \geq t\Lambda^*\left(\int_0^t \dot{\phi}_s \frac{ds}{t}\right) = t\Lambda^*\left(\frac{\phi_t - \phi_0}{t}\right) \,,$$

with equality for $\phi_s = sx/t$. Thus, (24) follows by the definitions (2) and (23). Since $\Lambda^*(\cdot)$ is nonnegative, so is $V(x,t)$. By the first equality in (2), which holds also for $t = 0$, $V(x,t)$, being the supremum of linear functions is convex and lower semicontinuous on $\mathbb{R}^d \times [0,\infty)$. ■

Recall that the *quasi-potential* associated with a set $E \subset \mathbb{R}^d$ is defined as

$$V_E = \inf_{x\in E, t\geq 0} V(x,t) \,.$$

The following theorem relates properties of the function $V(\cdot,\cdot)$ and of the set A, with conditions (C–1) and (C–2).

Theorem 5 *Suppose that A is a closed set with the following properties.*
(P–1) $V_A = V_{A^o} \in (0,\infty)$ (where A^o denotes the interior of A).
(P–2) There is a unique pair $x^* \in A$, $t^* \in (0,\infty)$ such that $V_A = V(x^*,t^*)$. Moreover, the straight line $u^*_s = (s/t^*)x^*$ is the unique path with respect to (23) for which the value of $V(x^*,t^*) = V_A$ is achieved.
(P–3)

$$\lim_{\eta\to 0}\lim_{r\to\infty} V_{co(A)^\eta \cap \{x:|x|>r\}} > 2V_A \,,$$

where $co(A)$ denotes the closed convex hull of A.
Then, (C–1) and (C–2) hold with $V^* = V_A$, $L^* = t^*$ and x^*, and u^* as stated above.

Proof: Proceeding to the verification of (C–1), set

$$\Psi = \{\psi \in D([0,T]) : \psi_t - \psi_\tau \in A \text{ for some } \tau \leq t \in [0,T]\} \,,$$

$$\Phi_\delta = \{\psi \in D([0,T]) : \psi_t - \psi_\tau \in A \text{ for some } \\ \tau \leq t \in [0,T] \,,\;\; t-\tau \in [0,t^*-\delta] \cup [t^*+\delta, T]\} \,,$$

and

$$\Psi_\delta = \{\psi \in D([0,T+t^*]) : \psi_t - \psi_\tau \in A, \sup_{0\le s\le t^*} |\psi_{s+\tau} - \psi_\tau - u^*_s| \ge \delta \quad \text{for some} \quad \tau \le t \in [0,T]\} \,.$$

Observe that

$$\begin{aligned} \mathrm{P}(T_\epsilon \le T) &= \mu_\epsilon(\Psi)\,, \\ \mathrm{P}(|L_\epsilon - t^*| \ge \delta \text{ and } T_\epsilon \le T) &\le \mu_\epsilon(\Phi_\delta)\,, \\ \mathrm{P}(\sup_{0\le s\le t^*} |U^\epsilon_s - u^*_s| \ge \delta \text{ and } T_\epsilon \le T) &\le \mu_\epsilon(\Psi_\delta)\,. \end{aligned}$$

Therefore, in view of the LDP of $\{\mu_\epsilon\}$, the estimates of (C–1) are consequences of the following lemma.

Lemma 6 *Assume (P–1)-(P–3). Then, for all $T > t^*$*

$$V_A = \inf_{\psi\in\overline{\Psi}} I_T(\psi) = \inf_{\psi\in\Psi^o} I_T(\psi)\,, \tag{25}$$

while for all $\delta > 0$,

$$\inf_{\psi\in\overline{\Phi_\delta}} I_T(\psi) > V_A\,, \tag{26}$$

and

$$\inf_{\psi\in\overline{\Psi_\delta}} I_{T+t^*}(\psi) > V_A\,. \tag{27}$$

Proof: Throughout, let $\|\cdot\|$ denotes the sup norm over the relevant bounded time interval. Starting with the proof of (25), determine $\psi^n \in \Psi$ such that $\|\psi^n - \psi\| \to 0$ with $\psi \in C_0([0,T])$. Accordingly, there exist $\tau_n \le t_n \in [0,T]$ and $x_n = (\psi^n_{t_n} - \psi^n_{\tau_n}) \in A$. With $[0,T]$ a compact set and possibly passing to a subsequence, we may take $\tau_n \to \tau \in [0,T]$ and $t_n \to t \in [\tau,T]$. It follows that $y_n \to y$ where $y_n = (\psi_{t_n} - \psi_{\tau_n})$ and $y = (\psi_t - \psi_\tau)$. Moreover, $|x_n - y_n| \to 0$ and since A is a closed set, $y \in A$, implying that $\psi \in \Psi$. Consequently, $\overline{\Psi} \cap C_0([0,T]) \subseteq \Psi$, and since $\{\phi : I_T(\phi) < \infty\}$ is a subset of $C_0([0,T])$, we have

$$\inf_{\psi\in\overline{\Psi}} I_T(\psi) = \inf_{\psi\in\Psi} I_T(\psi)\,. \tag{28}$$

Since $T > t^*$,

$$V_A = \inf_{x\in A, t\in[0,T]} V(x,t) = \inf_{x\in A, \tau\le t\in[0,T]} \; \inf_{\{\phi:\phi_{t-\tau}=x\}} I_{t-\tau}(\phi)\,.$$

Let the map $S_\tau : D([0,t-\tau]) \mapsto D([0,T])$ be defined via $\phi \mapsto \psi$ where

$$\psi_s = \begin{cases} sb & s \in [0,\tau) \\ \phi_{s-\tau} + \tau b & s \in [\tau, t) \\ \phi_{t-\tau} + \tau b + (s-t)b & s \in [t,T] \,. \end{cases}$$

Then, with $\Lambda^*(b) = 0$, clearly $I_{t-\tau}(\cdot) = I_T(S_\tau(\cdot))$ and hence also

$$V_A = \inf_{x\in A, \tau \le t \in [0,T]} \; \inf_{\{\phi : \phi_{t-\tau} = x\}} I_T(S_\tau(\phi)) = \inf_{\psi \in \Psi} I_T(\psi)\,.$$

The set

$$\tilde{\Psi} = \{\psi \in D([0,T]) : \psi_t - \psi_\tau \in A^o \text{ for some } \tau \le t \in [0,T]\}\,,$$

is open, for if $\psi \in \tilde{\Psi}$ then there exist $\tau \le t \in [0,T]$, $x \in A^o$ and $\eta > 0$ such that $x = \psi_t - \psi_\tau$ and $B_{x,2\eta} \subseteq A^o$, and consequently

$$\| \phi - \psi \| < \eta \quad \Rightarrow \quad \phi \in \tilde{\Psi}\,.$$

Since $\tilde{\Psi} \subseteq \Psi$ it follows that

$$\inf_{\psi \in \Psi^o} I_T(\psi) \le \inf_{\psi \in \tilde{\Psi}} I_T(\psi) = \inf_{x \in A^o,\, t \in [0,T]} V(x,t)\,,$$

and the proof of (25) is complete by showing that for all $T > t^*$,

$$\inf_{x \in A^o, t \in [0,T]} V(x,t) = V_A\,. \tag{29}$$

To this end, observe that $\nabla\Lambda(0) = E(X_1) = b$ and hence $\Lambda^*(z) > 0$ for $z \neq b$. Moreover, $\Lambda^*(\cdot)$ is a good rate function, so also

$$a = \inf_{|z| \le |b|/2} \Lambda^*(z) > 0\,.$$

Hence by (2), for all $r > 0$,

$$\inf_{|x| \le r} \; \inf_{t \ge 2r/|b|} V(x,t) \ge \inf_{t \ge 2r/|b|} \; \inf_{|x| \le \frac{|b|}{2} t} t\Lambda^*\left(\frac{x}{t}\right) \ge 2ra/|b|\,. \tag{30}$$

Consequently, by (P–1) and (P–3), there exists an $r < \infty$ such that

$$V_A = V_{A^o} = \inf_{x \in A^o, |x| \le r, t \le r} V(x,t)\,. \tag{31}$$

Consider an arbitrary sequence (x_n, t_n) satisfying $x_n \in A^o$, $|x_n| \le r$, $t_n \in [0,r]$ and $V(x_n,t_n) \to V_A$. Such a sequence admits at least one limit point, say (x,t), and by the lower semicontinuity of $V(\cdot,\cdot)$

$$V_A = \lim_{n\to\infty} V(x_n,t_n) \ge V(x,t)\,.$$

However, $x \in A$ and $t < \infty$ implying by (P–2) that $x = x^*$, $t = t^*$ (and for all $T > t^*$ eventually $t_n \in [0, T]$). When combined with (31) the conclusion of (29) is assured.

Now suppose that (26) is false for some $\delta > 0$. Then, since $I_T(\cdot)$ is a good rate function, there exists $\phi \in \overline{\Phi_\delta}$ with $I_T(\phi) \leq V_A < \infty$. Consequently, paraphrasing the reasoning leading to (28), one may find a $\phi \in \Phi_\delta$ such that $I_T(\phi) \leq V_A$. Fix $\tau \leq t \in [0, T]$ such that both $|t - \tau - t^*| \geq \delta$ and $\phi_t - \phi_\tau \in A$. Then,

$$V_A \geq I_T(\phi) \geq I_{t-\tau}(\phi_{s+\tau} - \phi_\tau) \geq V(\phi_t - \phi_\tau, t - \tau) \ ,$$

and hence by (P–2), $t - \tau = t^*$ resulting with a contradiction.

Fix $\delta > 0$, $\psi^n \in \Psi_\delta$ and $\psi \in C_0([0, T+t^*])$ such that $\| \psi^n - \psi \| \to 0$. There exist $\tau_n \leq t_n \in [0, T]$ such that $\psi^n_{t_n} - \psi^n_{\tau_n} \in A$, and

$$\sup_{0 \leq s \leq t^*} |\psi^n_{s+\tau_n} - \psi^n_{\tau_n} - u^*_s| \geq \delta \, .$$

The same argument as above yields (on a subsequence) $t_n \to t$, $\tau_n \to \tau$ and $(\psi^n_{t_n} - \psi^n_{\tau_n}) \to (\psi_t - \psi_\tau) = y \in A$. Moreover, since $\psi \in C_0([0, T + t^*])$ and $\tau_n \to \tau$,

$$\sup_{0 \leq s \leq t^*} |\psi^n_{s+\tau_n} - \psi^n_{\tau_n} - (\psi_{s+\tau} - \psi_\tau)| \to 0 \, .$$

Therefore, $\sup_{0 \leq s \leq t^*} |\psi_{s+\tau} - \psi_\tau - u^*_s| \geq \delta$, i.e., $\psi \in \Psi_\delta$.

Suppose that (27) is false. Then, since $I_{T+t^*}(\cdot)$ is a good rate function, there exists $\tilde{\psi} \in \overline{\Psi_\delta}$ with $I_{T+t^*}(\tilde{\psi}) \leq V_A < \infty$, and by the above argument also $\tilde{\psi} \in \Psi_\delta$. Fix $\tau \leq t \in [0, T]$ such that both $\tilde{\psi}_t - \tilde{\psi}_\tau \in A$ and

$$\sup_{0 \leq s \leq t^*} |\tilde{\psi}_{s+\tau} - \tilde{\psi}_\tau - u^*_s| \geq \delta \, . \tag{32}$$

Consequently,

$$V_A \geq I_{T+t^*}(\tilde{\psi}) \geq I_{t-\tau}(\tilde{\psi}_{s+\tau} - \tilde{\psi}_\tau) \geq V(\tilde{\psi}_t - \tilde{\psi}_\tau, t - \tau) \, .$$

Thus, by (P–2), $t - \tau = t^*$, $\tilde{\psi}_t - \tilde{\psi}_\tau = x^*$ and $\tilde{\psi}_{s+\tau} - \tilde{\psi}_\tau = u^*_s$ contradicting (32). It follows that (27) must hold. ■

Turning now to the proof of (C–2), observe that by Chebycheff's inequality, for any $\lambda \in \mathbb{R}^d$, and any compact, convex $K \subset \mathbb{R}^d$,

$$\begin{aligned} \mathrm{P}(Y^\epsilon_t \in K) = \mathrm{P}(\epsilon X_{t/\epsilon} \in K) &\leq E\left[\exp(\langle \lambda, X_{t/\epsilon} \rangle - \inf_{x \in K} \langle \lambda, x \rangle / \epsilon)\right] \\ &= e^{[t\Lambda(\lambda) - \inf_{x \in K} \langle \lambda, x \rangle]/\epsilon} \end{aligned}$$

Hence, by the min-max theorem (c.f. [ET76], pg. 174),

$$\epsilon \log \mathrm{P}(Y_t^\epsilon \in K) \le - \sup_{\lambda \in \mathbb{R}^d} \inf_{x \in K} [\langle \lambda, x \rangle - t\Lambda(\lambda)] = - \inf_{x \in K} V(x,t)$$

This inequality extends to every convex, closed K by intersecting it with a sequence of balls centered at the origin and of radii that monotonically increase to ∞. In particular, applying the above to the closed, convex sets $\mathrm{co}(A^\eta)$, it follows that (C–2) holds as soon as

$$\lim_{\eta \to 0} \lim_{C \to \infty} \inf_{x \in \mathrm{co}(A^\eta),\, t \ge C} V(x,t) > 2V_A \quad .$$

The latter inequality holds by combining (30) and (P–3) (recall that $\mathrm{co}(A^\eta) \subseteq \mathrm{co}(A)^\eta$). ■

Remark: As evident in the above proof, even when (P–2) fails, both (C–2) and the estimate (8) for all T large enough hold as soon as (P–1) and (P–3) hold. Hence, these suffice for (5) to hold true.

Proof of Theorem 1: In view of Theorems 2 and 5, and the above remark it suffices to show that the conditions of the theorem imply that (P–1) and (P–3) hold true, and if $\Lambda^*(\cdot)$ is finite everywhere, then (P–2) holds as well.

We shall start by proving (P–1). The existence of a point $\rho z \in A^o$ such that $\Lambda^*(z) < \infty$ implies that $V_A \le V_{A^o} \le V(\rho z, \rho) < \infty$. With $\Lambda^*(z)$ having compact level sets and unique minimum at $z = b$, it follows that $a_\rho = \inf_{z \notin B_{b,\rho}} \Lambda^*(z) > 0$ for all $\rho > 0$ (where $B_{b,\rho}$ denotes the ball of radius ρ centered at b). As $b \ne 0$, for $\rho > 0$ small enough $B_{b,\rho} \subseteq K_\delta$. Consequently, for some $a = a_\rho$,

$$V(x,t) \ge at \qquad \forall x \in A$$

Moreover, by (2),

$$V(x,t) \ge |x| - t \sup_{|\lambda|=1} \Lambda(\lambda) \ ,$$

and hence $V(x,t) \ge (2V_A + 1)$ for all $t \le (2V_A + 1)/a$ once $|x| > r$ for some r large enough. Combining the above estimates it follows that V_A is the infimum of $V(x,t)$ over $(x,t) \in A \cap \overline{B}_{0,r} \times [0, C]$ for some finite r, C large enough. The existence of the pair $x^* \in A$, $t^* \in (0, \infty)$ now follows by the compactness of the latter set and the lower semicontinuity of $V(\cdot,\cdot)$. Since $x^* \in A$, it follows that $x^*/t^* \ne b$, and consequently $V_{A^o} \ge V_A > 0$. Consider the point $\rho z \in A^o$ such that $z \in \mathcal{D}_{\Lambda^*}$. For all $\alpha \in (0,1]$ both $\phi_\alpha = \alpha \rho z + (1-\alpha) x^* \in A^o$ and $z_\alpha = \alpha z + (1-\alpha) x^*/t^* \in \mathcal{D}_{\Lambda^*}$ Note that $V_{A^o} \le V(\phi_\alpha, t_\alpha) = t_\alpha \Lambda^*(z_\beta)$, where $t_\alpha = \alpha\rho + (1-\alpha)t^*$ and $\beta = \alpha\rho/t_\alpha \in (0,1]$. As $\alpha \searrow 0$, both $t_\alpha \to t^*$ and $\Lambda^*(z_\beta) \to \Lambda^*(x^*/t^*)$ (see [Roc70] Corollary 7.5.1). Consequently, $V_{A^o} = V_A$.

With $\Lambda(\cdot)$ finite everywhere, it follows by dominated convergence that $\Lambda(\cdot)$ is differentiable everywhere, and hence $\Lambda^*(\cdot)$ is strictly convex in the relative interior of its domain (see [Roc70]

Corollary 26.4.1). Consequently, (P–2) holds as soon as x^*/t^* is in this set. In particular, (P–2) holds when $\Lambda^*(\cdot)$ is finite everywhere.

Turning now to the proof of (P–3), observe that $\mathrm{co}(A)^\eta \cap \{x : |x| > r\}$ excludes the cone $K_{\delta'}$ for $\delta' \leq \delta - 2\eta/r$. Hence, (P–3) follows paraphrasing the argument used when proving the existence of (x^*, t^*). ■

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