# Polyhedron Realization for Shape Transformation

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**Abstract**

Polyhedron realization is the transformation of a polyhedron into a convex polyhedron with an isomorphic vertex neighborhood graph. We present in this paper a novel algorithm for polyhedron realization, which is general, practical, efficient, and works for any 0-genus polyhedron. We show how the algorithm can be used for finding a correspondence for shape transformation. After the two given polyhedra are being realized, it is easy to merge their *vertex neighborhood graphs* into a common graph. This graph is then induced back onto the original polyhedra. The common vertex neighborhood allows the interpolation of the corresponding vertices.

**Key words:** Shape transformation, metamorphosis, correspondence, polyhedron realization.

## 1 Introduction

Metamorphosis is the gradual evolution of a source object, through intermediate objects, into a target object. It has numerous applications, including scientific visualization and animation sequences in the film and advertising industries. Much of the work done in the area focused on two-dimensional metamorphosis (e.g., [2], [4], [10], [21], [22], [25]). Three-dimensional morphing sequences are harder to compute, compared to their two-dimensional counterparts. They have, however, many advantages over the two-dimensional sequences. For instance, changes in the viewing position do not require a repeated computation of the entire sequence.

Several approaches to three-dimensional metamorphosis have been explored. Hong et al. [12] base their method on matching the faces whose centroids are closest. Bethel and Uselton [3] add degenerate vertices and faces until a common vertex-neighborhood graph is achieved. Chen and Parent [5] extend a transformation for piecewise linear two-dimensional contours to three dimensions. In their method, morphing is done between polyhedra which are represented by sets of three-dimensional planar contours. Kaul and Rossignac [15] use the Minkowski sum of the models to obtain a smooth transition. Kent et al. [16] [17] create a correspondence between zero-genus polyhedra, by mapping them onto a sphere and merging their vertex-neighborhood graphs. Kanai et al. [14] utilize a similar algorithm by using the harmonic maps of the polyhedra. Several voxel-based methods were also explored ([6], [7], [11], [13], [18], [20]).

Morphing a three-dimensional polyhedron into another is usually done in two steps: finding a correspondence between the polyhedra and interpolating the corresponding vertices. In this paper we focus on the problem of finding a correspondence. We follow the approach introduced in [17], where two zero-genus polyhedra are first “inflated” like balloons until they become spherical. The one-to-one correspondence is then found between points on the surfaces of the two spheres and induced back onto the original polyhedra. The algorithm in [17] works very well for several classes of polyhedra. It remained open in that paper whether there exists an algorithm that is applicable to any zero-genus polyhedra, or even to wider classes of polyhedra. The major contribution of the current paper is devising a novel polyhedron realization algorithm that works for any zero-genus polyhedron. Such an algorithm was not known before.

"Inflating" a polyhedron into a convex polyhedron rather than into a surface of a sphere has a few advantages. First, it is general and works for any zero-genus polyhedron rather than for specific classes. Second, "inflating" a polyhedron into a sphere might result with a non-convex polyhedron. Convexity, however, is a desired property. Third, our algorithm does not involve the intersection of spherical arcs, and thus the calculations can become more robust. Finally, if a sphere is required, it can be easily obtained from a convex polyhedra.

We say that a $3-$dimensional polyhedron, $P$, is *realized* by another $3-$dimensional polyhedron, $P'$, if $P'$ is convex, and the two polyhedra have isomorphic vertex-neighborhood graphs. In other words, there exists a one-to-one correspondence between the vertex sets and the edge sets of the two polyhedra.

Steinitz [9] proved that a graph is realizable if and only if it is planar and three-connected. In other words, our algorithm can be considered as a different proof of Steinitz's theorem for the special case where the faces are triangular. This is the most common case for real data. Our proof is constructive, and gives rise to an algorithm which is efficient, avoids circular arcs, and is easy to implement. We implemented the algorithm and achieved very good results.

The realization algorithm we propose consists of two phases: simplification and re-attachment. During the simplification phase, vertices are detached from the vertex-neighborhood graph of the polyhedron one by one, and the graph is re-triangulated. This step is repeated until a 4-clique results. The second phase starts by creating a tetrahedron, the convex polyhedron which realizes a 4-clique. The vertices are re-attached to the polyhedron, in the reverse order to their detachment, while maintaining the polyhedron's convexity. The vertex-neighborhood graph of each convex polyhedron, created in the process, is isomorphic to a graph encountered during the simplification stage. Thus, the last polyhedron's vertex-neighborhood graph is isomorphic to the input vertex-neighborhood graph.

In three-dimensional metamorphosis, after the input polyhedra are being realized, the resulting convex polyhedra are merged on the boundary of one of them. The merged edge set is later used for refining the original polyhedra. The result of the mutual refinement is the desired correspondence, which is utilized during the interpolation phase of the shape transformation process.

The remainder of the paper is organized as follows. The next section presents a high-level description of the shape transformation algorithm that uses polyhedron realization. Section 3 describes the realization algorithm, and proves its correctness. Section 4 describes how the two realized polyhedra can be merged into one structure. Section 5 demonstrates our results. We conclude in section 6.

# 2 The Shape Transformation Algorithm

In this section we describe the general scheme of using a realization algorithm for establishing a correspondence between the vertices of two given polyhedra. Our scheme follows the scheme described in [17]. However, while in [17] the polyhedra are "inflated" into spheres, we realize them into convex polyhedra. As explained above, our algorithm has a few benefits. It works for any zero-genus polyhedron, it guarantees convexity, and it does not involve the intersection of spherical arcs. We begin with a few definitions.

**Definition 2.1 Polyhedron vertex-neighborhood graph** *is a graph $G = (V, E)$ where $V$ is the set of vertices of the polyhedron, and there exists an edge $e \in E$ that connects two vertices $v_i$ and $v_j \in V$ iff there exists an edge between the two vertices in the polyhedron.*

**Definition 2.2 Graph refinement:** *$\tilde{G} = (\tilde{V}, \tilde{E})$ is a refinement of the graph $G = (V, E)$ if $V \subseteq \tilde{V}$ and an edge $(v_i, v_j) \in E$ is replaced by a path $(v_i, u_1), (u_1, u_2), \ldots, (u_{n-1}, v_j)$ in $\tilde{E}$ such that $\forall u_i, u_i \in \tilde{V} - V$.*

**Definition 2.3 Polyhedron refinement:** *A polyhedron $\tilde{P}$ is a refinement of a polyhedron $P$, if $\tilde{P}$ and $P$ are congruent, and the vertex-neighborhood graph of $\tilde{P}$ is a graph refinement of the vertex-neighborhood graph of $P$.*

**Definition 2.4 Polyhedron realization:** *A polyhedron $P_{|P|}$ is said to realize a polyhedron $P$, if $P_{|P|}$ is convex and the vertex-neighborhood graphs of $P_{|P|}$ and $P$ are isomorphic.*

We are given a pair of 0-genus polyhedra, $P$ and $Q$. The algorithm for finding a correspondence between $P$ and $Q$ consists of five phases, and proceeds as follows.

1. Create a convex polyhedron, $P_{|P|}$, whose vertex-neighborhood graph is isomorphic to $P$'s, using the realization algorithm described in the next section.

2. Create a convex polyhedron, $Q_{|Q|}$, whose vertex-neighborhood graph is isomorphic to $Q$'s, in a similar way.

3. Refine the vertex-neighborhood graphs of $P_{|P|}$ and $Q_{|Q|}$ in order to produce a joint vertex-neighborhood graph. A few merge algorithms were proposed in the past. In [1], two triangulations of corresponding planar polygons are merged. This is also done in [14] when two harmonic maps are being merged. In [17], the faces of polyhedra whose vertices reside on the sphere are merged. Our algorithm, which is described in Section 4, merges the boundaries of convex polyhedra.

   The main idea of the algorithm in to project the vertices and the edges of $Q_{|Q|}$ onto the boundary of $P_{|P|}$. The projected edges of $Q_{|Q|}$ are then intersected with the edges of $P_{|P|}$. These are the edges of the refined polyhedron $P_{|P|}$. After the faces of $P_{|P|}$ are found, the vertices, edges, and faces of the refined polyhedron are projected onto $Q_{|Q|}$, to form its refined polyhedron.

4. Induce the merged vertex-neighborhood graph found in (3) onto $P$, to form a polyhedron, $\tilde{P}$. The polyhedron $\tilde{P}$ is a refinement of $P$, since its vertex-neighborhood graph is isomorphic to the merged one, and thus a refinement of $P$'s vertex-neighborhood graph, and it is congruent to $P$.

5. Repeat the previous phase for $Q$, creating a new polyhedron, $\tilde{Q}$, which is a refinement of $Q$.

At the end of this process $\tilde{P}$ and $\tilde{Q}$ are refinements of $P$ and $Q$ respectively. Moreover, their vertex neighborhood graphs are isomorphic. Therefore, $\tilde{P}$ and $\tilde{Q}$ can be used for establishing a correspondence between $P$ and $Q$.

The next sections elaborate on each of the above phases of the algorithm. Section 3 focuses on the realization algorithm (steps (1)–(2)). Section 4 describes the merging process of two given vertex neighborhood graphs, and explains how the merged graph can be induced back onto the original polyhedra, to produce their refinements (steps (3)–(5)).

# 3 The Polyhedron Realization Algorithm

Given a zero–genus polyhedron, $P$, we present an algorithm for constructing a zero-genus convex polyhedron, $P_{|P|}$, that realizes $P$. We assume that the input polyhedron has triangular faces. If not, we triangulate the faces in linear time. The algorithm consists of two phases: simplification and re-attachment.

1. **Simplification** – Simplify the vertex-neighborhood graph by removing a low-degree vertex from it and re-triangulating the resulting graph. This operation is repeated until a 4-clique graph, representing a tetrahedron, results. During this phase, a hierarchy of planar graphs, $G_{|P|} = G(P), G_{|P-1|}, ..., G_4 = K_4$, is constructed, such that $G_{i+1}$ has one vertex more than $G_i$, for $4 \leq i < |P|$.

2. **Re-attachment** – Given the above graph hierarchy, proceed bottom up, and construct a convex polyhedron, $P_i$, for each graph $G_i$, so that $G(P_i)$ and $G_i$ are isomorphic, for $4 \leq i \leq |P|$. This procedure first constructs

a tetrahedron, $P_4$, so that $G(P_4) \cong G_4 = K_4$. Suppose that a convex polyhedron, $P_i$, is constructed, such that $G(P_i) \cong G_i$. $P_i$ is then used, jointly with $G_{i+1}$, to construct a convex polyhedron, $P_{i+1}$, such that $G(P_{i+1}) \cong G_{i+1}$, $4 \leq i < |P|$.

Once the convex polyhedron $P_{|P|}$ is constructed, our goal is achieved: $G(P_{|P|}) \cong G_{|P|} = G(P)$.

Note that the simplification phase is quite similar to the way the Dobkin-Kirkpatrick hierarchy [8] is constructed. The re-attachment phase, however, is very different. In the Dobkin-Kirkpatrick hierarchy the polyhedron is known to be convex, and thus the vertices need not be re-positioned. In our case, re-positioning is a major consideration. We elaborate on each of the two phases below.

### 3.1 Phase 1: Simplification

Given a triangular graph, $G_i$, representing a polyhedron, we describe in this section an algorithm for constructing a triangular graph, $G_{i-1}$, with one vertex less. To do it, we repeat the following two steps.

1. Find a vertex, $v$, of degree 3, 4, or 5, and remove it from the graph.

2. Re-triangulate the resulting face.

This procedure is graphically described in Figure 1. It is repeated until a 4-clique results. We now prove the correctness of the simplification phase.

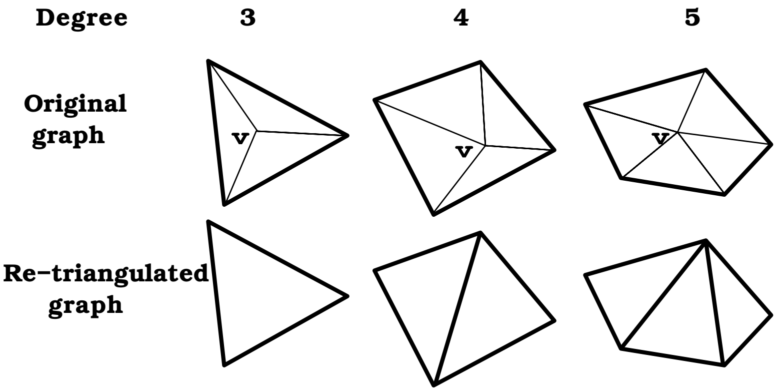



Figure 1: Simplification

**Lemma 3.1** *A planar triangular graph must contain at least one vertex of degree smaller than 6.*

**Lemma 3.2** *If a vertex $v$, of degree four, is removed from a planar triangular graph, $G_i$, then (1) The neighbors of $v$ in $G_i$ cannot be all inter-connected (in the new graph), and (2) A diagonal can be added, between two of $v$'s neighbors, so that the graph remains planar.*

**Proof:** Suppose, on the contrapositive, that the four vertices form a clique (contradicting (1)). Together with $v$, the five vertices used to form a clique before the removal of $v$. Since a five-vertex clique cannot be planar, it contradicts our assumption that we started from a planar graph.

Let us denote the four neighbors of $v$ as $v_1, v_2, v_3$, and $v_4$, so that $v_n$ and $v_{(n+1)mod4}$ are neighbors. Since we started from a triangulated graph, and $v$ was connected to all four vertices, the edges $v_1$-$v_2$, $v_2$-$v_3$, $v_3$-$v_4$, $v_4$-$v_1$ exist. Since they are not all connected to each other, one diagonal in the quadrilateral does not exist in the graph. It is obvious that the missing diagonal can be added so that planarity is not violated. $\Box$

**Lemma 3.3** *If a vertex $v$, of degree five, is removed from a planar triangular graph, $G_i$, then (1) There exist at most three diagonals between the neighbors of $v$ in $G_i$ and (2) Two diagonals, which share a common vertex, can be added between $v$'s neighbors so that the graph remains planar.*

**Proof:** We show first that there can exist at most three diagonals connecting the five vertices after we remove the vertex $v$. Suppose on the contrapositive that there are at least four diagonals. In this case, at most one pair of vertices, $v_k$ and $v_j$, are not connected by a diagonal in the resulting pentagon. Consider four vertices, among which only one is $v_k$ or $v_j$. They are all inter-connected by diagonals in the resulting pentagon. In the original graph, these four vertices, together with $v$, formed a five-clique. This contradicts planarity.

Having eliminated the possibility of four or five diagonals, we assume that there are three diagonals, and show that there exists a vertex through which none of the diagonals pass. Suppose on the contrapositive that the diagonals cover all the vertices. There must exist a vertex through which two diagonals pass, otherwise three diagonals would cover six vertices. Without loss of generality, assume it is $v_5$, and its diagonal-neighbors are $v_2$ and $v_3$. See Figure 2. By our assumption, the third diagonal must connect the remaining two vertices $v_1$ and $v_4$.

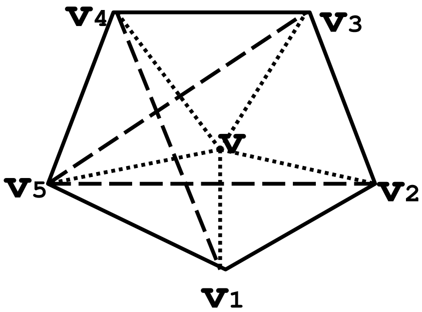



Figure 2: A pentagonal face with three diagonals covering all the vertices.

We now consider the original graph, before the detachment. We partition the vertices into two groups, $A = \{v_1, v_3, v_5\}, B = \{v, v_2, v_4\}$. Each member of A is connected to each member of B, and each member of B is connected to each member of A. Thus, the original graph contains a fully-connected bipartite graph known as $K_{3,3}$, and therefore planarity is contradicted.

If there exist only one or two diagonals, they cannot cover all five vertices, and a vertex through which two new diagonals can pass, exists. It is not hard to show that the two missing diagonals can be added in a planar fashion. □

**Theorem 3.4** *Given a triangular planar graph $G_i$, whose number of vertices is at least four, a vertex $v$ of degree 3, 4, or 5 can be found and removed, and the resulting graph can be re-triangulated so that the graph remains planar.*

**Proof:** By Lemma 3.1, there exists at least one vertex of degree smaller than 6. Since the graph is triangular, the degree cannot be 2. Thus, there exists at least one vertex of degree 3, 4, or 5, which can be detached. If the detached vertex is of degree 3, we do nothing. Obviously, the graph is still planar. Otherwise (the vertex is of degree 4 or 5), by Lemmas 3.2 and 3.3, diagonals can be added so that the graph remains planar. The resulting graph is also triangular. To see this, we distinguish between the different cases. If a vertex of degree 3 is detached, the former neighbors of $v$ form a triangle. If a vertex of degree 4 is detached, a quadrilateral is formed, and by Lemma 3.2 a diagonal can be added to form two triangles. Finally, if a vertex of degree 5 is detached, a pentagon is formed, and by Lemma 3.3 two adjacent diagonals can be added to form three triangles. □

**Lemma 3.5** *If the simplification procedure is repeated until there are four vertices left, the resulting graph is a 4-clique.*

Figures 3 – 4 illustrate the simplification phase by providing a few snapshots of an epcot being simplified, starting from a 50-vertex epcot (at the top left) and ending with a tetrahedron (at the bottom right). Note that even though the algorithm works on planar graphs, and not on the three-dimensional structures, for clarity, the snapshots shown below illustrate the three-dimensional structures of the intermediate objects.

Figure 3: Epcot simplification – wire-frame

Figure 4: Epcot simplification - faceted shading

### 3.2 Phase 2: Re-Attachment

We are given a hierarchy of vertex-neighborhood graphs, $G_{|P|} = G(P), G_{|P|-1}, ..., G_4$, which is the output of the simplification phase. The goal is to construct a sequence of convex polyhedra, $P_{|P|}, P_{|P|-1}, ..., P_4$, such that $G(P_i)$ is isomorphic to $G_i$, $4 \leq i \leq |P|$.

The re-attachment phase proceeds bottom up. A convex polyhedron, $P_4$, for $G_4$, is first constructed. Since $G_4$ is a $4$-clique, it can be realized by a tetrahedron. Suppose that a convex polyhedron, $P_i$, has been already constructed for the vertex-neighborhood graph $G_i$. We show next how $P_i$ can be used, along with $G_{i+1}$, to construct a convex polyhedron $P_{i+1}$, whose vertex-neighborhood graph is isomorphic to $G_{i+1}$, $4 \leq i < |P|$. At the end of this process, $P_{|P|}$ is the realized $P$.

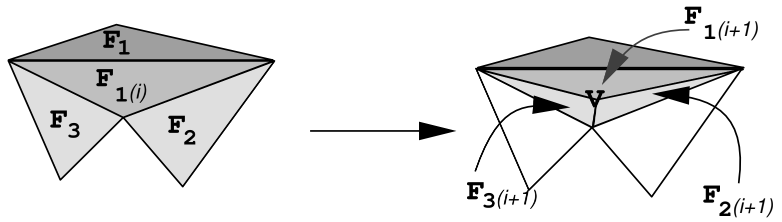
The major consideration is how to add the vertex $v$, which was detached during the simplification phase, to $P_i$, in order to form $P_{i+1}$. We distinguish between three cases according to the degree of $v$.

In case $v$ is of degree three, during the simplification process its cone of faces was replaced by a triangular face, $F_{1(i)}$. Thus $F_{1(i)}$ in $P_i$ needs to be replaced by a cone of three faces in $P_{i+1}$, as shown in Figure 5. Denote the three neighboring faces of $F_{1(i)}$ in $P_i$, $F_1$, $F_2$ and $F_3$.

In case $v$ is of degree four, By Lemma 3.2, $G_i$ was created by removing $v$ from the graph $G_{i+1}$ and adding a diagonal, thus forming two new faces $F_{1(i)}$ and $F_{2(i)}$, as illustrated in Figure 6. Denote the four neighboring faces of $F_{1(i)}$ and $F_{2(i)}$ in $G_i$, $F_1$, $F_2$, $F_3$, and $F_4$.

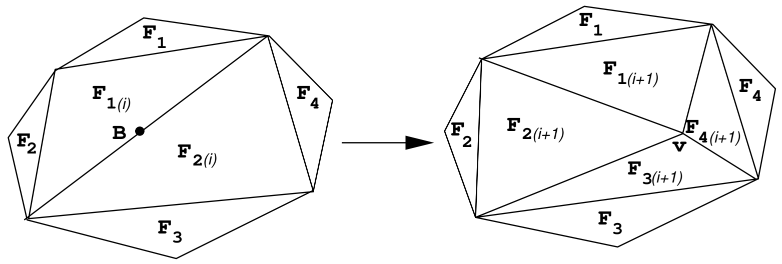
In case $v$ is of degree five, By Lemma 3.3, $G_i$ was created by removing $v$ from the graph $G_{i+1}$ and adding two adjacent diagonals, thus forming three new faces $F_{1(i)}$, $F_{2(i)}$ and $F_{3(i)}$, as illustrated in Figure 7. Denote the five neighboring faces of the new faces in $G_i$, $F_1$, $F_2$, $F_3$, $F_4$ and $F_5$.

By the induction hypothesis, a convex polyhedron $P_i$ whose vertex-neighborhood graph is $G_i$ has been created. We need to show how to add $v$ to $P_i$ in order to form $P_{i+1}$. To do it, we define a ray $r$ along which $v$ is located, as will be described below.. The ray $r$ is defined by its base $B$, and its direction $N$. To find $B$ and $r$, we follow the procedure below.




(a) $P_i$ (b) $P_{i+1}$

Figure 5: Attachment of $v$ of degree three to $P_i$

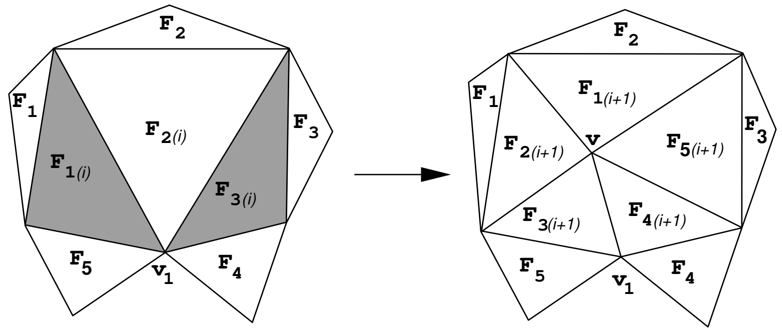



(a) $P_i$ (b) $P_{i+1}$

Figure 6: Attachment of $v$ of degree four to $P_i$

1. **Determine $B$ and $N$:**
   There are three possible cases:
   (a) if $v$ is of degree 3, as in Figure 5, then
      $B =$ the center of mass of the face $F_{1(i)}$.
      $N =$ the normal of $F_{1(i)}$.
   (b) if $v$ is of degree 4, as in Figure 6, then
      $B =$ the midpoint of the edge shared by $F_{1(i)}$ and $F_{2(i)}$.
      $N =$ the average of $F_{1(i)}$'s normal and $F_{2(i)}$'s normal.
   (c) if $v$ is of degree 5, as in Figure 7, then
      $v_1 =$ the vertex through which the two triangulation edges pass.
      $I_p =$ the intersection point of the planes on which $F_{1(i)}, F_2$ and $F_{3(i)}$ reside.
      $B =$ the midpoint of the segment $v_1 - I_p$.
      $N =$ the average of the normals of $F_{1(i)}$ and $F_{3(i)}$.
2. **Determine $r$:**
   Let $r$ be a ray whose base is $B$ and whose direction is $N$.
3. **Transform the polyhedron if needed:**
   If $v$ is of degree 5 and $B$ is below $F_2(i)$ then
   Apply the appropriate projective transformation as described below.
   Recalculate $r$ as described above.

So far we described the creation of $r$. We now explain how $v$ is located along $r$. Consider the neighboring faces of $v$'s cone of faces, $\{F_j \mid 1 \leq j \leq d\}$, where $d$ is the degree of $v$. Consider also the planes on which these faces reside. Calculate the intersection points between these planes and the line that $r$ defines




(a) $P_i$ (b) $P_{i+1}$

Figure 7: Attachment of $v$ of degree five to $P_i$

$\{I_j = Plane(F_j) \cap Line(r) \mid 1 \le j \le d\}$. Find the $d$ factors $\{t_j \mid I_j = B + t_j N,\ 1 \le j \le d\}$. Let

$$t = \begin{cases} 1 & \text{if } \forall j, t_j < 0 \\ 0.5 * \min\{t_j \mid t_j > 0\} & \text{otherwise} \end{cases} \tag{1}$$

Position $v$ at $B + tN$. Remove the faces $\{F_{j(i)} | 1 \le j \le d-2\}$ , and construct $d$ new faces between $v$ and its neighbors, as illustrated in Figures 5–7. This completes the construction of $P_{i+1}$.

In order to prove that $v$ is positioned correctly, it is sufficient to show that the triangulation faces (those created by the triangulation, during the detachment of $v$) are visible to $v$, and they are the only visible faces. In this case, according to beneath-beyond convex hull algorithm algorithm in [19], $v$ should be added to the convex hull, and the triangulation faces should be detached and replaced by the new faces.

**Lemma 3.6** *In case $v$ is of degree three, it is positioned correctly. That is, $v$ is in the positive normal direction relative to $F_{1(i)}$ and in the negative normal direction relative to $F_j$, for $\{j | 1 \le j \le 3\}$.*

**Proof:** Since $t$ is positive, $v$ is in the positive normal direction relative to $F_{1(i)}$. It is left to show that $v$ is in the negative normal direction relative to the neighbors of $F_{1(i)}$. We distinguish between two types of faces, according to the sign of $t_j$. If $t_j$ is positive, $v$ is in the positive normal direction of $F_j$ if and only if $t > t_j (> 0)$. If $t_j$ is negative, $v$ is in the positive normal direction if and only if $t < t_j (< 0)$. Our choice of $t$ prevents a positive normal direction. In the first case this is because $t < t_j$, and in the second case this is because $t$ and $t_j$ do not have the same sign. Therefore the only face, relative to which $v$ is in the positive normal direction, is $F_{1(i)}$. Since by our construction it is removed, the resulting $P_{i+1}$ is convex. $\Box$

**Lemma 3.7** *In case $v$ is of degree four, it is positioned correctly.*

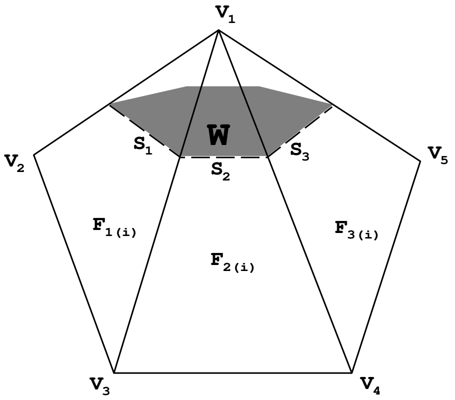
**Proof:** We need to show that $v$ is in the positive normal direction relative to $F_{1(i)}$ and $F_{2(i)}$, and that it is in the negative normal direction relative to $F_1$, $F_2$, $F_3$ and $F_4$. Since $N$ is the sum of the two normals of $F_{1(i)}$ and $F_{2(i)}$, it is clearly in acute angles to both. For this reason, any point on $r$ is in the positive normal direction relative to both faces, and thus for these faces every choice of a positive $t$ is acceptable. Moreover, since $t$ does not have both the same sign as $t_j$ and a larger absolute value, then similarly to the case of degree three, $v$ is in the negative normal direction relative to $\{F_1, F_2, F_3 F_4\}$. $\Box$

In case $v$ is of degree five, there are two possibilities. If we are lucky, $B$ is above $F_{2(i)}$. In this case, $B$ can be used to construct $P_{i+1}$, as was done in the case of degree $4$. Otherwise, we need to transform the polyhedron $P_i$ prior to the positioning of $B$. We discuss the required projective transformation below.

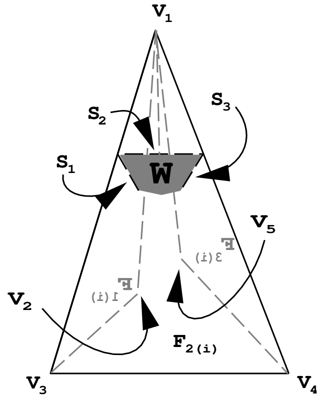
**Lemma 3.8** *In case $v$ is of degree five and $B$ is in the positive normal direction relative to $F_{2(n)}$, $v$ is positioned correctly.*

**Proof:** The proof is similar to the previous case. Along with the assumption that $B$ is in the positive normal direction relative to $F_{2(n)}$, we get the required result. $\square$

**The projective transformation:** There are two possible configurations, as shown in Figure 8. Figure 8(a) illustrates an "open" configuration in which the sum of dihedral angles between $F_{1(i)}$, $F_{2(i)}$ and $F_{2(i)}$, $F_{3(i)}$ is larger than $180°$. In Figure 8(b), illustrating a "closed" configuration, this sum is less than $180°$.



(a) open



(b) closed

Figure 8: Possible configurations

Recall that we assume that $B$ is below $F_{2(i)}$. We describe in this section the transformation for the case where the configuration is "closed". The case of an "open" configuration is similar.

To define the transformation, we find a plane, $W$ (the grey plane in Figure 8), which intersects $P_i$ only in the faces adjacent to $v_1$. Such a plane can be easily found. The intersections of the plane with faces of the polyhedron are segments on $W$, denoted as $S_1, S_2, S_3$ (corresponding to $F_{1(i)}, F_{2(i)}, F_{3(i)}$ respectively). We first define a planar transformation on this plane, and then extend the required transformation to three dimensions.

After finding $W$, we apply an affine transformation $Aff$ such that: (1) $Aff$ transforms $W$ to the plane $Z = 0$. (2) $Aff$ transforms $S_1$ to the $Y$ axis, to part of the segment $((0,0,0),(0,1,0))$. (3) $Aff$ transforms $S_3$ to the $X$ axis, to part of the segment $((0,0,0),(1,0,0))$. (4) $Aff$ transforms the intersection of the lines on which $S_1, S_3$ reside (denoted as $I_{S_1,S_3}$), to the origin. The situation on $Aff(W)$ (i.e., on $Z = 0$) is illustrated in Figure 9(a).

To complete the definition of the affine transformation, it is left to specify the source of the $Z$ axis. To do it, we define a new plane, a *separating plane*, that (1) separates $I_{S_1,S_3}$, the intersection of the lines on which $S_1$ and $S_3$ reside, from the polyhedron and (2) is parallel to $S_2$. Such a plane can be easily found. Once it is found, we can define the source of the $Z$-axis to be a vector on the separating plane which is not on $W$. From now on, we assume that the affine transformation has already been applied.

Figure 9(c) illustrates the goal of the projective transformation we are seeking. It should "open" the faces $F_{1(i)}$ and $F_{3(i)}$. And in two dimensions (on $Z = 0$), it should "open" the segments $Aff(S_1)$ and $Aff(S_3)$. In other words, the purpose of the projective transformation is to transform $Aff(S_1)$ and $Aff(S_3)$ such that the point of intersection of their lines is transformed to the other side of $Aff(S_2)$ (and in three dimensions, to the other side of $F_{2(i)}$). If this is achieved, then the polyhedron is transformed to another polyhedron whose configuration is "open" and $B$ is indeed above $F_{2(i)}$.

In two dimensions, two planar transformations are defined as illustrated in Figure 9. The first is a planar projective transformation (Figure 9(b)), and the second is a planar reflection around the line $x + y = 1$ (Figure 9(c)). The planar projective transformation is defined by $T(V) = \frac{V(1-\delta)}{V_x+V_y-\delta}$ where $V$ is a 2-dimensional vector, and $\delta$ is small enough to separate the origin from $P_i \cap W$. To illustrate that this transformation performs the required

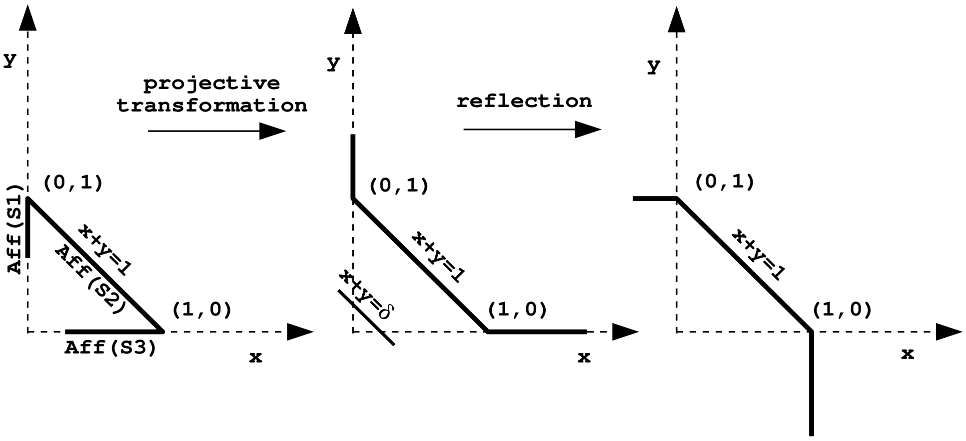



Figure 9: The projective transformation

opening, note that: (1) The line $x + y = 1$ is unchanged by the transformation. (2) The segment $((0, \delta), (0, 1))$ is transformed to $((0, \infty), (0, 1))$. (3) The open segment $((\delta, 0), (1, 0))$ is transformed to $((\infty, 0), (1, 0))$. The extension to three dimensions is easily obtained by $T(V) = \frac{V(1-\delta)}{V_x+V_y-\delta}$ where the equation $x + y = \delta$ represents the separating plane.

**Lemma 3.9** *The above projective transformation, $T$, transforms $I_P$, the point of intersection of the three planes on which the faces $F_{1(i)}, F_2, F_{3(i)}$ reside, and thus $B$, from one side of the plane of $F_{2(i)}$ to the other side.*

**Proof:** It can be easily shown that when $T$ is applied to the convex polyhedron $P_i$, a convex polyhedron results, if the polyhedron resides on one side of the plane, which is defined by the denominator of the projective transformation. This requirement is satisfied by our choice of the separating plane. The resulting polyhedron also has an isomorphic vertex graph, because the vertices of the transformed convex hull are images of the vertices of the original convex hull. Moreover, the transformation $T$ preserves both the boundary of $P_i$ (which is is transformed to the boundary of $T(P_i)$), and the intersection points between planes. It remains to be shown that this intersection, $T(I_p)$, resides above $T(F_{2(i)})$.

Let $L_F$ be the line of intersection of the two faces $F_{1(i)}$ and $F_{3(i)}$. The three points $v_1$, $I_{S_1,S_3}$, and $I_p$ reside on this same line, $L_F$. Now we examine $T(L_F)$ more closely. $T$ is continuous on $L_F$, except for the intersection of $L_F$ with the separating plane. This intersection is a single point, since the points on $L_F$ were established to be on both sides of the separating plane — $v_1$ on the same side as the polyhedron, $I_{S_1,S_3}$ on the other side. Moreover, $I_{S_1,S_3}$ and $I_P$ reside on the same side of the separating plane.

$T$ is continuous on the whole side of the separating plane which contains $I_{S_1,S_3}$ and $I_p$. We conclude that $T(v_1)$ is not between $T(I_{S_1,S_3})$ and $T(I_p)$, since the segment between $I_{S_1,S_3}$ and $I_p$ is transformed continuously. Thus $T(I_{S_1,S_3})$ and $T(I_p)$ are on the same side of $T(L_F)$ relative to $T(v_1)$.

$I_{S_1,S_3}$ is transformed by $T$ from one side to the other side of $T(S_2)$, and thus of $T(F_{2(n)})$, because the intersection of $S_1$ and $S_3$ before the projective transformation is at the origin, and after applying $T$ at $(1, 1, 0)$. Since $I_p$ and $I_{S_1,S_3}$ were transformed to the same side of $T(L_F)$ relative to $T(v_1)$, they were also transformed to the same side of $T(F_{2(n)})$, thus correcting the position of $I_p$. $\square$

**Theorem 3.10** *Given a convex polyhedron $P_i$, after re-attaching $v$ as described above, the resulting polyhedron $P_{i+1}$ is convex.*

**Proof:** By our construction, there are three possible cases: $v$ of degree three, four, or five. If $v$ is of degree three, by Lemma 3.6, the resulting polyhedron is convex. If $v$ is of degree four, by Lemma 3.7, the resulting polyhedron is convex. If $v$ is of degree five, there are two cases. If $B$ is in the positive normal direction relative to $F_{2(n)}$, by Lemma 3.8, the resulting polyhedron is convex. Otherwise, we apply the projective transformation as defined above. By Lemma 3.9, after applying this transformation, the case is reduced to the previous case, and thus a convex polyhedron can be constructed. □

**Theorem 3.11** *The realization algorithm described above runs in $O(N^2)$ time, where N is the number of vertices in the polyhedron.*

# 4 Merging Vertex Neighborhood Graphs

Given two convex polyhedra $P_{|P|}$ and $Q_{|Q|}$, our goal is to construct two convex polyhedra $\tilde{P}$ and $\tilde{Q}$, such that: (1) The vertex neighborhood graphs of $\tilde{P}$ and $\tilde{Q}$ are isomorphic to each other. (2) $\tilde{P}$ is a refinement of $P_{|P|}$. (3) $\tilde{Q}$ is a refinement of $Q_{|Q|}$. The algorithm works as follows.

1. **Initialization of $\tilde{P}$ and $\tilde{Q}$:**
   - Let $\tilde{P} \leftarrow P_{|P|}$ and $\tilde{Q} \leftarrow Q_{|Q|}$.
   - Move the centers of mass of $\tilde{P}$ and $\tilde{Q}$ into the origin.
2. **Update the vertex set of $\tilde{P}$ :**
   For each vertex $v$ of $\tilde{Q}$
   - Shoot a ray from the origin through $v$.
   - If the ray does not intersect $\tilde{P}$ in a vertex, add the intersection to the vertex set of $\tilde{P}$.
3. **Create the edge set of $\tilde{P}$ and update its vertex set:**
   For each edge $e$ of $\tilde{Q}$
   - Create the plane $Pl$ that passes through $e$ and through the origin $O$.
   - The intersection of $Pl$ and $\tilde{P}$ is a two-dimensional convex polygon. Sort the vertices according to their order in the polygon.
   - The polygon consists of two polygonal arcs between the vertices that correspond to the endpoints of $e$. Represent the edge $e$ in $\tilde{P}$ by the shorter arc.
   - Traverse the polygonal arc, adding its vertices and its edges to the sets of vertices and edges of $\tilde{P}$ respectively.
   - Replace every edge of $\tilde{P}$ which is split this way by its parts.
4. **Create the face set of $\tilde{P}$** (given the vertices and the edges found above).
5. **Update the vertex set of $\tilde{Q}$:**
   For each vertex $w$ of $\tilde{P}$:
   - Shoot a ray from the origin through $w$.
   - If the ray does not intersect $\tilde{Q}$ in a vertex add the intersection to the vertex set of $\tilde{Q}$.
6. **Create the edge set of $\tilde{Q}$:**
   For each edge $e$ of $\tilde{P}$
   - Create an edge between the two vertices in $\tilde{Q}$, which correspond to the vertices of $e$.

7. **Create the face set of $\tilde{Q}$.**
8. **Determine the location of the vertices of one polyhedron on the surface of the other:** This mapping is done using barycentric coordinates.

Note that the major difference between the merge algorithm described in [17] and ours is that the above algorithm works directly on polyhedra and does not involve the intersection of circular arcs. This is possible in our case since we work on convex polyhedra. As a result, the computation becomes more efficient and robust.

# 5 Implementation and Results

The algorithm is implemented in C and runs on SGI workstations. Though the complexity of the realization process is $O(N^2)$, without the projective transformation, the complexity is $O(N)$. The projective transformation is seldom necessary. As a result, the algorithm is extremely efficient and runs very fast.

Figures 10 – 11 show a couple of results of the realization algorithm. In Figure 10 an epcot and its realized epcot are being presented. In Figure 11 a deformed cube and its realized polyhedron is shown. The cube was created by "pushing" one face inside, so that it is $\epsilon$-away from its opposite face. This face cannot remain planar, so we allow it to "bend", and we re-triangulate the resulting model. The deformed-cube example, though small in size, illustrates our realization algorithm very well. The cube does not belong to any of the classes supported by previous algorithms. It is non-convex, not star-shaped, not an object of revolution, and not an extruded object.

(a) An epcot

(b) A realized epcot

Figure 10: Realization of an epcot

Figure 12 illustrates the results of the merge algorithm. In Figure 12(a) the vertex neighborhood graphs of a house and an icosahedron are merged on the surface of the icosahedron. In Figure 12(b) two deformed cubes are merged on the surface of one of them. The pair of cubes have opposite faces pushed in.

Figures 13–15 present a couple of morph sequences created by utilizing the realization algorithm. Figures 13–14 show the transformation between a pair of deformed cubes with opposite faces pushed in. During interpolation, the deformed face is pushed out, while the opposite one is pushed in. We use both wire-frame and faceted shading in order to illustrate the above process. Figures 15 – 16 transform a house into an icosahedron. While Figure 15 displays a few snapshots from the morph sequence in a smooth shading, Figure 16 shows some snapshots in a faceted shading. Note how the stairs and the roof disappear in the icosahedron's surface.

(a) A deformed cube

(b) A realized deformed cube

Figure 11: Realization of a deformed cube

(a) A house and an icosahedron

(b) Two deformed cubes

Figure 12: Merging the vertex neighborhood graphs

# 6 Conclusion

This paper presents a new algorithm for realizing a three-dimensional zero-genus polyhedron. Our algorithm is general, efficient, and easy to implement. We demonstrated how the algorithm can be used for finding a correspondence between two three-dimensional polyhedra for metamorphosis.

There are a few directions for future research. One interesting direction engages human intervention. The current algorithm involves almost no user control. While an automatic technique is desirable in many applications, at certain times, it is beneficial to provide finer controls. Second, an intriguing problem is to find an algorithm for"inflating" non zero-genus polyhedra. Finally, a challenging problem is to find a morph algorithm between polyhedra that can guarantee that self–intersection does not occur during interpolation.

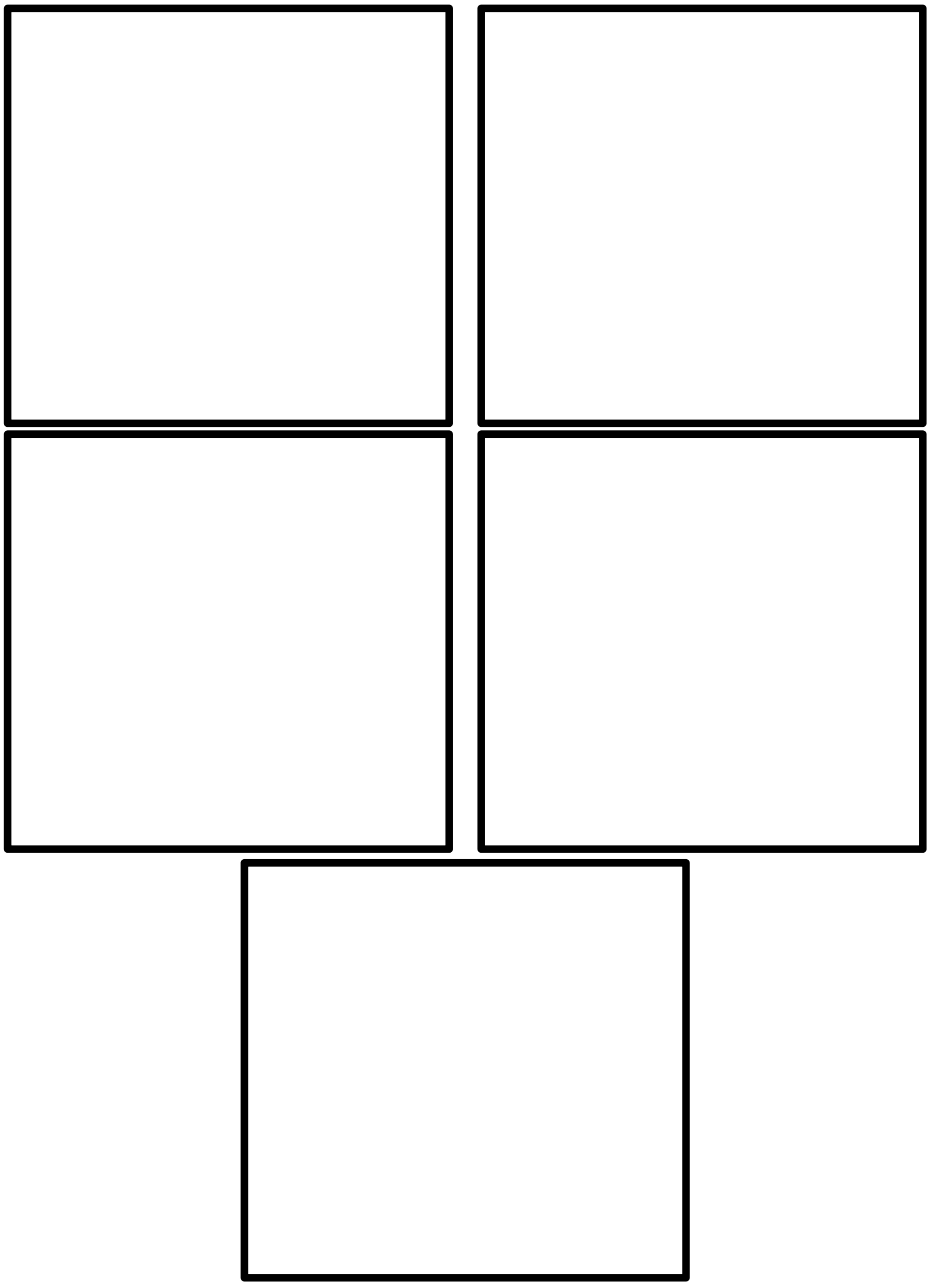


Figure 13: Shape Transformation between the deformed cubes – wire-frame

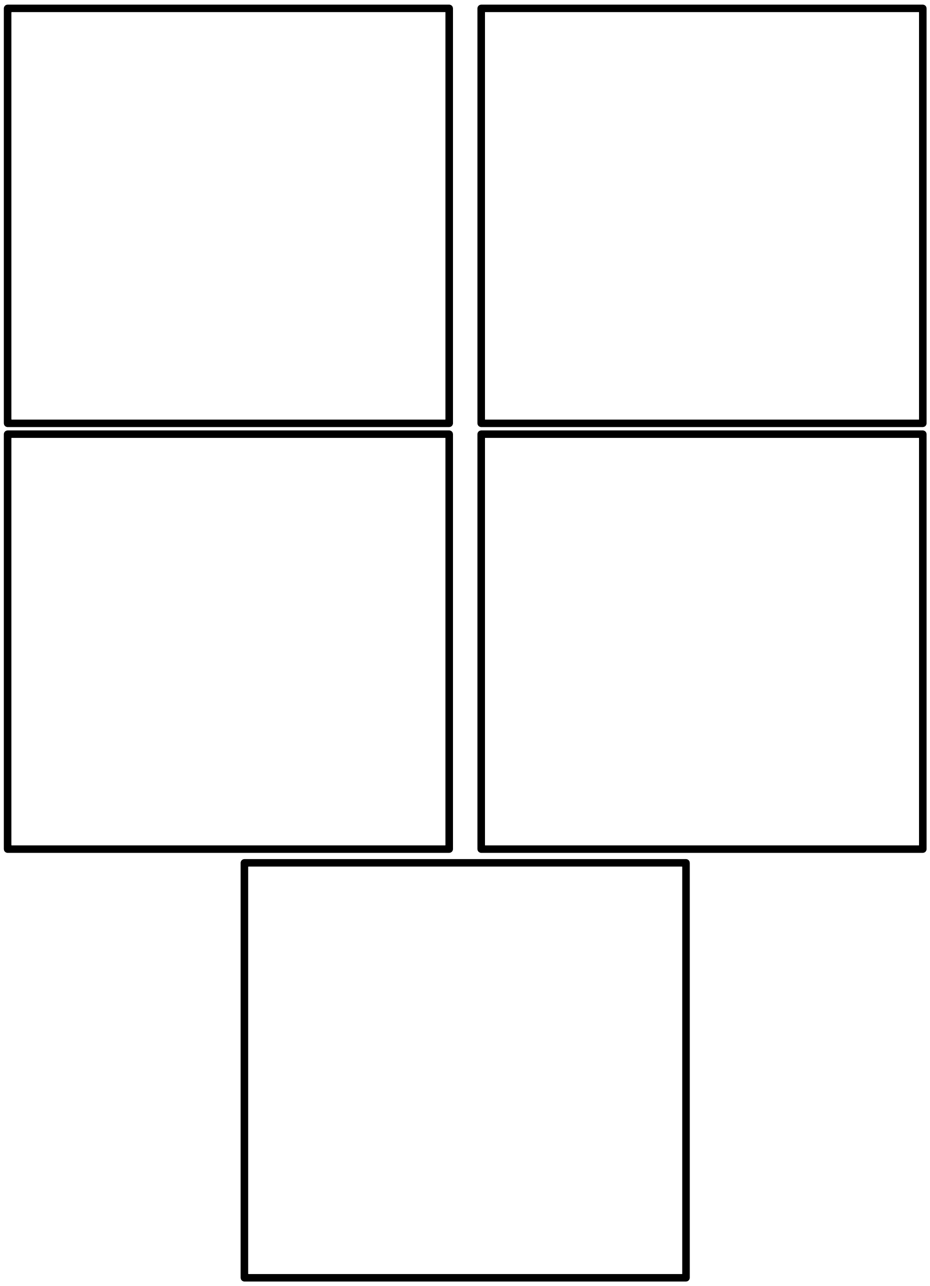


Figure 14: Shape Transformation between the deformed cubes – faceted shading

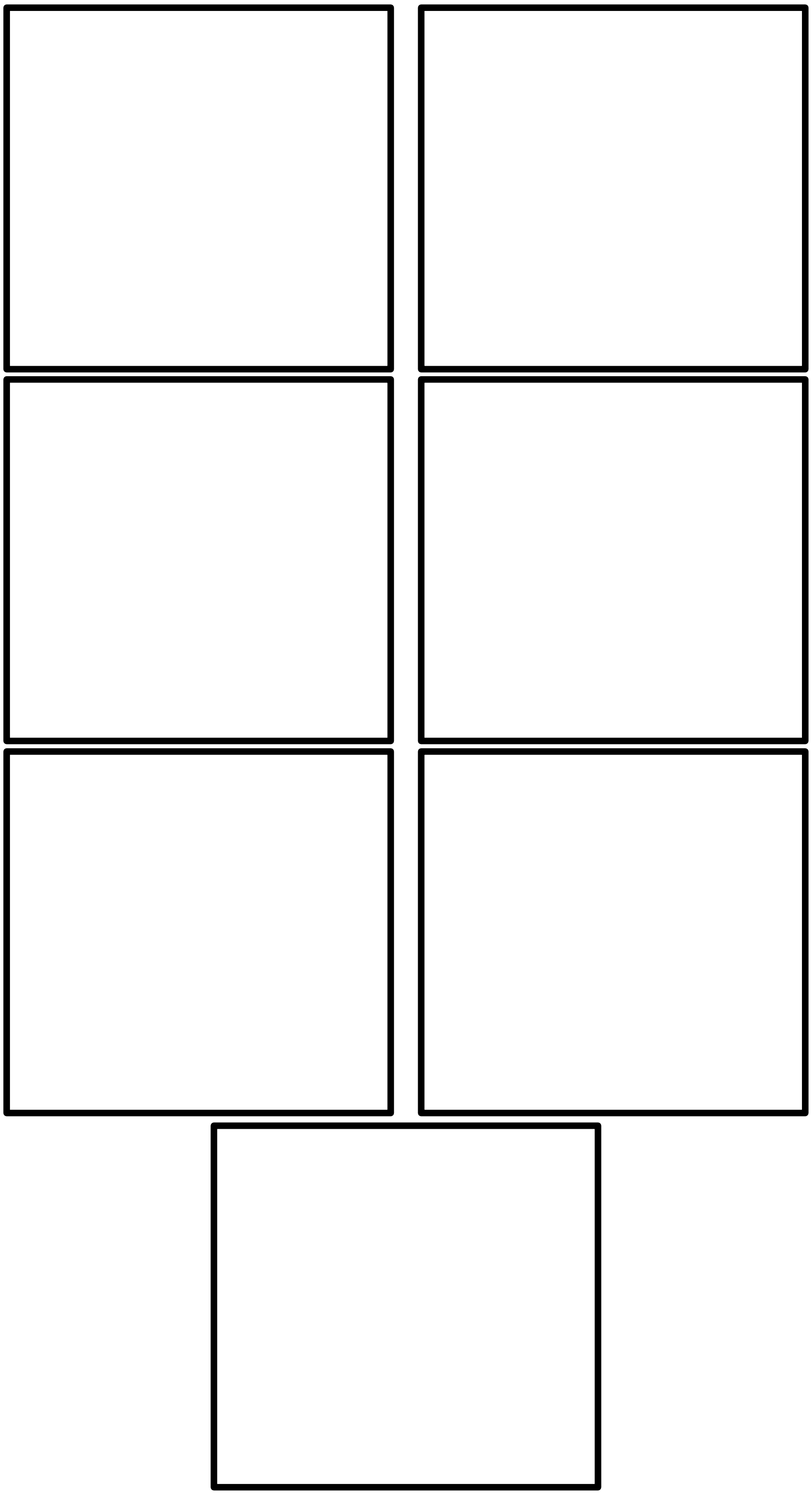


Figure 15: Shape Transformation between a house and an icosahedron - smooth shading

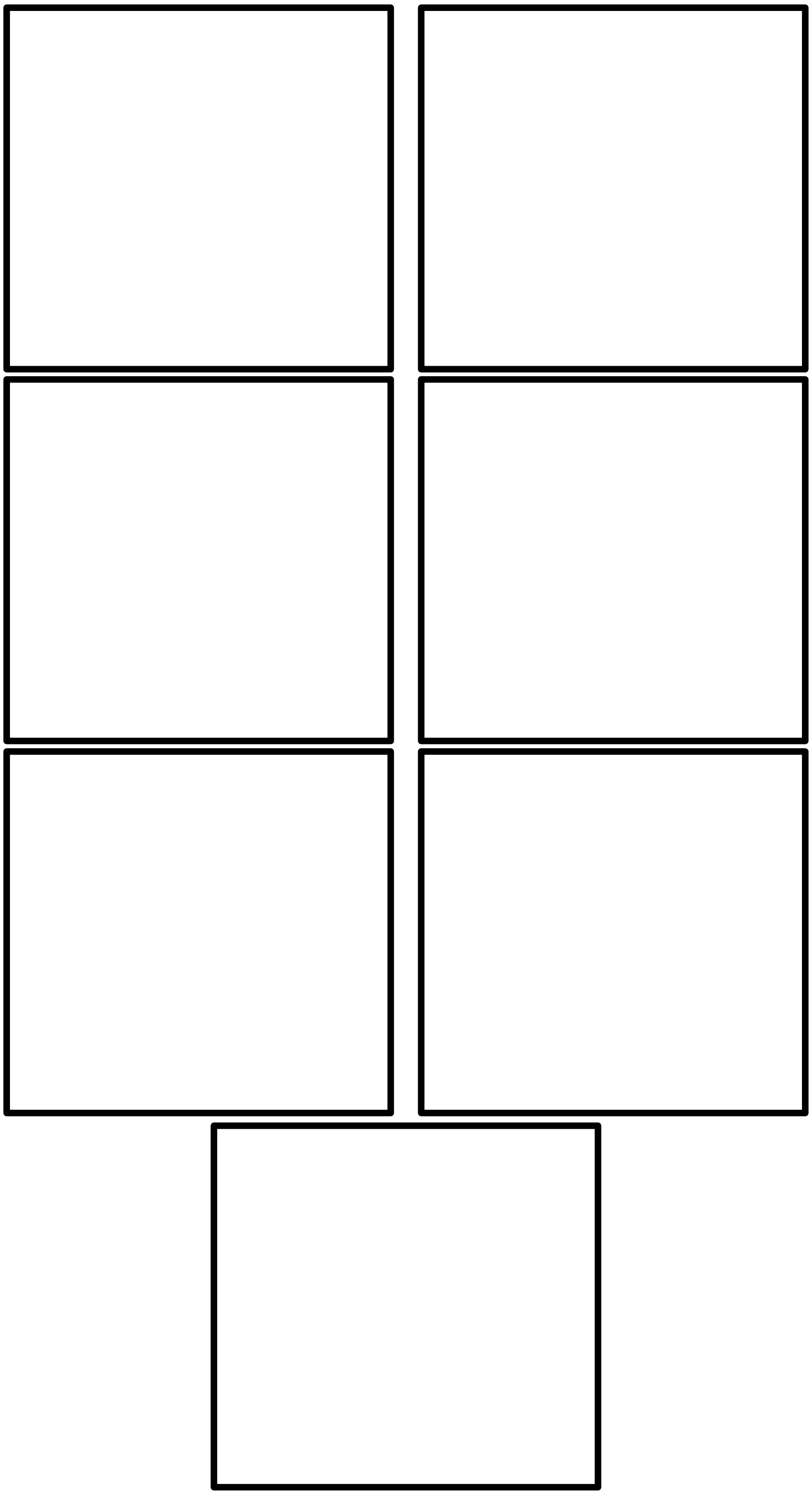


Figure 16: Shape Transformation between a house and an icosahedron – faceted shading

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