

The Variable-Increment Counting Bloom Filter

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Abstract—Counting Bloom Filters (CBFs) are widely used in networking device algorithms. They implement fast set representations to support membership queries with limited error, and support element deletions unlike Bloom Filters. However, they consume significant amounts of memory.

In this paper we introduce a new general method based on *variable increments* to improve the efficiency of CBFs and their variants. Unlike CBFs, at each packet arrival, the hashed counters are incremented by a hashed variable increment instead of a unit increment. Then, to query a packet, the exact value of a counter is considered and not just its positiveness. We present two simple schemes based on this method. We demonstrate that this method can always achieve a lower false positive rate and a lower overflow probability bound than CBF in practical systems. We also show how it can be easily implemented in hardware, with limited added complexity and memory overhead. We further explain how this method can extend many variants of CBF that have been published in the literature. Last, using simulations, we show how it can improve the false positive rate of CBFs by up to an order of magnitude given the same amount of memory.

I. INTRODUCTION

A. Motivation

CBF (Counting Bloom Filter) variants are increasingly used in networking device algorithms, in fields as diverse as accounting, monitoring, load-balancing, caching, policy enforcement, routing, filtering, security, and differentiated services. For instance, for a given flow, they can determine whether it has at least one packet currently queued (set membership), how many of its packets are queued (counter representation), whether it is in a given state (state representation), or where to forward its packets (IP lookups) [2]–[7].

CBFs are often used because they can be easily implemented in hardware. In particular, element insertions, deletions and queries can be implemented in CBFs using a *constant complexity* that is essentially independent of the number of elements for a given bits-per-element ratio. However, CBFs also consume *significant amounts of memory*. For instance, using four bits per entry and ten entries per element yields a needed memory space in bits that is 40 times larger than the number of inserted elements.

This paper is about a general method to improve the memory efficiency of CBFs with limited added hardware complexity. We introduce a novel method based on variable increments to reduce the amount of memory used by CBFs for a given false positive rate. This method can also implement element insertions, deletions and queries using a constant complexity per element. It has a low hardware implementation overhead when compared to CBFs, and can replace CBFs as a sub-

module in any algorithm implementation without required outside changes.

B. Intuition for Variable Increments

Let's provide some intuition for variable increments by comparing Bloom Filters (BFs), Counting Bloom Filters (CBFs) and Variable-Increment Counting Bloom Filters (VI-CBFs).

A Bloom Filter (BF) is a well-known simple data structure used to represent a set of n elements $S = \{x_1, \dots, x_n\}$ elements from a universe U using an array of m bits [8]. However, it is not designed to support deletions of elements, which are often needed in networking device algorithms.

As illustrated in Figure 1(a), BF uses k uniformly-distributed hash functions over the range $\{1, \dots, m\}$ of its m -bit filter. For each element $x \in S$, k hash entries are calculated using the hash functions and the corresponding bits are set to one. For instance, in the figure, the bits of x and y are set to one. In order to check whether an element z is in S , we check whether all of its k corresponding bit locations $h_i(z)$ are set to one. If this is not the case, we know that z is not in S . If all of them are set, as in Figure 1(a), we state that $z \in S$, although this might be a *false positive* error. For each $z \notin S$, the *false positive rate*, i.e. the probability of a false positive error, is $(1 - p_0)^k$, where $p_0 = (1 - 1/m)^{nk}$ is the probability that a specific bit is still zero after the insertion of n elements. Since BF does not support deletions of elements, it cannot for instance be used to represent the current set of packets of a flow in a router where flows might dynamically change.

The Counting Bloom Filter (CBF) suggested by Fan et al. [2] is a generalization of BF, in which each hash entry contains a counter with a fixed size of b bits, instead of a single bit in BF. Unfortunately, while supporting deletions, CBF also needs large amounts of memory space (i.e. b times the memory space consumed by BF), which is often valuable in networking devices.

As shown in Figure 1(b), to insert an element, all the corresponding counters are incremented by one. Likewise, to delete it, all of its counters are decremented. To determine if an element $z \in S$, we check if all of its hashed entries are positive. For instance, in Figure 1(b), we state that $z \in S$, which might be a false positive as in BF. Given only insertions, the false positive rate of CBF is the same as for BF with the mentioned increase in memory space. CBF might also suffer from counter overflows with a probability that depends on its counter size b , although $b = 4$ is sufficient to practically obtain a negligible overflow probability [2].

We now want to introduce the use of variable increments. We notice that CBF does not store much information in

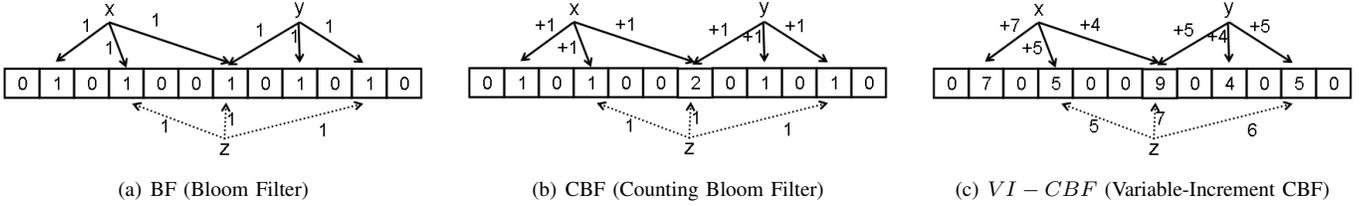


Fig. 1. Comparison of the concepts behind BF, CBF and our proposed $VI-CBF$, using $S = \{x, y\}$ and a query of element $z \notin S$. In this example, BF and CBF yield false positives while $VI-CBF$ does not.

its counter values. When querying an element, it does not distinguish between any counter values greater than zero, and only considers their positiveness. In this paper, we use the *specific value* of a counter in order to give a more complete answer to the query.

The Variable-Increment Counting Bloom Filter, denoted as $VI-CBF$, is a generalization of CBF that uses variable increments to update each entry. We first define a set of possible variable increments D . Then, for each counter update by an element, we hash the element into a value of D and use it to increment the counter. Likewise, to delete an element, we decrement by its hashed value in D . Last, to determine if an element $z \in S$, we check in each of its counters if its hashed value in D could be part of the sum. If this is not the case in at least one counter, then necessarily $z \notin S$. Otherwise, as for BF and CBF, we state that $z \in S$, which might be a false positive.

Figure 1(c) illustrates $VI-CBF$ with $D = \{4, 5, 6, 7\}$. First, x and y increment their corresponding counters by their corresponding hashed values in D . For instance, x increments its first counter value by $7 \in D$. Consider now a query of whether z is in S . The second hashed entry of z has counter value 9, while for this entry z hashes to increment $7 \in D$. Since $9 - 7 = 2$ cannot be presented as a sum of elements of D , the increment 7 cannot be part of the sum 9, and we deduce that necessarily $z \notin S$, avoiding the false positive that occurred in CBF (Figure 1(b)). Note that we could have known that as well from the third hashed entry of z (since $6 > 5$), but not from its first entry (because x and z hash to the same increment, so the sum of 5 can be composed of the increment 5).

C. Related Work: Applications of the Variable-Increment Method

The variable-increment method is a *generic* approach that can actually be implemented to improve or extend *most variants* of CBF. In this paper, we detail and evaluate its application to both CBF and $ML-HCBF$.

- **CBF** [2]: As presented above.
- **ML-HCBF** (MultiLayer Hashed CBF) [7]: This algorithm uses a hierarchical compression of CBF filters to achieve better performance. We explain in Section V how the same hierarchical compression idea could be combined with $VI-CBF$, and show in Section V how the combination can achieve even better results.
- **VL-CBF** (Variable Length CBF) [9]: This algorithm uses a variable-length coding, such as the Huffman coding, to represent counters with a variable number of bits. A

similar coding can be used for efficient representation of counters in $VI-CBF$. Unfortunately, even using index tables, a lookup in $VL-CBF$ might be 100 times slower than in the standard BF. Thus, it cannot be implemented at line rate.

- **SBF** (Stateful BF) **ACSM** (Approximate Concurrent State Machine) [3]: This scheme enables the representation of dynamically-changing states of flows. Using variable increments, it is possible to represent the set of two states hashing into the same entry as the sum of their hashed values, instead of simply storing a DK (Don't Know) value as currently done.
- **Counter Braids** [5]: This efficient counter architecture incrementally compresses flow counters as it uses a hierarchy of counters braided via a random graph. It later uses an iterative reconstruction scheme for the recovery of flow counters. The decoding of all flow counters is required to obtain any single flow counter. By using variable increments in each of its stages, it would provide more information to its reconstruction scheme, which may help ensure better guarantees on the termination of the decoding scheme.
- **Fingerprint-based Schemes** [3], [4]: Fingerprint-based schemes typically use d -left hashing to obtain balanced allocations of elements into buckets, and then use hashed fingerprints within each bucket to store information associated with each flow. While fingerprint-based schemes belong to a different family of algorithms than CBF, they may also be complemented using the variable-increment idea. For instance, a fingerprint-based scheme may store up to h states for each flow, or allow to store up to h flow mini-fingerprints together with each main fingerprint. By summing these state values, a variable-increment idea may decrease the number of bits required to store these values. In addition, it may behave more gracefully when there are more than h states, by temporarily losing some information, but being able to recover it with high probability upon deletion (i.e. decrement) of some of the states.

There are a few additional variants to CBF that the generic variable-increment idea does not necessarily improve. In particular, in [10], counters are used for the estimation of the multiplicities of individual items. Several schemes are introduced, and only some of them support deletions.

Finally, there are many works on B_h sequences [11]–[14], yet none have been applied to network applications.

D. Contributions

This paper presents an improved Counting Bloom Filter technique based on variable increments. This technique can also implement element insertions, deletions and queries using a constant complexity per element. We suggest two schemes based on CBFs with variable increments.

We first present the $B_h - CBF$ scheme. This scheme is based on B_h sequences. To the best of our knowledge, *this is the first time that B_h sequences are used in network applications*. Intuitively, a B_h sequence is a set of integers with the property that for any $h' \leq h$, all the sums of h' elements from the set are distinct. Therefore, given a sum of h' elements, we can determine whether an element of the B_h sequence is a part of the sum. In the $B_h - CBF$ scheme we have in each hash entry a pair of counters: one with fixed increments, and another one with variable increments that are selected from the B_h sequence. We illustrate the $B_h - CBF$ scheme and compute its false positive rate.

Then, we present the $VI - CBF$ scheme. In this scheme, each hash entry only contains a single counter, as illustrated above. We analytically show that the $VI - CBF$ scheme can always achieve a lower false positive rate and a lower overflow probability bound than CBF in practical systems.

We also provide detailed implementation considerations for these schemes. We discuss the complexity and throughput of the schemes, and show that their complexity overhead is lower than would be expected, especially for the $VI - CBF$ scheme, which can avoid using any lookup table.

Further, although in each operation both schemes require calculating $2k$ hash functions instead of k in CBF, we show that the relative increase in the number of required random bits is very small. For instance, the $VI - CBF$ scheme typically needs $k \cdot (\lceil \log_2(m) \rceil + 1)$ bits instead of $k \cdot (\lceil \log_2(m) \rceil)$ in CBF.

Last, we evaluate all schemes using simulations. We show that this method can reduce the false positive rate of the original CBF by up to an order of magnitude, or alternatively reduce the memory requirements for a requested false positive rate by 33%.

II. THE $B_h - CBF$ SCHEME

A. B_h Sequences

In this section we introduce the $B_h - CBF$ scheme, a *variable-increment* CBF (Counting Bloom Filter) based on B_h sequences [11]. We start with the formal definition of B_h sequences. B_2 sequences are also called Sidon sequences [12].

Definition 1 (B_h Sequence): Let $D = \{v_1, v_2, \dots, v_\ell\} \subseteq \mathbb{N}^*$ be a sequence of positive integers. Then D is a B_h sequence iff all the sums $v_{i_1} + v_{i_2} + \dots + v_{i_h}$ with $1 \leq i_1 \leq \dots \leq i_h \leq \ell$ are distinct.

Example 1: Let $D = \{v_1, v_2, v_3, v_4\} = \{1, 4, 8, 13\} \subseteq \mathbb{N}^*$. We can see that all the 20 sums of 3 elements of D are distinct: $1 + 1 + 1 = 3, 1 + 1 + 4 = 6, 1 + 1 + 8 = 10, 1 + 1 + 13 = 15, 1 + 4 + 4 = 9, 1 + 4 + 8 = 13, 1 + 4 + 13 = 18, 1 + 8 + 8 = 17, 1 + 8 + 13 = 22, 1 + 13 + 13 = 27, 4 + 4 + 4 = 12, 4 + 4 + 8 = 16, 4 + 4 + 13 = 21, 4 + 8 + 8 = 20, 4 + 8 + 13 = 25, 4 + 13 + 13 = 30, 8 + 8 + 8 = 24, 8 + 8 + 13 = 29, 8 + 13 + 13 =$

$34, 13 + 13 + 13 = 39$. Therefore, D is a B_3 sequence. However, $4 + 4 + 4 + 4 = 16 = 1 + 1 + 1 + 13$, therefore D is not a B_4 sequence.

We can observe that for any $h' \in [1, h]$ the sums of exactly h' elements of the B_h sequence are also distinct.

Observation 1: If $D = \{v_1, v_2, \dots, v_\ell\}$ is a B_h sequence and $h' \in [1, h]$ then D is also a $B_{h'}$ sequence.

Proof: The case of $h' = h$ is trivial. If for some $h' \in [1, h - 1]$ we have the same sum of h' elements with two different sets of elements of D , we can simply add to each of the sums the same arbitrary $h - h'$ new elements. We obtain two non-distinct sums of exactly h elements from the B_h sequence. Contradiction to Definition 1. ■

Example 2: We look again at the B_3 sequence D from Example 1. For $h' = 1$, every two elements of D are different. So D is a B_1 sequence. For $h' = 2$, we can see that all the 10 sums of 2 elements of D are distinct: $1 + 1 = 2, 1 + 4 = 5, 1 + 8 = 9, 1 + 13 = 14, 4 + 4 = 8, 4 + 8 = 12, 4 + 13 = 17, 8 + 8 = 16, 8 + 13 = 21, 13 + 13 = 26$. Therefore, D is a B_2 sequence as well.

B. Scheme Principles

We now introduce the $B_h - CBF$ scheme to represent $|S| = n$ elements using m entries. While in CBF, each entry contains a single counter with fixed increments of one, in $B_h - CBF$, each entry contains a *pair of counters*: The first counter, with fixed increments of one, counts the number of elements hashed into this entry (as in CBF). The second counter, with variable increments, provides a weighted sum of these elements. Its variable increments are selected from a pre-determined B_h sequence $D = \{v_1, v_2, \dots, v_\ell\}$.

Figure 2 illustrates how these counters are used and updated given element insertions and queries. Note that in Section IV we further show how to implement these operations efficiently in hardware. $B_h - CBF$ uses two sets of k hash functions. The first set $H = \{h_1, \dots, h_k\}$ uses k hash functions with range $\{1, \dots, m\}$, i.e. it points to the set of entries. The second set $G = \{g_1, \dots, g_k\}$ uses k functions with range $\{1, \dots, \ell\}$, i.e. it points to the set D .

Insertion — Upon insertion, an element x is hashed into the k hash entries pointed by $\{h_1, \dots, h_k\}$. At each entry $h_i(x)$, $B_h - CBF$ updates the pair of counters as follows. The first counter, with fixed increments, is incremented by one. The second counter, with variable increments, is incremented by the element $v_{g_i(x)}$ of the B_h sequence D , where g_i is the corresponding hash function of G .

Upon deletion, the counters are decremented similarly.

Example 3: Figure 2(a) illustrates the insertion of two elements u and v . It uses the B_3 sequence $D = \{v_1, v_2, v_3, v_4\} = \{1, 4, 8, 13\}$ from Example 1, and each element is hashed into $k = 2$ entries. In this example, $\{h_1(u), h_2(u)\} = \{1, 3\}$ and $\{g_1(u), g_2(u)\} = \{3, 2\}$. The packet belonging to flow u is hashed into entries 1, 3 using hash functions h_1, h_2 . It increments the first counter of each entry by one. It also increments the second counters of these entries by $(v_{g_1(u)}, v_{g_2(u)}) = (v_3, v_2) = (8, 4)$, respectively. Similarly, a packet of flow v is hashed into entries 5 and 7 and increments their variable-increment counters by 13 and 4, respectively.

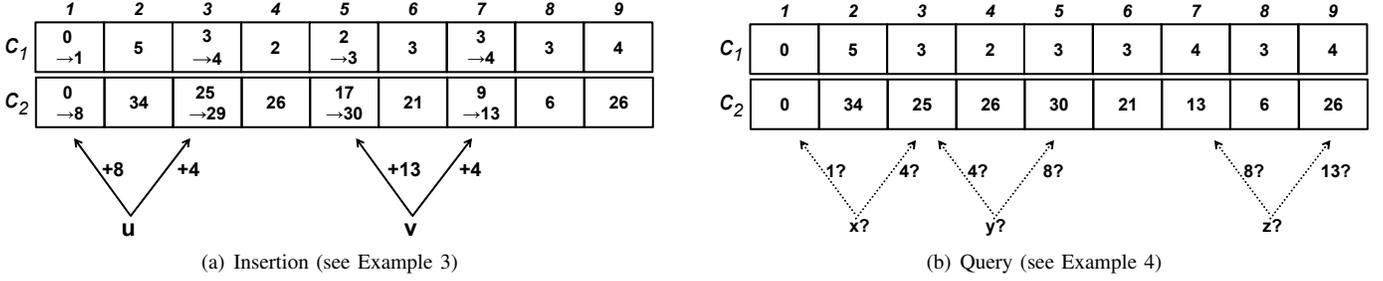


Fig. 2. Examples of insertion and query in $B_h - CBF$ with the B_3 sequence $D = \{1, 4, 8, 13\}$.

Query — To query whether an element y is in S , $B_h - CBF$ uses the unique properties of the B_h sequence. For $i \in [1, k]$, let $c_1(i)$ and $c_2(i)$ be the values of the fixed-increment and variable-increment counters in entry $h_i(y)$, respectively. $B_h - CBF$ asks whether y might have been inserted in this entry in the past. Namely, if y is hashed into value $v_{g_i(y)}$, it asks whether $c_2(i)$ can be a sum of $c_1(i)$ elements including $v_{g_i(y)}$. Specifically, in each entry $h_i(y)$, it considers several cases depending on the value of $c_1(i)$:

- If $c_1(i) = 0$, then as in CBF, $B_h - CBF$ determines that $y \notin S$.
- If $c_1(i) \in [1, h]$, $B_h - CBF$ considers the exact values of both the counters. In this case, no more than h elements were hashed into the hash entry $h_i(y)$. Therefore, $c_2(i)$ is a sum of $c_1(i) \leq h$ elements of D . Since D is a B_h sequence, $B_h - CBF$ can deduce the elements of D used in the sum $c_2(i)$ (i.e. in the insertions of these elements into this hash entry). In particular, $B_h - CBF$ can determine whether $v_{g_i(y)}$ is part of the sum $c_2(i)$. Otherwise, necessarily $y \notin S$.
- If $c_1(i) > h$, $B_h - CBF$ considers the entry as useless. The value of $c_2(i)$ is not used and we cannot determine that $y \notin S$ based on the current hash entry.

If $B_h - CBF$ cannot determine that $y \notin S$ based on any of the k hash functions, it determines that $y \in S$. With some probability, $B_h - CBF$ might be wrong and yield a false positive. More precisely, in case $y \notin S$, the false positive occurs if for each of the k hash functions, either there are more than h elements hashed into the corresponding hash entry, or the corresponding element of the B_h sequence was used by another element hashed into the same entry.

Example 4: As illustrated in Figure 2(b), we look at three queries for the $B_h - CBF$ introduced in Example 3. In this figure, we assume that for x, y, z the hash entries selected by the hash functions h_1 and h_2 are provided by their left and right outgoing arrows, respectively. For each element $a \in \{x, y, z\}$, we denote the values of the two counters in the hash entry $h_i(a)$ by $c_1^a(i), c_2^a(i)$.

First, in order to determine whether $x \in S$, we start by looking at the hash entry $h_1(x)$. Since $c_1^x(1) = 0$, $B_h - CBF$ can determine immediately that $x \notin S$. We now consider y . In its first hashed entry, $h_1(y)$, the number of elements is $c_1^y(1) = 3$. Since we use a B_h sequence D with $h = 3$, then $B_h - CBF$ can determine the components of the weighted sum by definition of the B_h sequence. In this case, $B_h - CBF$ can deduce that $c_2^y(1) = 25$ is comprised of $4 + 8 + 13 = 25$. Since y increments this counter by $v_{g_1(y)} = 4$, $B_h - CBF$

cannot exclude $y \in S$ based on this hash entry. However, since $c_1^y(2) = 3 \leq h$ as well, the variable-increment counter $c_2^y(2) = 30 = 4 + 13 + 13$ is not comprised of $v_{g_2(y)} = 8$. Thus, $B_h - CBF$ can also determine that $y \notin S$. Finally, applying the above method to z , $B_h - CBF$ cannot deduce that $z \notin S$ based on the two hash entries since $c_1^z(1), c_1^z(2) = 4 > h$. Therefore, $B_h - CBF$ determines that $z \in S$. This might of course yield a false positive if $z \notin S$. (In fact, using a more careful examination, it is possible to determine that indeed $z \notin S$, as detailed below with the *improved* $B_h - CBF$ scheme.)

C. False Positive Rate

We now provide the false positive rate of $B_h - CBF$. As shown in the literature, in real-world systems, practical hash functions usually work as if they were fully random [15]. We assume in our proofs that the hash functions map items to random numbers uniformly distributed over their given range, and that the inserted elements and the elements in the query are independent.

Theorem 1: The false positive rate of $B_h - CBF$ is given by:

$$FPR = \left(1 - \sum_{j=0}^h \binom{nk}{j} \left(\frac{\ell-1}{\ell m} \right)^j \left(1 - \frac{1}{m} \right)^{nk-j} \right)^k. \quad (1)$$

Proof: Let X denote the number of elements hashed into an arbitrary entry. The probability of the event $X = j$ is given by

$$\Pr(X = j) = \binom{nk}{j} \left(\frac{1}{m} \right)^j \left(1 - \frac{1}{m} \right)^{nk-j}. \quad (2)$$

When exactly $X = j$ elements are hashed into an entry, a specific value of the B_h sequence is not used by any of them with probability $\left(1 - \frac{1}{\ell} \right)^j$. Therefore, the false positive rate FPR , i.e. the probability that for an element $y \notin S$ we cannot deduce from each of the k hash entries that $y \notin S$, is

$$FPR = \left(1 - \sum_{j=0}^h \Pr(X = j) \left(1 - \frac{1}{\ell} \right)^j \right)^k, \quad (3)$$

yielding the result. \blacksquare

Note that by adopting a more complex query scheme, we can further decrease the false positive rate. This *improved* $B_h - CBF$ scheme analyzes the hash entry even when there

are more than h hashed elements. In such a case, the B_h sequence definition does not directly help anymore, because the elements in the sum $c_2(i)$ are not necessarily unique when there are more than h elements. However, by examining all the possible variable increments that can lead to the sum $c_2(i)$, we can still conclude that an element cannot have been inserted into this entry. For instance, in Figure 2(b), the first hashed entry of z indicates that there are $c_1^z(1) = 4 > h$ values totaling $c_2^z(1) = 13$. In addition, the first hashed value of z is 8. But 8 cannot be possible part of this sum, since the other $c_1^z(1) - 1 = 4 - 1 = 3$ values of the sum would need to total $c_2^z(1) - v_{g_1(z)} = 13 - 8 = 5$, and this is impossible, as shown in Example 1. So $z \notin S$. More details and analysis can be found in Section VI.

III. THE VI - CBF SCHEME

A. Scheme Description

The $B_h - CBF$ scheme suggested above uses two counters per entry instead of a single counter. This nearly doubles the needed number of bits (neglecting the differences in counter sizes). Consequently, we introduce *VI - CBF* (Variable-Increment Counting Bloom Filter), which also uses variable increments but only relies on a single variable-increment counter per entry, without the additional counter that indicates the number of hashed elements.

Specifically, as previously illustrated in Figure 1(c), we use again an array of m entries to represent $|S| = n$ elements. In each array entry, the single variable-increment counter is updated exactly like the second counter in the $B_h - CBF$ scheme, using variable increments selected from a set $D = \{v_1, v_2, \dots, v_\ell\}$. We use again two sets of k hash functions, $H = \{h_1, \dots, h_k\}$ and $G = \{g_1, \dots, g_k\}$. Upon insertion, at each corresponding array position $h_i(x)$, the counter is incremented by the element $v_{g_i(x)}$ of the set D . Likewise, upon deletion, counter $h_i(x)$ is decremented by $v_{g_i(x)} \in D$.

We now want to find an appropriate set D for the *VI - CBF* scheme. A problem in the *VI - CBF* scheme is that *it cannot directly use B_h sequences anymore*. This is because the B_h sequence definition requires to know the number of elements in a sum. However, unlike the $B_h - CBF$ scheme, the *VI - CBF* scheme cannot obtain it because it does not have a small counter that provides the number of elements hashed into a given entry.

B. A First Option for D : \tilde{B}_h Sequences

According to the last observation, we now suggest to use more complicated sequences, and call them \tilde{B}_h sequences. Informally, the \tilde{B}_h sequences are the sequences that can still help distinguish the elements in a sum *even when the exact number of elements is unknown*. As long as there are at most h elements in this sum, we can know it from the sum and determine the elements.

Definition 2 (\tilde{B}_h Sequence): Let $D = \{v_1, v_2, \dots, v_\ell\} \subseteq \mathbb{N}^*$ be a sequence of positive integers. Then D is a \tilde{B}_h sequence iff all the sums $v_{i_1} + v_{i_2} + \dots + v_{i_{h'}}$ with $1 \leq h' \leq h$ and $1 \leq i_1 \leq \dots \leq i_{h'} \leq \ell$ are distinct and differ from all the sums of more than h elements of D .

Example 5: Let $D = \{v_1, v_2\} = \{3, 4\} \subseteq \mathbb{N}^*$. Then all the sums of up to two elements are distinct, and also different from all the sums of at least three elements: the sums of at most two elements are 3, 4, $3 + 3 = 6$, $3 + 4 = 7$, $4 + 4 = 8$, while the sums of at least three elements are at least 9. Therefore, D is a \tilde{B}_2 sequence.

C. A Simple Option for D : $D_L = [L, 2L - 1]$

In the general case, to query whether an element is hashed into an entry, the implementation of the $B_h - CBF$ and *VI - CBF* schemes requires the use of a predetermined two-dimensional binary table based on the set D (see Section IV). However, we will now present a set $D = D_L$ that *does not need such a lookup table*. Therefore, D_L is easier to implement in hardware.

In the next subsections, we first analyze the *VI - CBF* scheme given $D = D_L$. In Section IV we show that for this case, no additional memory is required. We then provide an exact calculation of the false positive rate of this detailed scheme. We also show that it *always* improves the false positive rate of CBF given a number $m \geq 20$ of memory entries and a number n of inserted elements.

Let $L \geq 2$ be a positive integer of the form $L = 2^i$. We define the set D_L of size L as $D_L = [L, 2L - 1] = \{L, L + 1, \dots, 2L - 1\}$.

We now want to compute the false positive rate of the *VI - CBF* scheme. First, if an element $y \notin S$ hashes into an entry counter $h_i(y)$ of value c , we want to determine the probability that we will be able to tell that $y \notin S$ given c . Note that the entry counter value c is defined as a sum of elements of D_L . To do so, we distinguish different values of c using the following lemma.

Lemma 1: Let y be an element whose i -th hash function $h_i(y)$ hashes into an entry of value c . If $(c - v_{g_i(y)}) \in (-\infty, -1] \cup [1, L - 1]$ then $y \notin S$.

Proof: We distinguish different values of c :

- If $c = 0$, then the number of elements in the sum is zero, and therefore $y \notin S$.
- If $c \in [L, 2L - 1]$, we can deduce that c is composed of a single element of D_L , because the minimal value of a sum of two or more elements is $L + L = 2L$. Further, this element is of course c . Therefore if $v_{g_i(y)} \neq c$, then $y \notin S$.
- If $c \in [2L, \dots, 3L - 1]$, we must have that c is a sum of two elements, because the maximal value of one element is $2L - 1$ and the minimal value of three elements is $3L$. For instance, $c = L + (c - L)$, or $c = (L + 1) + (c - L - 1)$, etc. Therefore, c can be comprised of any of the elements $\{L, L + 1, \dots, c - L\}$, but not of any $x \in \{c - L + 1, \dots, 2L - 1\}$, since in such a case $(c - x) < L$. So if $v_{g_i(y)} \notin \{L, L + 1, \dots, c - L\}$ (i.e. $(c - v_{g_i(y)}) < L$), then $y \notin S$.
- If $c \geq 3L$, c can be comprised of any of the elements in D_L , since $(c - v_{g_i(y)}) > L$ for any $v_{g_i(y)} \in [L, 2L - 1]$.

Summarizing the cases above, we cannot exclude that $y \in S$ if $c = v_{g_i(y)}$ or $(c - v_{g_i(y)}) \geq L$, hence the result. ■

In the next theorem we present the false positive rate of this case.

Theorem 2: The false positive rate of the $VI - CBF$ scheme using $D = D_L$ is given by:

$$FPR = \left(1 - \left(1 - \frac{1}{m} \right)^{nk} - \frac{L-1}{L} \binom{nk}{1} \frac{1}{m} \left(1 - \frac{1}{m} \right)^{nk-1} - \frac{(L-1)(L+1)}{6L^2} \binom{nk}{2} \left(\frac{1}{m} \right)^2 \left(1 - \frac{1}{m} \right)^{nk-2} \right)^k. \quad (4)$$

Proof: Let $y \notin S$ be an input to a query. As usual, since there are k hashed entries, the false positive rate FPR is given by

$$(1-p)^k, \quad (5)$$

where p is the probability that by considering one of the k hash entries used by hash function h_i , we can determine that $y \notin S$. Let X denote again the number of elements hashed into this entry and let c be the resulting counter value, i.e. the weighted sum of X elements of D_L . We distinguish several cases based on X .

As explained earlier, if $X = 0$ then clearly $y \notin S$.

If $X = 1$, then c has one of the L values of D_L , each with the same probability of $\frac{1}{L}$. Therefore c has one of the $L-1$ values that differ from $v_{g_i(y)}$ w.p. $\frac{L-1}{L} \cdot \Pr(X=1)$, in which case we can deduce that $y \notin S$.

If $X = 2$, there are L^2 possible ordered pairs of increments. Further, there are L^3 combinations of the values of the two increments and the corresponding increment $v_{g_i(y)}$ of the examined element, each having an equal probability of $1/L^3$. We now distinguish depending on the value of c . First, as explained in the proof of Lemma 1, if $c \in [2L, \dots, 3L-1]$, there are exactly $c-2L+1$ options to obtain a sum of c with two addends of D_L : The possibilities are $\{(L) + (c-L), (L+1) + (c-L-1), \dots, (c-L) + (L)\}$. Also, the sum of c cannot be comprised of $(3L-c-1)$ of the possible values D_L . Therefore, out of the L^3 combinations above, in

$$\begin{aligned} & \sum_{c=2L}^{3L-1} (c-2L+1)(3L-c-1) \\ &= \sum_{i=0}^{L-1} (i+1)(L-i-1) = \sum_{i=1}^{L-1} i(L-i) \end{aligned}$$

of them, we can determine that $y \notin S$. In addition, if $c \geq 3L$, then the value of X is not necessarily known and c can be comprised of any of the L values of D_L . Thus, the current entry is not used to determine that $y \notin S$.

If $X \geq 3$ then $c \geq 3L$ and the current entry is not used again to determine that $y \notin S$.

Combining all cases for X , we obtain the following formula for the probability p that we can determine $y \notin S$ using c .

$$p = \Pr(X=0) + \frac{L-1}{L} \Pr(X=1) + \frac{1}{L^3} \Pr(X=2) \sum_{i=1}^{L-1} i(L-i).$$

We use the formula $\sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1)$ and get

$$\sum_{i=1}^{L-1} i(L-i) = L \sum_{i=1}^{L-1} i - \sum_{i=1}^{L-1} i^2 = \frac{1}{6}(L-1)L(L+1). \quad (6)$$

Simplifying p , and using Equations (2) and (5), we obtain the result. \blacksquare

D. Improving The False Positive Rate of CBF

We now demonstrate that $VI - CBF$ can always improve the false positive rate of CBF as the system is scaled. For a fair comparison, we assume that the two schemes use the same amount of memory for the same number of inserted elements. In addition, since this false positive rate computation does not take into account the counter overflow probability, we also show that the $VI - CBF$ scheme always obtains a lower overflow probability bound than CBF.

Assume that CBF uses four bits per counter (a common assumption, initially suggested by Fan et al. [2]). Let α denote the memory *bit-per-element ratio*, so that for every number of elements n , the memory size is αn bits with $m = \lceil \frac{\alpha n}{4} \rceil$ counters. Then the following theorem compares the performances of $VI - CBF$ and CBF as n is scaled and *both schemes use the same memory size*.

Theorem 3: While keeping the same bit-per-element ratio $\alpha > 0$ (and, as a consequence, also the same total memory size), $VI - CBF$ satisfies the following properties when compared to CBF:

- (i) $VI - CBF$ obtains a lower false positive rate than CBF with m counters for any $m \geq m_0 = 20$.
- (ii) When α goes to infinity, i.e. the two systems are made increasingly efficient, the ratio of their false positive rates goes to 0.
- (iii) $VI - CBF$ obtains a lower counter overflow probability bound than the classical bound for CBF from [2].

Proof: We first consider a CBF with m counters, n elements and k hash functions. Its false positive rate $FPR(CBF)$ is optimized when $k = \frac{m}{n} \ln(2)$ [2] and equals

$$\begin{aligned} FPR(CBF) &= \left(1 - \left(1 - \frac{1}{m} \right)^{kn} \right)^k \\ &= \left(1 - \left(1 - \frac{1}{m} \right)^{\ln(2)m} \right)^k > \left(\frac{1}{2} \right)^k. \quad (7) \end{aligned}$$

Incidentally, when k is not integer, it is possible to let each flow use either $k_1 = \lfloor k \rfloor$ hash functions with probability $p_k = \lfloor k \rfloor - k$ (i.e., as determined by an additional hash function), or $k_2 = \lceil k \rceil$ w.p. $1 - p_k$. The probability that an entry bit is equal to zero remains unchanged, but the resulting false positive rate is a weighted sum of those for k_1 and k_2 . The proof then stays unchanged, simply using the weighted sums instead. In the remainder, we assume that k is integer.

As mentioned in [2], using four bits per counter in CBF suffices to obtain a negligible counter overflow probability of

$$\begin{aligned} \Pr \left(\max_i c(i) \geq 16 \right) &\leq m \left(\frac{enk}{16m} \right)^{16} \\ &= m \left(\frac{e \ln(2)}{16} \right)^{16} \approx 1.37 \cdot 10^{-15} \cdot m. \quad (8) \end{aligned}$$

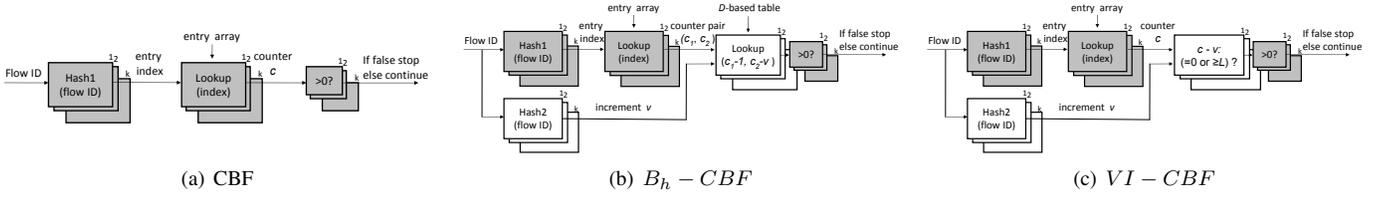


Fig. 3. Logical View of Hardware Implementation. Components that also appear in CBF are presented in gray, while white ones are new.

Let's now consider the $VI - CBF$ scheme with m' counters, $n' = n$ elements and k hash functions. Note that we use the same k as in CBF scheme for simplicity, but might further improve the false positive rate with a different k . We choose the parameter L of this scheme to be $L = 4$, and therefore $D_L = \{L, L + 1, \dots, 2L - 1\} = \{4, 5, 6, 7\}$. The maximal counter increment is $\max(D_L) = 2L - 1 = 7$. Instead of using four bits per counter as in CBF, we suggest to use eight bits per counter. For simplicity, we also assume that m is even. Therefore, given the same total number of memory bits, we use $m' = \lceil \frac{4m}{8} \rceil = \lceil \frac{m}{2} \rceil = \frac{m}{2}$ counters, i.e. half the number of counters since the counters are twice as large. Let $r = 1/2$ denote the ratio in the number of counters, such that $m' = mr$. Likewise, let p denote again the probability that by considering one of the k hash entries used by hash function h_i , we can determine that $y \notin S$ and let X denote again the number of elements hashed into this entry.

Comparing Equation (4) on the false positive rate of $VI - CBF$ to Equation (7) on the false positive rate of CBF, we can see that in order to demonstrate that the false positive rate of $VI - CBF$ is lower than the false positive rate of CBF, we just need to prove that $p > \frac{1}{2}$, where

$$\begin{aligned} p &= \left(1 - \frac{1}{m'}\right)^{n'k} + \frac{L-1}{L} \binom{n'k}{1} \frac{1}{m'} \left(1 - \frac{1}{m'}\right)^{n'k-1} \\ &\quad + \frac{(L-1)(L+1)}{6L^2} \binom{n'k}{2} \left(\frac{1}{m'}\right)^2 \left(1 - \frac{1}{m'}\right)^{n'k-2} \\ &= \Pr(X=0) + \frac{L-1}{L} \Pr(X=1) + \frac{(L-1)(L+1)}{6L^2} \Pr(X=2). \end{aligned}$$

Distinguishing the elements in the formula of p , we get

$$\begin{aligned} \Pr(X=0) &= \left(1 - \frac{1}{m'}\right)^{n'k} = \left(1 - \frac{1}{mr}\right)^{m \ln(2)} \\ &\geq \left(1 - \frac{1}{m_0 r}\right)^{m_0 \ln(2)}, \end{aligned}$$

where the last inequality holds for $m \geq m_0$. Likewise,

$$\begin{aligned} \Pr(X=1) &= \binom{n'k}{1} \frac{1}{m'} \left(1 - \frac{1}{m'}\right)^{n'k-1} \\ &= m \ln(2) \cdot \frac{1}{mr} \cdot \frac{mr}{mr-1} \cdot \left(1 - \frac{1}{mr}\right)^{m \ln(2)} \\ &= \frac{m \ln(2)}{mr-1} \cdot \left(1 - \frac{1}{mr}\right)^{m \ln(2)} > \frac{\ln(2)}{r} \cdot \left(1 - \frac{1}{mr}\right)^{m \ln(2)} \\ &\geq \frac{\ln(2)}{r} \cdot \left(1 - \frac{1}{m_0 r}\right)^{m_0 \ln(2)}, \end{aligned}$$

where again the last inequality holds for $m \geq m_0$. And again for $m \geq m_0$,

$$\begin{aligned} \Pr(X=2) &= \binom{n'k}{2} \left(\frac{1}{m'}\right)^2 \left(1 - \frac{1}{m'}\right)^{n'k-2} \\ &= \frac{m \ln(2)(m \ln(2) - 1)}{2} \cdot \left(\frac{1}{mr}\right)^2 \cdot \left(\frac{mr}{mr-1}\right)^2 \cdot \left(1 - \frac{1}{mr}\right)^{m \ln(2)} \\ &= \frac{m \ln(2)(m \ln(2) - 1)}{2(mr-1)^2} \cdot \left(1 - \frac{1}{mr}\right)^{m \ln(2)} \\ &\geq \frac{m \ln(2)(m \ln(2) - 1)}{2(mr)^2} \cdot \left(1 - \frac{1}{mr}\right)^{m \ln(2)} \\ &= \frac{\ln(2)(\ln(2) - \frac{1}{m})}{2r^2} \cdot \left(1 - \frac{1}{mr}\right)^{m \ln(2)} \\ &\geq \frac{\ln(2)(\ln(2) - \frac{1}{m_0})}{2r^2} \cdot \left(1 - \frac{1}{m_0 r}\right)^{m_0 \ln(2)}. \end{aligned}$$

For $L = 4$, $r = 1/2$, $m_0 = 20$, we get

$$\begin{aligned} p &= \Pr(X=0) + \frac{L-1}{L} \Pr(X=1) + \\ &\quad \frac{(L-1)(L+1)}{6L^2} \Pr(X=2) \geq \left(1 - \frac{1}{m_0 r}\right)^{m_0 \ln(2)} \cdot \left(1 + \right. \\ &\quad \left. \frac{L-1}{L} \cdot \frac{\ln(2)}{r} + \frac{(L-1)(L+1)}{6L^2} \cdot \frac{\ln(2)(\ln(2) - \frac{1}{m_0})}{2r^2}\right) > 0.5. \end{aligned}$$

We denote by p_0 the last lower bound of p for these parameters, s.t.

$$\begin{aligned} p_0 &= \left(1 - \frac{1}{m_0 r}\right)^{m_0 \ln(2)} \cdot \left(1 + \frac{L-1}{L} \cdot \frac{\ln(2)}{r} \right. \\ &\quad \left. + \frac{(L-1)(L+1)}{6L^2} \cdot \frac{\ln(2)(\ln(2) - \frac{1}{m_0})}{2r^2}\right) > 0.5. \end{aligned}$$

Next, we look at the counter overflow probability of $VI - CBF$ of each counter $c(i)$. Let $\gamma(i)$ be the number of addends in the sum $c(i)$. Since the maximal increment of $c(i)$ is $2L - 1 = 7$, we must have that

$$\begin{aligned} \Pr\left(\max_i c(i) \geq 256\right) &\leq \Pr\left(\max_i \gamma(i) \geq \left\lceil \frac{256}{7} \right\rceil\right) \\ &= \Pr\left(\max_i \gamma(i) \geq 37\right) \leq m' \left(\frac{en'k}{37m'}\right)^{37} \leq m' \left(\frac{en'k}{32m'}\right)^{37} \\ &= mr \left(\frac{em \ln(2)}{32mr}\right)^{37} = mr \left(\frac{e \ln(2)}{16}\right)^{37}. \end{aligned}$$

Comparing to the classical overflow probability bound of CBF (from Equation (8)), since $\frac{e \ln(2)}{16} < 1$ and $r = 0.5$, we can see that the overflow probability bound of $VI - CBF$ is strictly lower.

To complete the proof we can see that when α goes to infinity, the optimal number of hash functions for CBF (k) also

goes to infinity. Thus, we have

$$\begin{aligned} \frac{FPR(VI - CBF)}{FPR(CBF)} &= \frac{(1-p)^k}{\left(1 - \left(1 - \frac{1}{m}\right)^{\ln(2)m}\right)^k} \\ &\leq \left(\frac{1-p_0}{1/2}\right)^k \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

IV. IMPLEMENTATION CONSIDERATIONS

In this section we discuss the implementation of the $B_h - CBF$ and $VI - CBF$ schemes in comparison with CBF. We consider several issues such as logic complexity, memory throughput and hashing complexity.

Pipeline Complexity — The insertion, deletion and query of each element can be organized in a pipeline manner, so that each operation is implemented at line rate. We assume for simplicity that there are no conflicts between elements at different steps of the pipeline. Note also that the set D is fixed, and therefore the pipeline is also fixed and there is no need to recompute online any of the values related to the B_h sequences.

Figure 3 illustrates the logical pipeline implemented to go through a query of packet x . It focuses on one of k parallel pipelines, shown on k parallel planes, and corresponding to the k hash entries.

Figure 3(a) presents the implementation of CBF in which the flow ID of the packet is hashed into one of the CBF array entries. The corresponding counter value c is considered. If it equals zero, CBF determines that $x \notin S$. Otherwise, CBF continues to check the next hash entry.

Figure 3(b) illustrates the implementation of $B_h - CBF$. Components that also appear in CBF are presented in gray, and the additional components in white. $B_h - CBF$ uses two hash functions instead of one. The first points to an entry in the $B_h - CBF$ array with the pair of counters (c_1, c_2) . The second points to an increment from the set D denoted by v .

In order to efficiently determine whether the weighted sum c_2 can be comprised of v using exactly c_1 addends, we suggest using a predetermined two-dimensional binary table A that is based on D . The bit $A[i][j]$ is set if there is a sum of i addends from D that equals j . We notice that the value c_2 can be comprised of v using c_1 addends for $c_1 \geq 1$ only if there is a sum of exactly $c_1 - 1$ addends that equals the value $c_2 - v$, i.e. if the bit value $A[c_1 - 1][c_2 - v]$ is set. The cases out of the table boundary ($c_1 = 0$ and $c_1 > h$) can easily be defined in the lookup procedure. In summary, if $c_1 > h$ or if the relevant bit value is set, $B_h - CBF$ continues to the next hash entry. Otherwise, it determines that $x \notin S$.

Note that $B_h - CBF$ enters table A only for cases where $c_1 \in [1, h]$. Therefore the first dimension index is $c_1 - 1 \leq h - 1$ and the maximal possible value of the weighted sum is at most $(c_1 - 1) \cdot \max(D) \leq (h - 1) \cdot v_\ell$. Thus, the memory size of the table A is at most $h \cdot ((h - 1) \cdot v_\ell + 1) \leq h^2 \cdot v_\ell$ bits.

Next, we consider the implementation of the $VI - CBF$ scheme given the set $D = D_L$. As explained in Section III-C a lookup table is not required in this case. As in the previous scheme, two hash functions are calculated. The first points to an

entry in the $VI - CBF$ array with a counter c and the second points again to an increment $v \in D$. Instead of the lookup operation, we just check whether $c - v = 0$ or $(c - v) \geq L$ (Lemma 1). If so, $VI - CBF$ continues to the next hash entry. Otherwise, it determines that $x \notin S$.

Memory Throughput — In order to increase the memory throughput, we can implement the two schemes using the ideas implemented in the Blocked Bloom Filter [16]. For each element, all hash functions are mapped into a single block in the memory, i.e. a single memory word. Although this technique suffers from a higher false positive rate, it is clearly energy-efficient and improves the memory throughput, because there is a single memory word access instead of up to k .

Hashing Complexity — In each operation of insertion, deletion or query, the two suggested schemes require the computation of $2k$ hash functions instead of k in CBF. However, the total required number of random bits is much less than twice the number in CBF. In order to point to a random element in D , only $\lceil \log_2(|D|) \rceil = \lceil \log_2(\ell) \rceil$ random bits are required. Since the selection of D with small cardinality (such as $\ell = 4$) is enough to have an improved false positive rate, as explained in the proof of Theorem 3, we can reduce to two the number of random bits generated by each of $G = \{g_1, \dots, g_k\}$ hash functions. In addition, given the same overall memory size, the number of counters in $VI - CBF$ is reduced by a factor of two, because they are twice larger, therefore they need one less bit for each counter selection. Thus, $VI - CBF$ typically needs $k \cdot (2 + (\lceil \log_2(m) \rceil - 1)) = k \cdot (\lceil \log_2(m) \rceil + 1)$ bits instead of $k \cdot (\lceil \log_2(m) \rceil)$ in CBF.

V. COMBINING WITH THE $ML-HCBF$

In this section we show that $VI - CBF$ can be combined with the MultiLayer Hashed CBF scheme ($ML-HCBF$) [7] to further decrease its memory requirements for a requested false positive rate.

Instead of using one level with a constant number of bits per counter, $ML-HCBF$ uses a hierarchical structure of several layers with narrower counters such that the number of counters is a decreasing function of the level number. The number of counters in the first level is a baseline number of counters, and their number in each additional level is based on the overflow probability of counters in the previous level. During insertion, a counter in the first level is examined. If it is not saturated, it is simply incremented. If it is saturated, the counter position is hashed to obtain an address of a counter in the next levels. If it is also saturated, the procedure continues in the next levels until a non-saturated counter is found.

A similar concept can be implemented using variable increments, as in $VI - CBF$. We denote it $ML-VI-HCBF$. We increment a counter in the first level by a variable increment till it reaches its maximal value. If it was saturated, we continue and increment another counter in the next level by the rest of the current increment, such that the sum of their increments equals the requested one. By using perfect hashing as in the original work, the actual value of a counter can be calculated as the sum of the corresponding counters in one or more levels. For instance, instead of using counters of 7 bits as for the

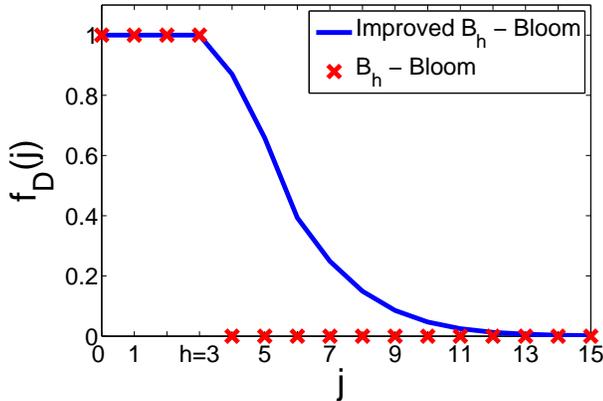


Fig. 4. The detection probability function $f_D(j)$ for the B_3 sequence $D = \{1, 4, 8, 13\}$.

$VI - CBF$, we can have a hierarchical structure of 4 levels with $(4, 3, 3, 3)$ bits per counter, i.e. only four bits per counter in the first level and three bits per counter in all the other levels.

VI. IMPROVEMENTS OF THE $B_h - CBF$ SCHEME

Detection with more than h elements — Consider a hash entry $h_i(y)$ with more than h hashed elements. In such a case, the B_h sequence definition does not help anymore, because the elements in the sum $c_2(i)$ are not necessarily unique. However, we can still obtain some information in certain cases. By examining all the possible variable increments that can lead to the sum $c_2(i)$, we can conclude that $y \notin S$ if $v_{g_i(y)}$ does not belong to any of these possible variable increments. Let v_i be an arbitrary element of $D = \{v_1, v_2, \dots, v_\ell\}$. Let c be an arbitrary sum of j elements of $D \setminus \{v_i\}$. With some probability, we can determine that c is not comprised of v_i . We denote by $f_D(j)$ the probability that we can indicate this fact based on the values c and j . The function $f_D(j)$ is called the *detection probability function*. Of course, if D is a B_h sequence, for any $j \in [0, h]$, there is only one sum of j elements that equals c . Thus, this sum must be the sum with the property that its elements are all in $D \setminus \{v_i\}$. Therefore, we must have that $f_D(j) = 1$ for $j \in [0, h]$. In the previously presented $B_h - CBF$ scheme, $f_D(j) = 0$ for $j > h$, since we cannot leverage the properties of the B_h sequence in such cases. In the improved scheme $f_D(j)$ is the probability that all the sums of j elements of D which equal c are not comprised of v_i . In all such cases, we can deduce that c is not comprised of v_i .

Using the last definition, we can now present the false

positive rate of the improved scheme.

$$\begin{aligned}
 & \left(1 - \sum_{j=0}^{\infty} \Pr(X = j) \left(1 - \frac{1}{\ell}\right)^j f_D(j)\right)^k \\
 &= \left(1 - \sum_{j=0}^h \Pr(X = j) \left(1 - \frac{1}{\ell}\right)^j\right)^k \\
 &= \left(1 - \sum_{j=h+1}^{\infty} \Pr(X = j) \left(1 - \frac{1}{\ell}\right)^j f_D(j)\right)^k \\
 &= \left(1 - \sum_{j=0}^h \binom{nk}{j} \left(\frac{\ell-1}{\ell m}\right)^j \left(1 - \frac{1}{m}\right)^{nk-j} - \sum_{j=h+1}^{\infty} \binom{nk}{j} \left(\frac{\ell-1}{\ell m}\right)^j \left(1 - \frac{1}{m}\right)^{nk-j} f_D(j)\right)^k.
 \end{aligned} \tag{9}$$

From the last statement, it is clear that this scheme has an improved false positive rate in comparison with the basic $B_h - CBF$ scheme. The false positive rate of the basic $B_h - CBF$ scheme presented in Equation (1), can be achieved from Equation (9) using $f_D(j) = 1$ for $j \in [0, h]$ and $f_D(j) = 0$ for $j > h$ since with this scheme we do not consider any hash entry with more than h hashed elements.

Example 6: We look again at the B_3 sequence D from Example 1. Figure 4 presents $f_D(j)$ for $j \in [0, 15]$. As explained earlier, for the basic $B_h - CBF$ scheme, we have that $f_D(j) = 1$ for $j \in [0, h]$ and $f_D(j) = 0$ for $j > h$. We now try to calculate $f_D(j)$ for our improved $B_h - CBF$ scheme. As explained earlier, $f_D(j) = 1$ for $j \in [0, h] = [0, 3]$. For instance, to calculate $f_D(4)$, we consider the $4^4 = 256$ possible combinations of four ordered elements of D . Among them there are, for each of the four values of D , 3^4 (ordered) combinations in which it does not appear. I.e. a total number of $4 \cdot (3^4) = 324$ cases of four addends $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4} \in D \setminus \{v_t\}$ for some $t \in [1, \ell]$. We denote by C the sum $v_{i_1} + v_{i_2} + v_{i_3} + v_{i_4}$. Among these cases, there are 282 cases in which any sum of four addends that equals C is not comprised of v_t . Therefore, $f_D(4) = \frac{282}{324} \approx 0.87$. We try to explain this results for a query of $y \notin S$. We look at the hash entry $h_i(y)$ with counters values $c_1(i)$ and $c_2(i)$. If $c_2(i)$ is not comprised of $g_i(y)$ and $c_1(i) = h + 1 = 4$, we can determine that $y \notin S$ based on the current hash entry with probability 0.87.

Example 7: Consider the query of the element z from Example 4, as illustrated in Figure 2(b). Although $c_1^z(1) = 4 > h$, we can notice that $c_1^z(2) - v_{g_1(z)} = 13 - 8 = 5$ cannot be comprised of exactly $c_1^z(1) - 1 = 4 - 1 = 3$ addends from the B_3 sequence D . Thus, we can determine that $z \notin S$, avoiding a false positive.

Counter Representation — We now suggest another possible improvement. As explained in the previous section the value of the second counter c_2 is at most v_ℓ times the value of the first counter c_1 . In order to reduce the overflow probability of c_2 we can update it differently. To show it, we define a new compressed counter c_3 , used instead of c_2 . The first counter c_1 is not changed. First, it is easy to see that for any arbitrary value c , if instead of incrementing c_2 by v_i , c_3 is incremented by $v_i - c$, the original value of c_2 can be calculated as $c_3 + c \cdot c_1$. Therefore, a first change can

be incrementing c_3 by $v_i - v_1$ instead of by v_i . With this change, the maximal increment is $v_\ell - v_1$, smaller than the original one and the counter overflow is decreased. In case that $v_\ell - v_1 \leq \frac{v_\ell}{2}$, the most significant bit of the counter can be reduced. However, since if $D = \{v_1, v_2, \dots, v_\ell\}$ is a B_h sequence then $D' = \{v_1 + c, v_2 + c, \dots, v_\ell + c\}$ is a B_h sequence as well, we can assume that the used B_h sequence has the property that $v_1 = 1$ (in order to minimize v_ℓ), so the improvement is not significant.

A second, more significant improvement is choosing c as the rounded value of the mean of the elements of D , i.e. $c = \text{round}(\frac{1}{q} \sum_{i=1}^q v_i)$. Now, c_3 is incremented by $v_i - c$ and might have also negative values. However, assuming insertion of independent elements, the mean of the value of this counter is close to zero (not exactly zero due to the rounding operation). Therefore, by representing a set of negative values as well as a set of positive values, we can use less bits for this counter without increasing the counter overflow probability.

VII. LOWER BOUNDS ON THE CODING SCHEMES

We now consider the $B_h - CBF$ scheme which uses a B_h sequence D and the $VI - CBF$ scheme with the option of a B_h sequence D . A fundamental view of these algorithms is that in each entry, using either one or two counters, they represent a multiset of up to h elements taken from a space of size $|D| = \ell$. This can provide us a lower bound on the needed number of bits.

As an example, consider again the $B_h - CBF$ scheme with the B_3 sequence $D = \{v_1, v_2, v_3, v_4\} = \{1, 4, 8, 13\}$ with $|D| = \ell = 4$. There are 20 multisets of exactly three elements from D , 10 multisets of two elements and of course ℓ multisets of size one. The empty set is also a subset of D . Thus there are 35 multisets of up to three elements from D . Therefore at least $\lceil \log_2(35) \rceil$ bits are required to represent the information in such cases.

More generally, in order to represent differently all these multisets, we must have a number of bits that equals at least the base-2 logarithm of the number of possible elements. In particular, since the number of multisets of up to h elements from a set of size ℓ equals $\phi(h, \ell) = \binom{h+\ell}{h}$, we obtain a lower bound of

$$\lceil \log_2 \phi(h, \ell) \rceil = \lceil \log_2 \binom{h+\ell}{h} \rceil \quad (10)$$

on the number of bits per entry. For instance, as we explained with $h = 3$ and $\ell = 4$, at least $\lceil \log_2 \phi(h, \ell) \rceil = \lceil \log_2 \binom{3+4}{3} \rceil = \lceil \log_2 \binom{7}{3} \rceil = \lceil \log_2 35 \rceil$ bits are required for each hash entry. However, while these provide attractive lower bounds, they might lack additional properties of $B_h - CBF$ that are essential for fast hardware implementation: in particular, during insertions and deletions, they first need to decode the value before re-encoding the next value, which is not needed in the schemes presented in this paper. Of course, special codes that can be incremented iteratively may be suggested.

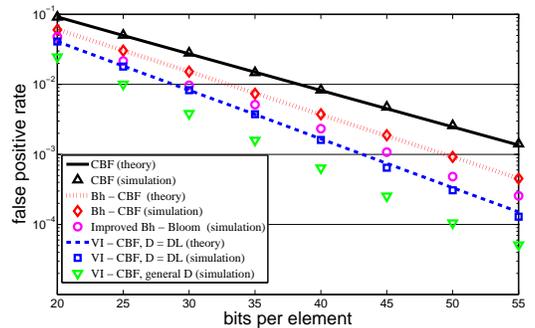


Fig. 5. Comparison of the false positive rates of CBF, $VI - CBF$ and $B_h - CBF$ schemes and the theoretical models from Theorem 1 and Theorem 2.

VIII. EXPERIMENTAL RESULTS

A. Trace-Driven Simulation

We conduct experiments using real-life traces recorded on a single direction of an OC192 backbone link [17], and rely on a 64-bit mix hash function [18] of the IP 5-tuple to implement the requested hash functions.

Figure 5 plots the false positive rate of various schemes as a function of the memory size (in bits per element). It also compares the obtained values with the values obtained by theory. For $B_h - CBF$, we use $D = \{1, 4, 13, 15\}$, and compare the false positive rate of the basic scheme with Theorem 1. We also present the results of the improved $B_h - CBF$ scheme that we introduced after Theorem 1. For $VI - CBF$, we first use $D = D_L$ with $L = 4$ (in comparison with Theorem 2), and later also use the more general set $D = \{8, 12, 14, 15\}$ (that was found to be a good one) to compare the two sets. We classically assume four bits per counter for CBF, and for the two $B_h - CBF$ schemes, twelve bits per hash entry (four and eight bits for the fixed-increment and variable-increment counters, respectively). For the $VI - CBF$, we use seven bits per counter for the $VI - CBF$ with $D = D_4$ and eight bits per counter for the general D .

First, the simulation results confirm the theory from Theorem 1 and Theorem 2. Further, all the suggested schemes improve upon CBF. In addition, the two variants of $VI - CBF$ outperform the $B_h - CBF$ and the improved $B_h - CBF$. $VI - CBF$ with the general set D yields the best performance, and the improvement is especially significant for larger numbers of bits per element. For instance, for 30 bits per element, the false positive rate for CBF, $B_h - CBF$, improved $B_h - CBF$, $VI - CBF$ with $D = D_L$ and $VI - CBF$ with a general D are 0.02803, 0.01521, 0.00970, 0.00825 and 0.00383, respectively, thus obtaining an improvement by a factor of 7. Likewise, for 50 bits we have 0.00258, 0.00092, 0.00048, 0.00031 and 0.00011 respectively, thus improving the result by over an order of magnitude. Of course, as mentioned, $VI - CBF$ with $D = D_L$ is easier to implement in hardware than $VI - CBF$ with a general D , hence there is a clear tradeoff between efficiency and complexity.

Alternatively, for the same false positive rate of CBF with 50 bits per element, $VI - CBF$ with a general D requires

	CBF	Spectral BF	dICBF	<i>ML-HCBF</i>	B_h -CBF	Improved B_h -CBF	<i>VI-HCBF</i> , $D = D_L$	<i>ML-VI-CBF</i> , $D = D_L$	<i>VI-CBF</i> , general D	<i>ML-VI-HCBF</i> , general D
Main Structure (KB)	14.1	8.12	5.2	7.14	12.07	11.31	10.97	6.27	9.28	5.80
Secondary Structure (KB)	-	-	-	0.41	-	-	-	0.33	-	0.82
Additional Tables (KB)	-	4	-	-	0.02	0.07	-	-	0.03	0.03
Total Size (KB)	14.1	12.12	5.2	7.55	12.09	11.38	10.97	6.60	9.31	6.65
False Positive Rate	10^{-3}	10^{-3}	$1.5 \cdot 10^{-3}$	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}

TABLE I

MEMORY REQUIREMENTS COMPARISON. THE RESULTS OF FOUR STATE-OF-THE-ART ALGORITHMS ARE PRESENTED ON THE LEFT SIDE OF THE TABLE, AS IN [7], AND THE RESULTS OF SIX SUGGESTED SCHEMES APPEAR ON THE RIGHT SIDE.

approximately 32 bits per element, hence a reduction of about a third in the memory requirement.

B. Comparison with State-of-the-Art Algorithms

We adopt the same settings as in [7] to examine the memory requirements of our suggested schemes in comparison with several well-known schemes such as CBF [2], Spectral BF [10], dICBF [4] and *ML-HCBF* [7]. Specifically, we compare the total memory size required to obtain a false positive rate of 10^{-3} when $|S| = n = 2000$ elements are represented.

For the new schemes, we consider the additional size if the lookup table, when required. For instance, as explained in Section IV, the B_h -CBF requires a table of size $h^2 \cdot v_\ell = 3^3 \cdot 15 = 135$ bits ≈ 0.02 KB. For the improved B_h -CBF we base, in this simulation, the query decision on counters with up to $2 \cdot h = 6$. Thus, the table size is $(2h)^2 \cdot v_\ell \approx 0.07$ KB. For *VI-CBF* scheme given the set $D = D_L$, a lookup table is not required. For the general set D , the size is at most $15 \cdot 15 = 225$ bits ≈ 0.03 KB.

The performance results of the previous schemes are taken from [7]. We present the results of six new schemes: B_h -CBF, the improved B_h -CBF, *VI-CBF* with $D = D_L$, *VI-CBF* with a general D and each of these two versions of *ML-VI-HCBF*, i.e. *VI-CBF* combined with *ML-HCBF*. For B_h -CBF and *VI-CBF*, we use the same set D and the same number of bits per counter as in the previous simulation. Table I summarizes the results.

In B_h -CBF, the required memory size is 12.09 KB, and it drops to 11.38 KB in the improved B_h -CBF. *VI-CBF* with $D = D_4$ requires 12842 counters of seven bits, while with the general D , 9500 counters of eight bits are used in addition to the lookup table. This yields a total memory size of 10.97 KB and 9.31 KB, respectively, i.e. improvements of 22.2% and 34.0% in comparison with CBF. All these schemes present lower memory requirements than CBF and the Spectral BF, but higher than those of the dICBF. However, dICBF may have additional issues, like overflows and complexity [7]. As mentioned in the Introduction, improving dICBF in the same way by using variable increments is left to future work.

For *ML-VI-HCBF* with $D = D_4$, we use four layers with 12842 counters in the first level (as in the original scheme), 802 in the second, 100 in the third and 6 in the fourth levels. In these four levels, 4, 3, 3 and 3 bits per counter were used, respectively. *ML-VI-HCBF* uses a total memory size of 6.60 KB, an improvement of 40% in comparison with *VI-CBF*,

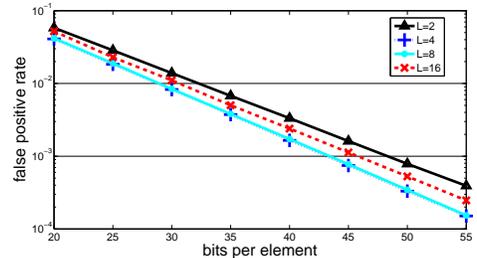


Fig. 6. False positive rate of the *VI-CBF* with $D = D_L$ for various L values.

and a 2x improvement (precisely, 53%) in comparison with CBF. Likewise, when the set $D = \{8, 12, 14, 15\}$ is considered, we have four levels of 5, 5, 5, 4 bits with an almost similar memory consumption of 6.65 KB. Both of these hierarchical schemes also perform better than *ML-HCBF*.

C. Optimizing the *VI-CBF* parameters

We want to examine the effect of the parameter L for the *VI-CBF* scheme with $D = D_L$ on the false positive rate. On the one hand, increasing L makes it easier to exclude membership of a non-member element based on one counter, while on the other hand it requires more bits per counter and thus reduces their number.

To do so, we first improve the analysis of the overflow probability presented in the proof of Theorem 3. We consider again a *VI-CBF* with m' counters, $n' = n$ elements, k hash functions, $L = 4$ and $D = D_L = \{v_1, v_2, \dots, v_\ell\} = \{L, L + 1, \dots, 2L - 1\} = \{4, 5, 6, 7\}$.

We present an improved upper bound for this probability (in comparison with the overflow probability of CBF from Equation (8), also if we use only 7 bits per counter in the *VI-CBF*). Using the same amount of memory as in CBF, we have here that $m' = mr$ for $r = 4/7$. To do so, we denote by $\gamma(i, j)$ (for $i \in [1, m']$, $j \in [1, \ell = 4]$) the number of addends in the sum $c(i)$ that equal v_j , s.t. $\gamma(i)$, the number of addends in the sum $c(i)$, holds $\gamma(i) = \sum_{j=1}^{\ell} \gamma(i, j)$. Using 7 bits per counter, the maximal value of a counter that does yield to an overflow is 127. We observe that if $c(i) \geq 128$, then $\gamma(i) \geq \lceil \frac{128}{v_\ell} \rceil = \lceil \frac{128}{7} \rceil = 19$. Further, if $c(i) \geq 128$ and $\gamma(i) = 19$, then $\gamma(i, 4) \geq 14$, i.e. at least 14 among the 19 addends in the sum $c(i)$ are $v_4 = 7$. Since all the elements of D are uniformly selected and the events,

$\{(\gamma(i, j) \geq 14, \gamma(i) = 19) | j \in [1, \ell = 4]\}$ are disjoint, we must have that $\Pr(\gamma(i, 4) \geq 14 | \gamma(i) = 19) \leq \frac{1}{\ell} = 0.25$.

We now have that

$$\begin{aligned}
\Pr\left(\max_i c(i) \geq 128\right) &\leq m' \Pr(c(i) \geq 128) \\
&\leq m' \cdot \left(\Pr(\gamma(i, 4) \geq 14, \gamma(i) = 19) + \Pr(\gamma(i) \geq 20)\right) \\
&= m' \cdot \left(\Pr(\gamma(i) = 19) \cdot \Pr(\gamma(i, 4) \geq 14 | \gamma(i) = 19) \right. \\
&\quad \left. + \Pr(\gamma(i) \geq 20)\right) \\
&\leq m' \cdot \left(0.25 \cdot \Pr(\gamma(i) = 19) + \Pr(\gamma(i) \geq 20)\right) \\
&\leq m' \cdot \left(0.25 \cdot \Pr(\gamma(i) \geq 19) + \Pr(\gamma(i) \geq 20)\right) \\
&\leq m' \cdot \left(0.25 \cdot \left(\frac{en'k}{19m'}\right)^{19} + \left(\frac{en'k}{20m'}\right)^{20}\right) \\
&= mr \cdot \left(0.25 \cdot \left(\frac{em \ln(2)}{19mr}\right)^{19} + \left(\frac{em \ln(2)}{20mr}\right)^{20}\right) \\
&= mr \cdot \left(0.25 \cdot \left(\frac{e \ln(2)}{19r}\right)^{19} + \left(\frac{e \ln(2)}{20r}\right)^{20}\right).
\end{aligned}$$

For $r = 4/7$, this upper bound of the overflow probability is lower than the bound in Equation (8).

We now assume a set with $n = 1024$ elements and variable numbers of bits per elements. We examine the values 2, 4, 8, 16 for L and for each value we use the optimal number k of hashed functions. Further, for each value of L , we use $4 + \lceil \log_2(v_\ell) \rceil = 4 + \lceil \log_2(2L - 1) \rceil$ bits per counter. For instance, for $L = 4$, we have $4 + 3 = 7$ bits per counter. The false positive rates of the optimal values of k are presented in Figure 6.

We can see that the performances of D_L with $L = 4$ and $L = 8$ are similar and are much better than the cases of $L = 2$ and $L = 16$. For example, for 30 bits per element we have false positive rates of 0.01388, 0.00825, 0.00841, 0.01105 for $L = 2, 4, 8, 16$, respectively.

D. Optimizing the $B_h - CBF$ parameters

In this section we provide several simulations results in order to examine the $B_h - CBF$ scheme. For instance, we show how (a) we can pick an optimal B_h sequence among all possible such sequences; (b) we can decode the information provided by the variable-increment counter in the case when there are more than h elements; (c) we can trade off the false positive rate against the overflow probability for a given number of bits using the entry sizes; and (d) we can choose an optimized coding of the counters by coding either their difference with their expected values.

We first try to find a B_h sequence that minimizes the false positive rate of the $B_h - CBF$ scheme. Based on Equation (1), for any values of k, m, n , the false positive rate is always decreased when h, ℓ are increased. Therefore, given k, m, n , the false positive rate is minimized when $h, \ell \rightarrow \infty$. Unfortunately, we encounter some constraints on their values. We also observe that according to this formula the false positive rate of the basic scheme depends only on the parameters h, ℓ of the B_h sequence and not on the specific values of $\{v_1, v_2, \dots, v_\ell\}$. Upon insertion of an element x , the counter with variable increment, is incremented by the element $v_{g_i(x)} \in D$, thus the increment is

TABLE II
 B_h VARIATIONS WITH THE CONSTRAINT $v_\ell \leq 15$

h, ℓ	B_h Example	Optimal k	FPR (for 30 bits per element)
$h = 1, \ell = 15$	[1,15]	2	0.04630
$h = 2, \ell = 5$	{1,2,4,8,13}	4	0.01963
$h = 3, \ell = 4$	{1,2,5,14}	5	0.01521
$h = 12, \ell = 3$	{1,2,15}	5	0.02902

at most v_ℓ . Since each of the counters has a fixed size we must bound this increment. For example, we can bound the maximal increment to be at most $2^4 - 1 = 15$, i.e. require that $v_\ell \leq 15$. In such a case, we can deduce that the variable-increment counter needs approximately four more bits than the typical counter size of four bits of CBF, i.e. it requires a total of about eight bits. Now, with such a constraint, we have a tradeoff: we can choose B_h sequence with large h and small ℓ or vice versa. In our case, there are four different possibilities: First, we can have a B_h sequence with $\ell = 15, h = 1$, i.e. a B_1 sequence of all the 15 elements of [1,15]. Second, we can use a B_2 sequence, which has a maximal size of $\ell = 5$. Similarly, there is a maximal size of $\ell = 4$ for $h = 3$. We can also find a B_{12} sequence of size $\ell = 3$. We assume the following parameters: $n = 1024$ and the number of bits per elements is $bpe = 30$, i.e. the total memory size is $n \cdot bpe$ bits. For each of these four possibilities, a minimal false positive rate is achieved for some value of k .

The theoretical results are summarized in Table II where an example of a B_h of each of the cases is also given. We can see that the minimal false positive rate is achieved for $h = 3, \ell = 4$ and $k = 5$.

To further reduce the false positive rate, we can also exploit the information provided by the variable-increment counter in the case when there are more than h elements as explained in Section VI. To implement this generalization, we suggest using another predetermined two-dimensional binary table with more entries than the original table.

With this improvement, the false positive rate is affected by the exact values of $\{v_1, v_2, \dots, v_\ell\}$. Thus, we want to examine the possible B_h sequences with $h = 3, \ell = 4, v_\ell \leq 15$. Without loss of generality, we assume that $v_1 = 1$. Otherwise, we can just have the same false positive rate using the B_h sequence $\{v_1 - (v_1 - 1), v_2 - (v_1 - 1), \dots, v_\ell - (v_1 - 1)\}$. There are 48 such B_h sequences. For each of them, the false positive rate is smaller than the result of Equation (1) that appears in Table II. For the parameters above, the minimal simulated false positive rate is achieved for the B_h sequence $D = \{1, 4, 13, 15\}$ and equals 0.00970, which is lower by 36% than the unoptimized value of 0.01521.

There exists a tradeoff between the false positive rate and the overflow probability. In fact, decreasing the number of bits per variable-increment counter enables to store more counters, thus decreasing the false positive rate, but also runs into overflow more frequently. For instance, we consider again the case of 30 bits per element and $k = 5$. Using four bits per fixed-increment counter and six bits per variable-increment counter (i.e. $4 + 6 = 10$ bits for each hash entry), we have a false positive rate of 0.00475 and a counter overflow probability of

0.00234. With three and five bits (respectively), there are more counters and the false positive rate falls to 0.00262 with the price of greater overflow probability of 0.05726. We notice that the overflow probability can be larger than the false positive rate, since it is calculated as the ratio of counters with overflow. A false answer to a query can be returned even if some of the k counters have encountered overflow.

In order to improve the presented tradeoff, we now suggest to use some more complicated counter representation scheme, explained in the last paragraph of Section VI. We again dedicate four and six bits to the two counters, i.e. four bits for the fixed-increment counter and six bits for the variable-increment counter. The last one has to represent negative values as well, and it can represent the 2^6 values in the range $[-(2^{6-1} - 1), 2^{6-1}] = [-31, 32]$. Other requested values lead to counter overflow. We notice that here, unlike the typical case, we might have a counter overflow that might be wrongly 'solved' by future insertions. In any such case, we consider it as an overflow. Using this technique and the parameters above, we have a similar false positive rate of 0.00473 and a significantly lower counter overflow probability of 0.00049.

IX. CONCLUSION

In this paper we presented a novel method based on *variable increments* to improve the efficiency of CBFs and their variants. We showed that it can be efficiently implemented in hardware with limited added complexity. We also demonstrated that this method can always achieve a lower false positive rate and a lower overflow probability bound than CBF in practical systems. More generally, we explained how this method can extend many variants of CBF published in the literature.

To our knowledge, this is the first time that B_h sequences are used in network applications. We believe that this is a first step towards a more general use, because they seem to fit increasingly-complex coding needs in networking applications. These applications often require both a scalability in the number of states to encode, yet also low hardware complexity—and indeed, as explained in this paper, B_h sequences efficiently compress sets of h states, yet can also be readily encoded and decoded at line rates using fixed translation tables.

X. ACKNOWLEDGMENT

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