Large Deviations for a Polling System with Exhaustive Service

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Research Thesis
Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Applied Mathematics

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Submitted to the Senate of the Technion -
Israel Institute of Technology

Tamuz 5765 Haifa July 2005

The Research Thesis Was Done Under the Supervision of

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The Generous Financial Help of the Technion
is Gratefully Acknowledged

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Abstract

This research thesis is dedicated to analysis of the polling system with exhaustive service discipline by means of the theory of large deviations. The discussed polling system has numerous applications, particularly in the field of digital communications.

We consider a model which expresses the polling system as a jump Markov process. While free Markov processes on Euclidean spaces have been extensively studied, this model presents a new challenge as it demonstrates an abrupt change in behavior every time one of the served queues becomes exhausted.

In order to deal with such discontinuities, we introduce a new topological space as the state space for the random process of cliens arrival and service, where discontinuities in behavior of the process are located on the boundaries. We apply the results which exist for the free processes, while taking special care of boundary areas, to deduce similar results for the considered model.

As a first step, we deduce the most probable path of the random process, and show that the probability to deviate from it in a significant manner decays exponentially, as we increase the scaling coefficient of the system.

Furthermore, we establish the notion of the rate function on the introduced topology, as a composition of rate functions for the free processes. An interesting outcome of our discussion is that in some cases the rate function doesn't have the usual structure of integral of a local rate function over time, but rather exhibits some "predefined strategy" of choice between free rate functions.

Finally, we establish the Large Deviations Principle for the discussed model, or namely, set upper and lower bounds on the order of exponential decay for probability of sets on the model state space.

The achieved results provide a base for further research of polling systems with exhaustive service. For example, one can now rather easily address the frequently posed questions about the probability of escape from the stable state, the dynamics of an escape path, and more.

List of Notations

1. Preface

"... The Ayalon River, which runs along the highway, overflowed as its water level rose to seven meters. The highway had to be closed for a day, hundreds of cars were stranded... "

Ha'aretz, 26 Oct. 2000

While such news do not surprise anyone anymore, we still ask ourselves, how did it come that the country's major highway, which was planned to flood only once in twenty years, did so right on the year of its opening, and on the two years following it.

Was it an excessive reliance on the Almighty? Was the global warming responsible for such a treacherous conduct of the local weather? Right now these questions have no definite answer, but they bring us to the point: the theory of Large Deviations enters where the law of averages fails.

The general notion of large deviations describes in some sense probabilities of very unlikely events in a given setting. Usually it provides means to determine asymptotic order of probability of some rare events on an exponential scale. Research concerning very small probabilities is perhaps as old as the theory of probability itself, but the foundations of the modern abstract theory of Large Deviations were largely laid by Varadhan [Var66]. The theory was greatly expanded in the 80's, and found numerous applications (e.g. digital communications and rare event simulations [Buc90], thermodynamics [Ell85]).

The application of large deviations to queues can be done by numerous methods. One of them, the "sample path" approach, involves representation of the queue length as a jump Markov process, and it is extensively covered by Shwartz and Weiss [SW95]. Another approach, which uses weak convergence, is presented by Dupuis and Ellis [DE97]. A recently published book by Ganesh et al. [GOW04] offers the "continuous mapping" approach, which utilizes simple models to propagate large deviations results onto more complex settings by means of continuous mappings on some topology. A review by Weiss [Wei95] provides more sources on various approaches to large deviations.

In this paper we establish the Large Deviations Principle for the polling system with two queues and an exhaustive service model. Our approach heavily relies on the sample path technique laid out in [SW95]. While the said book covers a wide range of settings, it still can't be applied directly to our model. The problematic point in our case is the presence of moments when the server abruptly changes its behavior, and such discontinuities are not resolved in the scope of the existing theory.

1.1. Overview.

The considered system consists of a server with two queues. Both queues are continuously filled by arriving clients, but the server can serve only one queue at any time. The exhaustive service model, which is the object of our research, simply states that the server serves each queue until it is empty, and only after that passes to serve the other queue. Accordingly, when the other queue is empty, the server passes back to the first queue, and so on.

The arrival of clients to each queue constitutes a Poisson process with some constant rate (not necessarily the same for both queues). The service of each queue, when it occurs, is also a Poisson process, and has a rate of its own. In general, Poisson processes give us a very good approximation of real-world random processes. e.g. in telecommunications or even customer service in a retail store.

As usual for large deviations, we don't focus on a specific random process, but rather study the asymptotic behavior of an infinite family of similar processes. In order to be able to speak about asymptotics, we perform a scaling of our arrival-service process on a time-space scale. For this purpose we represent our polling system as a two-dimensional random walk, where the number of clients waiting in each queue at any given moment constitutes the location of the walk on the appropriate dimension at that moment (see formal definition in Section 2.1). In this setting each arrival or service event is represented by a step of length 1 of the random walk in some direction. The scaling of this random walk is obtained by making jumps which are n times faster, but also n times shorter.

A common sense hints us that such scaling would bring the random walk to some sort of deterministic limiting behavior, just like randomly moving water molecules all go very deterministically to the kitchen sink. Indeed, we are able to show that our random walk converges in probability to some deterministic path, as the scaling coefficient n goes to infinity. This result is a sort of a law of large numbers, and it is known as "Kurtz Theorem".

In our case, we show that there is some deterministic "average" path, and the scaled random walk tends very strongly to stay near that path. More specifically, the probability of the scaled random walk to escape from some fixed small neighborhood of the average path vanishes exponentially with respect to the scaling coefficient.

The last sentence gives the reader a first taste of the idea of large deviations. Note that we considered the escape of some random walk from its average state, or namely a "large deviation" from the average. The theory states that such an escape is a very rare event, and its incidence decreases on an exponential scale, as we refine the steps of the motion. Nevertheless, this statement about escaping the vicinity of the most probable path is still a very coarse one. The multitude of possible escape paths can be further divided into areas, and a question may arise, whether an escape from the average state along some specific path α is more likely to happen than an escape along some other path β.

In order to answer this question we define an "escape cost" for each deterministic path. This cost is intended to reflect the relative chance of the random walk to stay near that path. In this context, the average path would undoubtedly have the lowest cost, as the probability of the random walk to stay near it is incomparably larger than anything else. The cost as a function of paths is called the "rate function". One of the main objectives in the course of study of any model from large deviations' point of view, is to define the rate function for this model and to prove that this definition accurately reflects the situation.

Once the rate function is obtained for the given model, it becomes a powerful tool to further study of the properties of that model. For example, one may consider two points A and B and ask, what was the most probable path of the random motion which started in A and arrived to B after some period of time. The answer to this question is that among all paths which connect A to B, the cheapest path would be the most probable one.

The statement which relates rate functions to probabilities is called "the Large Deviations Principle". This principle doesn't operate with single paths, but rather with open and closed sets of such paths. The cost of a set of paths is defined as the infimum of costs of all single paths comprising the set. The Principle of Large Deviations consists of two parts:

1. The probability of a sample random walk to belong to some open set *dominates* on the exponential scale the cost of that set. This is the so-called "lower bound" part of the principle.
2. The probability of a sample random walk to lie in some closed set *is dominated* on the exponential scale by the cost of that set. This statement constitutes the "upper bound".

The results we obtained for the polling system with exhaustive service, are presented in this paper in the following order:

In Chapter 2 we formally introduce the model of the exhaustive service. We also describe the geometry of the random motion and define the topology which governs the notions of open and closed sets of paths.

In Chapter 3 we find out how the average path of the random motion looks like, and prove that the scaled random motion stays near it with sufficiently large probability.

In Chapter 4 we present a deeper insight into the structure of the random walk, and define the rate function of our model. Furthermore, we state several useful properties of the rate function.

In Chapters 5 and 6 we state and prove the upper bound and the lower bound, which together establish the Large Deviations Principle for our model.

1.2. **The basics.** Numerous books provide the basis for the large deviations, e.g. [Var84]. We bring here some simple overview of the theory, based on the presentation in [DZ93, Ch. 1]. The latter also includes a concise review of the history of large deviations.

In the most general setting we consider a topological space Ω with a family of probability measures $\{\mu_\epsilon\}$ on it. We wish to characterize the behavior of the probability $\mu_\epsilon(\Gamma)$ for some measurable set $\Gamma \subseteq \Omega$, as ϵ tends to zero. Such characterization is usually obtained in the form of the Large Deviations Principle (LDP).

Definition 1 ([DZ93]). *A lower-semicontinuous mapping $I : \Omega \to [0, \infty]$ is called a "rate function".*

Definition 2 ([DZ93]). *The family of measures $\{\mu_\epsilon\}$ satisfies the Large Deviations Principle with some rate function I, if for any measurable $\Gamma \subseteq \Omega$*

$$\liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \geq - \inf_{x \in \Gamma^\circ} I(x), \tag{1a}$$

$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq - \inf_{x \in \overline{\Gamma}} I(x), \tag{1b}$$

where Γ° and $\overline{\Gamma}$ denote the interior and the closure of Γ respectively.

Whenever the righthand sides of (1a), (1b) coincide, the function I designates the rate of convergence of $\mu_\epsilon(\Gamma)$ on an exponential scale, and (1) can be loosely restated as

$$\mu_\epsilon(\Gamma) \simeq e^{-\frac{1}{\epsilon} I_\Gamma},$$

where $I_\Gamma = \inf_{x \in \Gamma^\circ} I(x) = \inf_{x \in \overline{\Gamma}} I(x)$.

In a typical problem in the field of large deviations, one has to find an appropriate function I which would satisfy LDP for the given family $\{\mu_\epsilon\}$. A simpler problem may ask to determine the points in Ω which "attract" the measure μ_ϵ, i.e. $x \in \Omega$ such that

$$\lim_{\epsilon \to 0} \epsilon \log \mu_\epsilon(B_x) = 0$$

for any open neighborhood B_x of x. One can see that such an attraction loosely tells that $\mu_\epsilon(B_x)$ vanishes at a subexponential rate, or perhaps doesn't vanish at all, and in this sense the measure tends to accumulate near x.

Let us provide a couple of examples, which demonstrate some applications of the large deviations.

The first example deals with one of the simplest settings in the large deviations theory, and it can be found in any introductory text in this field (see e.g. [Var84, Sec. 3]).

Example 1. Consider a sequence $\{X_n, n \in \mathbb{N}\}$ of i.i.d. random variables with finite mean, and let

$$S_n = X_1 + X_2 + \ldots + X_n.$$

We regard the distributions of the scaled variables S_n/n as a family of measures on the probability space. The LDP for this family is known as Cramér's Theorem, and it states that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \in \Gamma\Big) \geq -\inf_{x\in\Gamma^\circ} I(x),$$
$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \in \Gamma\Big) \leq -\inf_{x\in\overline{\Gamma}} I(x),$$

where the rate function $I(x)$ is calculated as follows:

$$M(\theta) = \mathbb{E}e^{\theta X_1}, \qquad \forall\theta \in \mathbb{R}$$
$$I(x) = \sup_\theta \Big(\theta x - \log M(\theta)\Big), \qquad \forall x \in \mathbb{R}.$$

Example 2. Simple birth-death process [SW95, Sec. 4.2].

Imagine a population of species which multiply and die according to some law. In this example we wish to define birth and death as two independent Poisson processes with constant rates. That means, at any given moment, a member of population would be born or die after some random exponentially distributed time. This model allows negative population.

We can associate the size of population with a random process $x(t)$ on the integer set, with each birth or death event corresponding to a jump in either positive or negative direction. Formally speaking, $x(t)$ is a process with generator

$$Lf(a) = \lambda\big(f(a+1) - f(a)\big) + \mu\big(f(a-1) - f(a)\big), \quad a \in \mathbb{R},$$

where λ and μ are the rates of the birth and death Poisson processes respectively.

Now let us define the family of scaled processes $\{x_n(t)\}$ by taking

$$x_n(t) = \frac{1}{n} x(nt), \quad n \in \mathbb{N}.$$

A scaled process x_n is basically the same jump process as x, but with jumps n times shorter, occurring at rates n times faster.

Assume for the sake of simplicity that the initial population is zero species (a very realistic assumption indeed), and the birth rate λ is larger than the death rate μ. As the time passes, the population size would then exhibit a drift to the right. Moreover, in any unit of time there would be on average λ births and μ deaths, amounting to population increase of $\lambda - \mu$ per unit.

This mean behavior is formally described by Kurtz theorem, which appears in a more generalized form in [SW95, Th. 5.3]. For our case the theorem states that the average path for the birth-death process is the linear function

$$x_\infty(t) = (\lambda - \mu)t,$$

and the probability of x_n to escape from the ϵ-neighborhood of x_∞ by some fixed time T satisfies

$$\mathbb{P}\Big(\sup_{0\le t\le T}|x_n(t)-x_\infty(t)|\ge\epsilon\Big)\le C_1e^{-nC_2(\epsilon)}, \tag{2}$$

where C_1 is some positive constant, and $C_2(\epsilon)\sim O(\epsilon^2)$.

Note the exponential decrease of the right-hand side of (2) with respect to n.

The rate function, or namely the cost, is defined for any path $r : [0,T]\to\mathbb{R}$ by the means of the local rate function $l_{\mathrm{b-d}}$ of the birth-death process:

$$l_{\mathrm{b-d}}(y)=\sup_{\theta\in\mathbb{R}}\big(\theta y-\lambda(e^{\theta}-1)-\mu(e^{-\theta}-1)\big)$$

$$I_0^T(r)=\begin{cases}\int_0^T l_{\mathrm{b-d}}\big(r'(s)\big)ds, & \text{if } r \text{ is absolutely continuous}\\ \infty, & \text{otherwise}\end{cases}$$

(see [SW95, (5.2)–(5.5)]). One can check that $l(\lambda-\mu)=0$, and thus $I_0^T(x_\infty)=0$. This stresses once again that following the average path bears zero cost.

The basic birth-death process presented above is subject to various generalizations. The most immediate one allows for multidimensional process, where at any moment a jump can occur in one of many directions, not necessarily axial. A further generalization talks about a multidimensional random process $\vec{x}(t)$ in terms of its generator

$$Lf(\vec{x})=\sum_{i=1}^{k}\lambda_i(\vec{x})\big(f(\vec{x}+\vec{e}_i)-f(\vec{x})\big), \tag{3}$$

where $\vec{e}_i$ is any multidimensional vector, and $\log\lambda_i(\vec{x})$ is a bounded and Lipschitz continuous function.

The latter generalization is extensively discussed in Chapter 5 of [SW95], and it stands as the basis for this entire paper. In particular, for the family of scaled processes $\vec{x}_n(t)$ the rate function is presented and the Large Deviations Principle is shown.

Our careful readers have undoubtedly noticed that all the above examples bear some similarities in their spirit. Indeed, all of them feature some sort of time-space scaling. This sort of scaling is necessary in order to bring things into right proportion. For instance, take X_n from Example 1 to have strictly positive support. You will immediately obtain that the support of S_n shifts to infinity, making any attempt to apply LDP directly to S_n meaningless.

Remark. While all the examples also involve a Markovian dependence of some kind, this is not necessarily the case. Large deviations treatments do exist for more general stationary processes, and they are discussed in [GV93] and others.

Let us now examine Example 2 a bit closer. It describes a sort of birth-death process, where some quantity (namely, a size of population) increases or decreases according to given rules. This model involves a discrete state set $\mathbb{Z}$ and a continuous time scale. Note that instead of considering ever-increasing time intervals, we achieve by scaling that the time of evolution T remains the same, but births and deaths occur at increasing rates and affect decreasing "units of life".

Now let us become a little more realistic. As a first step away from the model described in Example 2, we disallow negative populations. In order to achieve that, we just need to alter the rules, such that no death is allowed for a population of zero size. But the resulting scenario is still quite fantastic, since we allow a birth to happily occur, even when there are no species to give it. For this reason we introduce the server-queue model.

Example 3. M/M/1 [SW95, Ch. 11].

Consider a server which operates on a queue. Clients arrive and join the queue at some rate, and server serves the queue at some other rate, as long as the queue is not empty. When the queue is empty, the server naturally stands idle.

Observe, that if the arrival rate λ is larger that the service rate μ, then as the time passes, there would be more and more clients in the queue, and eventually the size of the queue will tend to infinity. On the other hand, if μ is larger than λ, then the number of clients will decrease from the initial state, and once the queue is exhausted, it will tend to stay nearly empty. The latter case demonstrates a so-called *stable* queue, with empty state being the steady state.

The M/M/1 queue is formally defined as a jump Markov process $x(t)$ on $\mathbb{Z}^+$ with jump directions

$$e_1 = +1, \text{ with rate } \lambda(x) = \lambda$$

$$e_2 = -1, \text{ with rate } \mu(x) = \begin{cases} \mu, & x > 0, \\ 0, & x = 0. \end{cases}$$

This definition can be alternatively expressed in the terms of generator

$$Lf(a) = \lambda(a)\big(f(a+1) - f(a)\big) + \mu(a)\big(f(a-1) - f(a)\big), \quad a \in \mathbb{R}.$$

Note that $\log \mu(a)$ is neither bounded nor Lipschitz continuous function, so the model M/M/1 can't be regarded as a specific case of (3). Discontinuities of jump rates as functions of queue size commonly appear in queue theory models, mainly as a result of server starting/finishing serving a particular queue or queue exhaustion. Their presence is what makes a particular model non-trivial.

Nevertheless, M/M/1 has been extensively studied, and all the regular questions were answered. Many of them can be addressed using the terms of the simple birth-death process (Example 2), due to the apparent relation between the two models. In particular, the local rate function l for M/M/1 can be defined in the terms of the local rate function $l_{\mathrm{b-d}}$ for the birth-death process (see [SW95, (11.6)]) as

$$l(x,y)=\begin{cases} l_{\mathrm{b-d}}(y), & \text{for } x>0 \text{ or } x=0 \text{ and } y>0\\ 0, & \text{for } x=0 \text{ and } y=0\\ \infty, & \text{for } x<0 \text{ or } x=0 \text{ and } y<0\end{cases}$$

The rate function I is defined as

$$I_0^T(r)=\begin{cases} \int_0^T l\big(r(s),r'(s)\big)ds, & \text{if } r \text{ is absolutely continuous,}\\ \infty, & \text{otherwise.}\end{cases}$$

The described model is one of the simplest models studied in the scope of large deviations applied to the queue theory. In what follows we shall briefly describe some of the many models based on this simple approach, and mention achieved results, whenever such results exist.

A typical model in the queue theory involves numerous servers and queues, and a flow of arriving clients. These interact between themselves with various modes of behavior. For example, a server may be defined to give preference to some "VIP" queue by serving it with first priority, regardless of whether there are clients in other queues. On the other hand, clients may always choose the shortest queue upon their arrival. Another complication which is quite common in the real world, is the need to reconfigure the server every time it changes the queue being serviced, causing an idle time on each such change.

1.3. **The polling system.** Polling systems have been extensively studied over past thirty years, and many their aspects were explored to great extent. A wide range of practical applications prompted authors to address issues such as stability, escape problems, comparisons between various policies, and more. Among examples of polling systems in engineering applications there are papers by Borst, Boxma and Levy on local network control [BBL95], by J. Mišić and V. Mišić on Bluetooth technology [MM03], and many others.

Among the multitude of polling disciplines, commonly considered are the exhausted and the gated policies in various forms. The interest in these two policies is reasonable, as they represent two basic approaches: the exhaustive service policy instructs the server to process a certain queue until it is empty, and the gated policy considers only those customers, who were present in a certain queue at the moment the server switched to it. The two mentioned basic policies are compared together and with other models in numerous surveys, using various criteria. Levy, Sidi and Boxma [LSB90] conclude that the exhaustive discipline is most efficient with respect to the total amount of unfinished work found in the system at any time. Another recent paper by Bischof [Bis01] compares policies applied to the model of single queue and non-negative setup and vacation times. According to it, the gated policy may sometimes have an advantage over the exhaustive one in terms of mean waiting time, for some choice of input parameters.

As one of leading candidates for the position of "the best policy", the exhaustive policy received some special attention from researchers. An aspect which is crucial to the applicability of a certain model is its stability. Several papers are devoted to the question of stability of the polling system with exhaustive service discipline. Coffman, Puhalskii and Reiman demonstrate in [CPR95] the asymptotic behavior of the total unfinished work, and address the waiting times in limit under some sort of scaling. Foss and Last [FL96] establish a simple criterion for stability of a multiple-queue system with zero switch-over times. In this paper we present in a rather heuristic manner a stability condition (14c) which is surprisingly similar to the one by Foss and Last.

Unlike the above fields, the area of large deviations, as they apply to the exhaustive policy, received rather little attention. Numerous results on related areas consider the limited polling, where the server moves to a new queue after serving some predefined number of customers. As an example one can mention Delcoigne and de La Fortelle [DdLF00], who study Markovian routing among queues and the extreme limit of just one customer per session. The model employed by these two authors bears some similarity to ours, but its transition policy is probabilistic in nature and is applied uniformly after each service event. Another paper by Massoulié [Mas99] addresses a family of models with constant service times.

In this paper we present the analysis of the exhaustive service model by the methods of large deviations. As we noted earlier, there exist numerous treatments of Markov random walks using large deviations. Yet, none of them can be easily applied to polling systems. There are two main reasons for that. First, the change in server state, when it leaves one queue and goes to another, creates a highly abrupt process, which cannot be generally addressed by methods developed for continuous changes in service rates. Second, at any moment the behavior of the system is dictated not only by the length of each queue, but also by the location of the server. This creates a complex topology on the event set, in the sense that one cannot easily say whether two states of the system are close or far from each other.

The main achievement of this paper is that is demonstrates how to overcome both these obstacles and to obtain the large deviations bounds for the exhaustive policy, and outlines the methods which may perhaps help in study of polling systems with other governing policies.

2. Introduction

We consider a polling system consisting of two queues served by a single server, with exhaustive service. Each queue is filled by a Poisson process, with arrival rates λ_0 for the first queue and λ_1 for the second. The server serves each queue until it is emptied, and then passes immediately (i.e. without setup delay) to serve another one. The service times are i.i.d. exponential with service rates μ_0 for the first queue and μ_1 for the second queue.

2.1. Geometry of the model.

Our first major task is to define a set of states and a topology that would properly represent our system. As the system consists of two queues, each holding some number of clients at any given moment, one might naturally think of a two-dimensional plane $\mathbb{R}^2$ as a possible state set. Of course, we note that the number of clients in each queue is always non-negative, so the aforementioned plane can as well be reduced just to its first quadrant, namely $(\mathbb{R}^+)^2$.

Furthermore, we observe that merely giving the state of both queues doesn't convey all possible information about the state of the system. The missing element is the information about the queue which is being served right now. By adding this information to the stack, we arrive at a need to employ a set of a kind $(\mathbb{R}^+)^2 \times \{0,1\}$ to fully describe each possible state of the polling system.

Each state of the system can be represented therefore by a triple of numbers $(x, y, s) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z}_2$, where x and y denote the level of each queue, and s denotes the queue being served (either 0 or 1).

Finally, we observe that the transitional states of the system, namely the states with at least one empty queue, in fact do not allow us to specify explicitly the queue being served. Indeed, at any such state one queue has just been exhausted, and the other is about to start being served. Thus, we can understand at the intuitive level that there is a need to establish an equivalence between states $(x, 0, 0)$ and $(x, 0, 1)$ or between $(0, y, 0)$ and $(0, y, 1)$ for any x, y. We therefore consider the set $\mathbb{D}$ of equivalence classes on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z}_2$:

$$\mathbb{D} = \left\{ \begin{array}{ll} \{(x,y,s)\}, & \forall x, y \in \mathbb{R}^+ \backslash \{0\}, s \in \mathbb{Z}_2; \\ \{(x,0,0),(x,0,1)\}, & \forall x \in \mathbb{R}^+; \\ \{(0,y,0),(0,y,1)\}, & \forall y \in \mathbb{R}^+ \end{array} \right\} \tag{4}$$

From now on we will permit a slight abuse of notation: a triple (x, y, s) would be identified with its equivalence class, and a pair (x, y) would be identified with the equivalence class $\{(x, y, 0), (x, y, 1)\}$ when either of x, y is zero.

The above definition effectively constructs $\mathbb{D}$ as a stack of two *panes* which we shall denote as follows:

$$(5) \qquad \begin{aligned} \mathbb{D}_0 &= \{(x,y,0),\ x,y \in \mathbb{R}^+\}, \\ \mathbb{D}_1 &= \{(x,y,1),\ x,y \in \mathbb{R}^+\}. \end{aligned}$$

For the sake of convenience, we shall also denote their common boundary as

$$(6) \qquad \partial\mathbb{D} = \{(x,y), \quad x,y \in \mathbb{R}^+ \text{ and } x = 0 \text{ or } y = 0\}.$$

Note that each of $\mathbb{D}_0, \mathbb{D}_1$ by itself is merely a positive quadrant in the Euclidean plane $\mathbb{R}^2$. Thus when two members of $\mathbb{D}$ belong to the same pane, they can be treated just as ordinary vectors, disregarding the third component. In this manner the addition and the multiplication by a constant can be readily defined for any $(x_1,y_1,s),(x_2,y_2,s) \in \mathbb{D}$:

$$\begin{aligned} (x_1,y_1,s) + (x_2,y_2,s) &= (x_1+x_2, y_1+y_2, s), \\ c \cdot (x_1,y_1,s) &= (cx_1, cy_1, s). \end{aligned}$$

Sometimes we shall attempt to add an element in $\mathbb{D}$ and a vector in $\mathbb{R}^2$. In this case the result would inherit its third component from the element of $\mathbb{D}$:

$$(x_1,y_1,s) + (x_2,y_2) = (x_1+x_2, y_1+y_2, s).$$

Remark. For two elements in $\mathbb{D}$ which belong to different panes, the notion of addition is meaningless. Therefore, and expression of the sort

$$(x_1,y_1,0) + (x_2,y_2,1)$$

with neither of the elements belonging to the boundary $\partial\mathbb{D}$, should never be employed.

In order to define a metric on the state set $\mathbb{D}$ (which in turn will induce a topology), we need to understand how the system gets from one state to another. The obvious situation involves two states which reside on the same pane. In this case, we can naturally induce on them the Euclidean metric $|\cdot|$ on the plane $\mathbb{R}^2$.

For two states which reside on a different panes, a more delicate treatment is required. We see, that in order to arrive from one state to another, one needs to change the queue being served, and this can only happen on the boundary. Therefore, we are tempted to define the distance between two such states as the length of some path that touches $\partial\mathbb{D}$ somewhere in-between. One may notice, that in order to get from some state in $\mathbb{D}_0$ to another state in $\mathbb{D}_1$, the system must pass through the y-axis, while in order to get back, it must pass through the x-axis. But since the distance is a symmetric function, there is no way we can incorporate this information in the topology. Thus, we define the distance between two such states just as the shortest path between them which touches the boundary $\partial\mathbb{D}$.

Definition 3. *d is the distance on $\mathbb{D}$ defined as follows:*

$$\tag{7a} \begin{aligned} d\left((x_0,y_0,0),(x_1,y_1,0)\right) &= d\left((x_0,y_0,1),(x_1,y_1,1)\right) \\ &= |(x_0,y_0)-(x_1,y_1)|, \end{aligned}$$

$$\tag{7b} \begin{aligned} d\left((x_0,y_0,0),(x_1,y_1,1)\right) &= d\left((x_0,y_0,1),(x_1,y_1,0)\right) \\ &= \min\left\{ \begin{array}{l} \min\limits_{x}\{|(x_0,y_0)-(x,0)|+|(x,0)-(x_1,y_1)|\}, \\ \min\limits_{y}\{|(x_0,y_0)-(0,y)|+|(0,y)-(x_1,y_1)|\} \end{array} \right\} \\ &= \min_{\vec{a}\in\partial\mathbb{D}}\{|(x_0,y_0)-\vec{a}|+|\vec{a}-(x_1,y_1)|\}. \end{aligned}$$

i.e. the shortest path from one point to another over the two-fold surface $\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{Z}_2$, glued at the axes.

Remark. The minima in (7b) are indeed attained, as shown later in Proposition 25.

The distance d obviously satisfies the axioms of positivity, symmetry and identity of indiscernibles. The triangle inequality for d is established below in Proposition 1. Thus the space $(\mathbb{D},d)$ is a metric space.

Proposition 1. *The distance d defined in* (7) *satisfies the triangle inequality.*

Proof. Let $A,B,C\in\mathbb{D}$. We need to show that

$$d(A,B)+d(B,C)\geq d(A,C).$$

Note that the relative placement of A, B, C can belong to one of the following four cases:

I. A, B, C belong to the same pane.
II. A and B belong to the same pane, and C belongs to the other one.
III. A and C belong to the same pane, and B belongs to the other one.
IV. B and C belong to the same pane, and A belongs to the other one. This case is similar to II.

In the event of either point belonging to the boundary, we can include it arbitrarily in any pane.

Before we proceed, let us recall that if two points X and Y belong to the same pane, they satisfy

$$d(X,Y)=|XY|;$$

and if two points X and Y belong to different panes, then by Proposition 25 there exists $Z\in\partial\mathbb{D}$ such that

$$d(X,Y)=|XZ|+|ZY|.$$

Now we shall address the three cases I-III separately.

I. In this case the distance d coincides with the Euclidean distance on the pane which accomodates A, B, C, and the triangle inequality obviously holds.

II. As we noted earlier, there exists $D \in \partial\mathbb{D}$ such that

$$d(B,C) = |BD| + |DC|.$$

Therefore,

$$d(A,B) + d(B,C) = |AB| + |BD| + |DC| \geq |AD| + |DC|,$$

and by the definition of $d(A,C)$

$$|AD| + |DC| \geq d(A,C).$$

III. Since B belongs to the pane different from the pane of A and C, there exist $D, E \in \partial\mathbb{D}$ such that

$$d(A,B) = |AD| + |DB|,$$
$$d(B,C) = |BE| + |EC|.$$

Therefore,

$$d(A,B) + d(B,C) = |AD| + |DB| + |BE| + |EC| \geq |AC| = d(A,C).$$

$\square$

2.2. **The random process.** Consider the set

$$\mathcal{X} = \{(x,y,s) \in \mathbb{D}, \quad x, y \in \mathbb{Z}^+\}.$$

Our polling system can be represented by a jump Markov process $\vec{z}(t)$ with the state space $\mathcal{X}$. It would be difficult to define $\vec{z}(t)$ as a "mixing" of Poisson processes of arrival and service, as the service of each queue is interrupted every time the client pool of that queue is exhausted. Therefore we prefer to characterize the process by its total event rate at any state and the probability of jump in a certain direction.

Let $\vec{a} \in \mathcal{X}$ be some state of $\vec{z}(t)$. Consider first the case of the non-empty first queue being served, namely $\vec{a} = (a_x, a_y, 0)$ with $a_x > 0$. At this state the process $\vec{z}(t)$ is a "local" sum of the two arrival processes with directions $(1,0)$ and $(0,1)$ and rates λ_0 and λ_1 respectively, and the service process of the first queue with direction $(-1,0)$ and rate μ_0. We can therefore define the generator L of $\vec{z}(t)$ as

$$\begin{aligned} Lf(\vec{a}) =& \lambda_0 f(\vec{a} + (1,0,0)) + \lambda_1 f(\vec{a} + (0,1,0)) \\ &+ \mu_0 f(\vec{a} + (-1,0,0)) - (\lambda_0 + \lambda_1 + \mu_0) f(\vec{a}), \\ &\qquad\qquad\qquad\qquad \vec{a} = (a_x, a_y, 0),\ a_x > 0. \end{aligned} \tag{8a}$$

Furthemore, if the second queue is being served, i.e. the state $\vec{a}$ is chosen to have $a_s = 1$ and $a_y > 0$, the processes in effect are the two arrival processes again, and the service process of the second queue with direction $(0, -1)$ and rate μ_1. Accordingly,

$$
\begin{aligned}
\text{(8b)} \quad Lf(\vec{a}) =& \lambda_0 f(\vec{a} + (1,0,1)) + \lambda_1 f(\vec{a} + (0,1,1)) \\
&+ \mu_1 f(\vec{a} + (0,-1,1)) - (\lambda_0 + \lambda_1 + \mu_1) f(\vec{a}), \\
&\qquad \vec{a} = (a_x, a_y, 1),\ a_y > 0.
\end{aligned}
$$

The last possibility is that of $\vec{a} = (0,0)$, namely when both queues are empty. In this case none of them can possibly be served, and

$$
\begin{aligned}
\text{(8c)} \quad Lf(\vec{a}) = \lambda_0 f(\vec{a} + (1,0)) + \lambda_1 f(\vec{a} + (0,1)) - (\lambda_0 + \lambda_1) f(\vec{a}), \\
\vec{a} = (0,0).
\end{aligned}
$$

2.3. **The free motions.** In the previous section we described the random process $\vec{z}(t)$, and noted that it changes its behavior aburptly each time it touches the boundary $\partial\mathbb{D}$. This prompts us to introduce the notion of the *free motion* to describe the behavior of $\vec{z}(t)$ during each service session, i.e. between two consecutive changes of served queue. The free motion with constant rates is well-explored and will greatly assist us in the development of the theory.

Consider the free process $\zeta(t)$ which describes the model with only the first queue being served forever with the rate μ_0, while clients arrive to both the first and the second queue with rates λ_0 and λ_1 respectively. We allow each queue to hold any number of clients, even negative.

The process $\zeta(t)$ can be defined in the terms of its generator

$$
\begin{aligned}
L_\zeta f(\vec{a}) =& \lambda_0 f(\vec{a} + (1,0)) + \lambda_1 f(\vec{a} + (0,1)) \\
&+ \mu_0 f(\vec{a} + (-1,0)) - (\lambda_0 + \lambda_1 + \mu_0) f(\vec{a}), \\
&\qquad \vec{a} \in \mathbb{R}^2 \times \{0,1\}.
\end{aligned}
$$

We return now to $\vec{z}(t)$ and observe that, given the same initial conditions and $s = 0$, $\vec{z}(t)$ would behave just like $\zeta(t)$ as long as the first queue is not empty. This observation can be formalized using the notion of *coupling* as explained below.

Recall that both $\vec{z}$ and ζ can be represented as counting processes for i.i.d. exponentially distributed random variables. Therefore we can couple $\vec{z}$ and ζ by considering a probability space with countable number of i.i.d. exponentially distributed waiting times, and by representing both $\vec{z}$ and ζ as deterministic functions of these times.

Furthermore, given some set of paths, all whose members are confined to the first pane without the boundary, the distribution of $\vec{z}$ and ζ is deterministic conditioned on that set, with respect to the waiting times, and hence probabilities to stay in the same set of paths coincide.

Strictly speaking, let $\vec{a} \in \mathbb{D}_0 \setminus \partial\mathbb{D}$ be the initial state, and let $A(t) \subseteq \mathbb{D}_0 \setminus \partial\mathbb{D}$, $t \in [0,T]$ be some measurable time-dependent subset of the first pane without the boundary. Then

(9a) $$\mathbb{P}_{\vec{a}}\big(\forall t \in [0,T] \quad \vec{z}(t) \in A(t)\big) = \mathbb{P}_{\vec{a}}\big(\forall t \in [0,T] \quad \zeta(t) \in A(t)\big).$$

In a similar fashion we can analyze the behavior of $\vec{z}(t)$ on the second pane $\mathbb{D}_1$, by coupling it to the free motion $\xi(t)$ defined by its generator

$$\begin{aligned} L_\xi f(\vec{a}) =& \lambda_0 f(\vec{a} + (1,0)) + \lambda_1 f(\vec{a} + (0,1)) \\ &+ \mu_1 f(\vec{a} + (0,-1)) - (\lambda_0 + \lambda_1 + \mu_1) f(\vec{a}), \\ &\qquad\qquad\qquad\qquad \vec{a} \in \mathbb{R}^2 \times \{0,1\}. \end{aligned}$$

The result similar to (9a) would now take the form

(9b) $$\mathbb{P}_{\vec{a}}\big(\forall t \in [0,T] \quad \vec{z}(t) \in A(t)\big) = \mathbb{P}_{\vec{a}}\big(\forall t \in [0,T] \quad \xi(t) \in A(t)\big).$$

for any $\vec{a} \in \mathbb{D}_1 \setminus \partial\mathbb{D}$ and $A(t) \subseteq \mathbb{D}_1 \setminus \partial\mathbb{D}$, $t \in [0,T]$.

Remark 4. While the coupling is a very general concept by itself, we intend to apply it only in the way shown above. Therefore, whenever the readers encounter in this paper a derivation based on "coupling", they should realize that it stands for the very specific coupling described above together with the result (9).

2.4. **The scaled process.** Once we obtained the strict definition of the random process $\vec{z}(t)$ in terms of its generator L, we can further consider the family of scaled processes $\{\vec{z}_n(t)\}$, as described in Section 4.3 of [SW95].

We know that the generator L_n for $\vec{z}_n(t)$ can be defined according to [SW95, (4.13)] as

(10)

$$L_n f(\vec{a}) = \begin{cases} n\Big(\lambda_0 f(\vec{a} + (\frac{1}{n}, 0)) + \lambda_1 f(\vec{a} + (0, \frac{1}{n})) \\ \quad + \mu_0 f(\vec{a} + (-\frac{1}{n}, 0)) - (\lambda_0 + \lambda_1 + \mu_0) f(\vec{a})\Big), \\ \qquad\qquad\qquad \text{for } \vec{a} = (a_x, a_y, 0),\ a_x > 0; \\ n\Big(\lambda_0 f(\vec{a} + (\frac{1}{n}, 0)) + \lambda_1 f(\vec{a} + (0, \frac{1}{n})) \\ \quad + \mu_1 f(\vec{a} + (-\frac{1}{n}, 0)) - (\lambda_0 + \lambda_1 + \mu_1) f(\vec{a})\Big), \\ \qquad\qquad\qquad \text{for } \vec{a} = (a_x, a_y, 1),\ a_y > 0; \\ n\Big(\lambda_0 f(\vec{a} + (\frac{1}{n}, 0)) + \lambda_1 f(\vec{a} + (0, \frac{1}{n})) \\ \quad - (\lambda_0 + \lambda_1) f(\vec{a})\Big), \\ \qquad\qquad\qquad \text{for } \vec{a} = (0,0), \end{cases}$$

and will yield the scaled process

$$\vec{z}_n(t) = \frac{1}{n}\vec{z}(nt).$$

For smooth functions f the limit of $L_n f(\vec{a})$ as n tends to ∞, exists at all points of $\mathbb{D}$ except the origin, and it is equal to (see [SW95, p. 76]):

$$(11)\qquad L_\infty f(\vec{a}) = \begin{cases} (\lambda_0 - \mu_0)\dfrac{\partial f}{\partial x} + \lambda_1 \dfrac{\partial f}{\partial y}, & \vec{a} = (a_x, a_y, 0),\ a_x > 0; \\ \lambda_0 \dfrac{\partial f}{\partial x} + (\lambda_1 - \mu_1)\dfrac{\partial f}{\partial y}, & \vec{a} = (a_x, a_y, 1),\ a_y > 0. \end{cases}$$

Now we are tempted to employ the theory developed in [SW95, Cor. 4.15] and to declare that the most probable path $\vec{z}_\infty(t)$ can be constructed as the solution of some differential equation derived from (11), and that it is a deterministic process. Indeed, by that corollary, the process $\vec{z}_\infty(t)$ should satisfy

(12a)

$$\frac{d}{dt}\vec{z}_\infty(t) = \begin{cases} \lambda_0 \cdot (1,0,0) + \lambda_1 \cdot (0,1,0) + \mu_0 \cdot (-1,0,0), \\ \qquad\qquad \text{if } \vec{z}_\infty(t) \in (0,\infty) \times [0,\infty) \times \{0\}; \\ \lambda_0 \cdot (1,0,1) + \lambda_1 \cdot (0,1,1) + \mu_1 \cdot (0,-1,1), \\ \qquad\qquad \text{if } \vec{z}_\infty(t) \in [0,\infty) \times (0,\infty) \times \{1\}. \end{cases}$$

Unfortunately, this attempt lacks the required rigorosity, as the expression (11) doesn't quite define a differential operator on $\mathbb{R}^2$. Yet we may take the liberty to neglect this objection and notice that (11) does define a generator which is *locally* a differential operator on $\mathbb{R}^2$ for all non-boundary points of $\mathbb{D}$, and therefore most of the theory can be applied to obtain at least the intuition for how $\vec{z}_\infty(t)$ does really behave.

The components of (12a) describe the behavior of each piece of $\vec{z}_\infty$ by a separate differential equation. The continuity of $\vec{z}_\infty$ is achieved by setting the initial value at each time interval to be equal to the final value of its predecessor.

Note that (12a) implies that $\vec{z}_\infty$ is generally piecewise linear. Jumping a bit forward, we also observe that $\vec{z}_\infty$ may tend to zero (see Figure 1), so it consists of a countable number of ever-shrinking linear pieces, whose total length is bounded. In Section 3.1 we will provide the so-called "stability conditions", which ensure such behavior.

In the course of the proof of Theorem 3, we will show that once $\vec{z}_n$ arrives near zero, it tends to stay there, under the same "stability conditions".

Passing to the limit $n \to \infty$, we obtain that if $\vec{z}_\infty(t_0) = (0,0)$ then $\forall t > t_0\ \vec{z}_\infty(t) = (0,0)$, i.e.

$$(12b)\qquad \frac{d}{dt}\vec{z}_\infty = (0,0), \quad \text{if } \vec{z}_\infty(t) = (0,0).$$

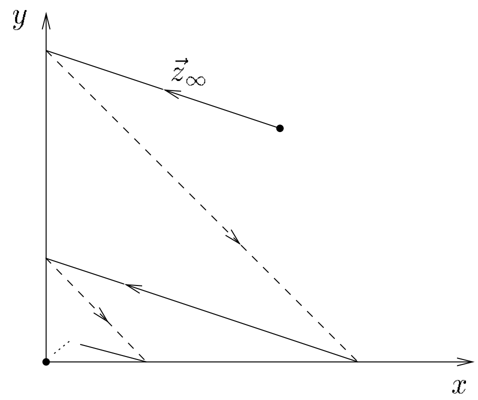


FIGURE 1. The most probable path.

To complete the treatment of scaled processes, we also introduce the scaled processes ζ_n and ξ_n generated by

$$L_{\zeta,n}f(\vec{a}) = n\Big(\lambda_0 f(\vec{a} + (\frac{1}{n}, 0)) + \lambda_1 f(\vec{a} + (0, \frac{1}{n})) \\ + \mu_0 f(\vec{a} + (-\frac{1}{n}, 0)) - (\lambda_0 + \lambda_1 + \mu_0) f(\vec{a})\Big),$$

$$L_{\xi,n}f(\vec{a}) = n\Big(\lambda_0 f(\vec{a} + (\frac{1}{n}, 0)) + \lambda_1 f(\vec{a} + (0, \frac{1}{n})) \\ + \mu_1 f(\vec{a} + (-\frac{1}{n}, 0)) - (\lambda_0 + \lambda_1 + \mu_1) f(\vec{a})\Big), \\ \vec{a} \in \mathbb{R}^2 \times \{0,1\},$$

respectively.

By [SW95, (5.7)], the limit processes ζ_∞ and ξ_∞ satisfy the differential equations

$$\frac{d}{dt}\zeta_\infty(t) = (\lambda_0 - \mu_0)(1,0) + \lambda_1(0,1), \tag{13a}$$

$$\frac{d}{dt}\xi_\infty(t) = \lambda_0(1,0) + (\lambda_1 - \mu_1)(0,1). \tag{13b}$$

and therefore define some motion with constant velocity across $\mathbb{R}^2$.

Of course, one must provide some initial conditions in order to determine ζ_∞ and ξ_∞ uniquely. In what follows, we shall denote the solution of (13a) with an initial condition $\zeta_\infty(0) = \vec{x}$, as $\zeta_\infty^{\vec{x}}$. The notation $\xi_\infty^{\vec{x}}$ shall be employed as well.

The discussion of coupling of $\vec{z}$ with either ζ or ξ remains of course valid for the scaled processes. The coupling equalities (9) can then be restated in an identical form, with $\vec{z}$, ζ and ξ replaced by $\vec{z}_n$, ζ_n and ξ_n respectively.

3. Most probable behavior

3.1. **Stability.** As we come to study the stability of the differential equation (12), it could be useful to give first the empirical view of the model in order to understand its expected behavior on a long-time scale.

We understand intuitively, that in a well-designed model the capacity of the server should be sufficient to empty both queues over time, and to maintain them in nearly empty state.

The most basic requirement from the system (expressed in terms of its arrival/service rates λ_0, λ_1, μ_0, μ_1) is for the server to be able to handle each queue separately. That said, for each queue its service rate must exceed its arrival rate. Clearly, this condition is essential in order to ensure at all that the server is able to exhaust one given non-empty queue before it passes to serve the other one. This requirement yields the conditions

$$\mu_0 > \lambda_0, \tag{14a}$$

$$\mu_1 > \lambda_1. \tag{14b}$$

Furthermore, in order to understand the mutual dynamics of the server and both queues, we consider the situation where there is a large number N of clients in the first queue, and the second one is empty. The server starts serving the first queue, and it takes roughly $\frac{N}{\mu_0-\lambda_0}$ time to exhaust it, as the server faces simultaneous arrivals at rate λ_0, as it works with the rate μ_0. Meanwhile, the second queue was populated by $\frac{N\lambda_1}{\mu_0-\lambda_0}$ clients. Now the server comes to serve the second queue, and naturally exhausts it in $\frac{N\lambda_1}{(\mu_0-\lambda_0)(\mu_1-\lambda_1)}$ time. During this time the first queue acquired again $\frac{N\lambda_1\lambda_0}{(\mu_0-\lambda_0)(\mu_1-\lambda_1)}$ clients, and we face once again the situation when the second queue is empty, and the first contains some large number of clients. We observe that a "good" system should have reduced its total load by now:

$$\begin{aligned}\frac{N\lambda_1\lambda_0}{(\mu_0-\lambda_0)(\mu_1-\lambda_1)} &< N\\ \lambda_1\lambda_0 &< (\mu_0-\lambda_0)(\mu_1-\lambda_1)\\ \mu_0\lambda_1+\mu_1\lambda_0 &< \mu_0\mu_1.\end{aligned}$$

This condition can be restated as

$$\frac{\lambda_0}{\mu_0}+\frac{\lambda_1}{\mu_1} < 1. \tag{14c}$$

Remark. Since λ_0, λ_1, μ_0 and μ_1 are all taken to be positive, one can immediately see that (14c) implies both (14a) and (14b).

We now claim that the list (14) of conditions implies the stability of the equation (12).

Proposition 2. *If the parameters λ_0, λ_1, μ_0, μ_1 satisfy the stability condition* (14)*, then the differential equation* (12) *is stable, and the origin* $(0,0)$ *is a stable critical point. Moreover, for each initial value* $\vec{z}_\infty(0)$*, there is a unique solution* $\vec{z}_\infty$*.*

Proof. If either coordinate of $\vec{z}_\infty(0)$ is positive, then (12a) can be solved uniquely, and the stability until the arrival to zero can be verified explicitly. We therefore only consider the case $\vec{z}_\infty(0) = (0,0,0)$.

It is our purpose to show the stability of (12) at the origin. We shall use the Liapunov stability theorem [BD97, Thm 9.6.1].

Consider the Liapunov function

$$V(x,y,s) = V(x,y) = \mu_1 x + \mu_0 y.$$

Trivially, V is continuous, positive definite on $\mathbb{D}$ and has continuous first partial derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$. Moreover, the derivative $\dot{V}$ of V with respect to the autonomous system (12) is negative definite, because

$$\begin{aligned}
\dot{V}(x,y,s) &= \frac{\partial V(x,y)}{\partial x} \cdot \frac{dz_{\infty,x}}{dt} + \frac{\partial V(x,y)}{\partial y} \cdot \frac{dz_{\infty,y}}{dt} \\
&\overset{(12)}{=} \begin{cases} \mu_1(\lambda_0 - \mu_0) + \mu_0\lambda_1, & \vec{z}_\infty(t) \in (0,\infty) \times [0,\infty) \times \{0\} \\ \mu_1\lambda_0 + \mu_0(\lambda_1 - \mu_1), & \vec{z}_\infty(t) \in [0,\infty) \times (0,\infty) \times \{1\} \end{cases} \\
&= \mu_1\lambda_0 + \mu_0\lambda_1 - \mu_0\mu_1 \\
&\overset{(14c)}{<} 0, \qquad \text{for any } x, y > 0.
\end{aligned}$$

Therefore, by the Liapunov stability theorem, $(0,0)$ is a stable critical point. □

Remark 5. In this paper we shall only deal with the stable case, so the stability conditions (14) are always assumed to hold. In order to assist the reader, we shall note explicitly for each significant result, whether it requires stability, and identify the points in the proof where the stability conditions are applied.

3.2. **The theorem.** In what follows, we shall formalize the above discussion and establish the most probable behavior scheme of the process $\vec{z}_n(t)$ starting at a given point. Indeed it will turn out that for very large n, $\vec{z}_n(t)$ behaves nearly deterministically in the sense that it follows the path $\vec{z}_\infty(t)$ with a very large probability.

Strictly speaking, we shall prove a variation of Kurtz theorem [Kur78] [SW95, Thm 5.3] for the given model:

Theorem 3. *Let* $\vec{z}(t)$ *be the random process defined by its generator* L *in* (8) *and let* $\{\vec{z}_n(t)\}$ *be the family of scaled processes, as defined by the generators* L_n *in* (10)*. Let* $\vec{z}_\infty(t)$ *be the limit process which is the solution of* (12) *with the initial condition*

$$\vec{z}_\infty(0) = \vec{a}_0.$$

Then for each $T \geq 0$ there exist a positive constant C_1 and a function C_2, both independent of $\vec{a}_0$, with

$$\lim_{\epsilon \to 0} \frac{C_2(\epsilon)}{\epsilon^2} \in (0, \infty) \quad \textit{and} \quad \lim_{\epsilon \to \infty} \frac{C_2(\epsilon)}{\epsilon} = \infty \tag{15}$$

such that for all $n \geq 1$ and $\epsilon > 0$

$$\mathbb{P}_{\vec{a}_0}\left(\sup_{0 \leq t \leq T} |\vec{z}_n(t) - \vec{z}_\infty(t)| \geq \epsilon \right) \leq C_1 e^{-nC_2(\epsilon)}. \tag{16}$$

Remark. As we noted previously, the theorem is being proven under the assumption of stability. In fact, this assumption is quite essential to our proof, especially in Section 3.8. Nevertheless, we believe that the statement of the theorem may stay valid even without such assumption, provided, of course, that the initial point $\vec{a}_0$ is chosen to differ from the origin.

3.3. **Notation.** In this section we will introduce the geometry of the problem, and establish the connection between our model and the model of free two-dimensional Poisson process.

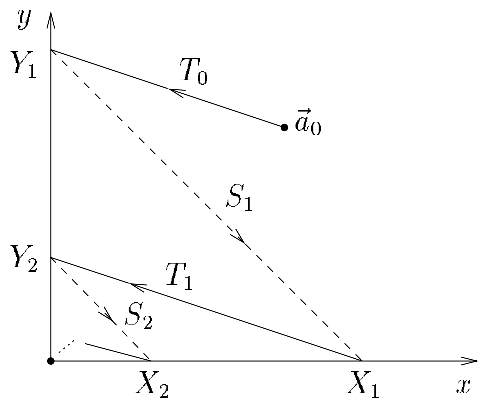


FIGURE 2. Geometric notation of $\vec{z}_\infty$.

We have already concluded from (12), that $\vec{z}_\infty(t)$ is a piecewise-linear path which flips between the panes $\mathbb{D}_0$ and $\mathbb{D}_1$, spending on them ever-shrinking amounts of time until it arrives to the origin $(0,0)$ in a finite time T_∞.

T_∞ can be easily computed from the simple observation that the projection of $\frac{d\vec{z}_\infty}{dt}$ on the axis $\widehat{(\mu_1, \mu_0)}$ is the same constant for both panes, so the projection of $\vec{z}_\infty(t)$ moves along that axis with a constant velocity until it arrives to the origin. For the sake of completeness we present here the exact value of T_∞:

$$T_\infty = \frac{\langle (a_{0,x}, a_{0,y}), (\mu_1, \mu_0) \rangle}{\langle \frac{d\vec{z}_\infty}{dt}, (\mu_1, \mu_0) \rangle} = \frac{a_{0,x}\mu_1 + a_{0,y}\mu_0}{\mu_0\mu_1 - \lambda_0\mu_1 - \lambda_1\mu_0}. \tag{17}$$

We also define the series of time intervals $\{T_0, S_1, T_1, S_2, T_2, \dots\}$, that $\vec{z}_\infty$ spends on a single pane (see Figure 2). T_i will denote the interval between i-th hitting of the x-axis and $i+1$-th hitting of the y-axis. Similarly, S_i will denote the interval between i-th hitting of the y-axis and i-th hitting of the x-axis. T_0 will denote the time from the beginning ($t = 0$) till the first hitting of the y-axis.

Remark. This setting assumes that $\vec{a}_0$ is located on $\mathbb{D}_0$. This assumption can be obviously taken without any loss of generality.

In addition, we define $\{Y_1, Y_2, Y_3, \dots\}$ to be the series of points at which $\vec{z}_\infty$ hits the y-axis, and $\{X_1, X_2, X_3, \dots\}$ to be the series of points at which $\vec{z}_\infty$ hits the x-axis.

Strictly speaking,

$$
\begin{aligned}
Y_1 &= z_{\infty,y}(T_0) \\
Y_i &= z_{\infty,y}(T_0 + \ldots + T_{i-1} \;+\; S_1 + \ldots + S_{i-1}) \\
X_1 &= z_{\infty,x}(T_0 + S_1); \\
X_i &= z_{\infty,x}(T_0 + \ldots + T_{i-1} \;+\; S_1 + \ldots + S_i).
\end{aligned}
\tag{18}
$$

The above quantities are related in a simple manner:

$$
\begin{aligned}
T_0 &= \frac{x_0}{\mu_0 - \lambda_0} \\
T_i &= \frac{X_i}{\mu_0 - \lambda_0}, \quad S_i = \frac{Y_i}{\mu_1 - \lambda_1}, \qquad i = 1, 2, \ldots
\end{aligned}
\tag{19}
$$

Obviously,

$$\sum_{i=0}^{\infty} T_i + \sum_{i=1}^{\infty} S_i = T_\infty.$$

3.4. **Overview of the proof.** Here we shall outline the proof in some detail and provide two auxiliary lemmas.

Our proof will be primarily based on the fact that as long as the random process $\vec{z}_n$ stays on one pane, it behaves like a free motion, and thus its probability to stay around some linear segment of $\vec{z}_\infty$ can be estimated using Kurtz theorem [SW95, Thm 5.3] for the appropriate free motion.

The straightforward approach meets, of course, several obstacles that must be addressed separately. For example, as $\vec{z}_n$ hits the boundary, it changes behavior. Furthermore, as $\vec{z}_n$ arrives to the vicinity of the empty state, it is expected to stay there and therefore changes panes very frequently. Naturally, the examination of this state would require a whole different approach.

For the sake of clarity, the proof will go in several parts:

3.5 *First step.* We define a stopping time as the first hit of the boundary by $\vec{z}_n$. Since until that time $\vec{z}_n$ remains on the same pane as it was in the beginning, it will behave like a free motion and therefore will stay close to $\vec{z}_\infty$ until then.

3.6 *Second step.* Here we examine the behavior of $\vec{z}_n$ between the first and the second "collision" with the boundary. The major difference between this step and the previous one is that while there $\vec{z}_n$ started at a specific point $\vec{a}_0$ at the zero time, now it starts at some random point in the vicinity of Y_1 and at some random time.

3.7 *Third step and further.* We observe that the third step, the fourth step, and so forth, can be evaluated very similarly to the second step. But now there arises a problem that the number of steps $\vec{z}_\infty$ makes until it arrives close to the origin, may be very large. Therefore we also take care to find a simple bound for the probability of $\vec{z}_n$ to stay close to $\vec{z}_\infty$ during all these steps.

3.8 *Staying around the origin.* This part shows that once $\vec{z}_n$ is near the empty state, it stays there with a very large probability.

3.9 *Finalizing the proof.* Although we have shown by now that the probability stated in the Theorem is bounded appropriately on each interval of time. we still need to obtain a uniform bound in the form of (16). This part is rather easy.

We wish to introduce two short notations related to the free motions ζ and ξ.

Let $\vec{x} \in \mathbb{R}^2$ be the initial state of the scaled process ζ_n, for all n. Recall the definition of $\zeta_\infty^{\vec{x}}$ as the solution of (13a) with the initial condition $\zeta_\infty(0) = \vec{x}$, as stated at the end of Section 2.4. We denote

$$\mathcal{P}(T, \epsilon, n) = \mathbb{P}_{\vec{x}}\left(\sup_{0 \le t \le T} |\zeta_n(t) - \zeta_\infty^{\vec{x}}(t)| \ge \epsilon \right). \tag{20a}$$

Naturally, $\mathcal{P}$ doesn't depend on the initial position $\vec{x}$, and therefore $\vec{x}$ can be chosen arbitrarily.

In a similar fashion we denote

$$\mathcal{Q}(T, \epsilon, n) = \mathbb{P}_{\vec{x}}\left(\sup_{0 \le t \le T} |\xi_n(t) - \xi_\infty^{\vec{x}}(t)| \ge \epsilon \right). \tag{20b}$$

Remark. $\mathcal{P}$ and $\mathcal{Q}$ denote, in fact, the probability of an appropriate free motion to escape the allocated region of width 2ϵ in the allocated time T.

By the mere monotonicity of measure (or set inclusion principle), $\mathcal{P}$ and $\mathcal{Q}$ decrease as ϵ increases, and increase as T increases.

Now we present a useful lemma concerning infinite series of $\mathcal{P}(T, \epsilon, n)$ and $\mathcal{Q}(T, \epsilon, n)$.

Lemma 4. *Let $\{T_k\}_{k=1}^{\infty}$ be a set of strictly positive real numbers with a finite sum*

$$S = \sum_{k=1}^{\infty} T_k < \infty.$$

Then for any $\epsilon > 0$ and $n \in \mathbb{N}$

$$\sum_{k=1}^{\infty} \mathcal{P}(T_k, \epsilon, n) \le \frac{\mathcal{P}(S, \frac{\epsilon}{2}, n)}{1 - \mathcal{P}(S, \frac{\epsilon}{2}, n)}, \tag{21a}$$

$$\sum_{k=1}^{\infty} \mathcal{Q}(T_k, \epsilon, n) \le \frac{\mathcal{Q}(S, \frac{\epsilon}{2}, n)}{1 - \mathcal{Q}(S, \frac{\epsilon}{2}, n)}. \tag{21b}$$

Proof. We shall prove only the result for $\mathcal{P}$'s; the one for $\mathcal{Q}$'s is similar.

Denote the partial sums

$$S_m = \sum_{k=1}^{m} T_i, \quad m \in \mathbb{N}.$$

Let $\epsilon > 0$ and $n \in \mathbb{N}$. We wish to prove by induction that for any $m \in \mathbb{N}$

$$\sum_{k=1}^{m} \mathcal{P}(T_k, \epsilon, n) \le \frac{\mathcal{P}(S_m, \frac{\epsilon}{2}, n)}{1 - \mathcal{P}(S, \frac{\epsilon}{2}, n)}. \tag{22}$$

The base of induction holds trivially, as for $m = 1$

$$\mathcal{P}(T_1, \epsilon, n) \le \mathcal{P}(T_1, \frac{\epsilon}{2}, n) \le \frac{\mathcal{P}(T_1, \frac{\epsilon}{2}, n)}{1 - \mathcal{P}(S, \frac{\epsilon}{2}, n)}. \tag{23}$$

We assume now that (22) holds for some specific $m \in \mathbb{N}$. We need to prove that it holds for $m + 1$, i.e.

$$\sum_{k=1}^{m+1} \mathcal{P}(T_k, \epsilon, n) \le \frac{\mathcal{P}(S_{m+1}, \frac{\epsilon}{2}, n)}{1 - \mathcal{P}(S, \frac{\epsilon}{2}, n)}. \tag{24}$$

Let $\vec{x} \in \mathbb{R}^2$ be the initial state of ζ_n, and let ζ_∞ be the "most probable path of ζ_n", i.e. the solution of (13a) with the initial condition $\zeta_\infty(0) = \vec{x}$. Let also $\vec{v} \in \mathbb{R}^2$ be some point such that $|\vec{v} - \zeta_\infty(S_m)| < \frac{\epsilon}{2}$.

By the Markov property of ζ_n,

$$\begin{aligned}
\mathbb{P}_{\vec{x}}\left(\sup_{t \le S_{m+1}} |\zeta_n(t) - \zeta_\infty(t)| < \frac{\epsilon}{2} \ \cap \ \zeta_n(S_m) = \vec{v} \right) \\
= \mathbb{P}\left(\sup_{S_m \le t \le S_{m+1}} |\zeta_n(t) - \zeta_\infty(t)| < \frac{\epsilon}{2} \ \middle| \ \zeta_n(S_m) = \vec{v} \right) \\
\times \mathbb{P}_{\vec{x}}\left(\sup_{t \le S_m} |\zeta_n(t) - \zeta_\infty(t)| < \frac{\epsilon}{2} \ \cap \ \zeta_n(S_m) = \vec{v} \right).
\end{aligned} \tag{25}$$

Consider the solution $\tilde{\zeta}(t)$ of (13a) which satisfies

$$\tilde{\zeta}(S_m) = \vec{v}.$$

It can be easily seen from Figure 3, that

$$\left\{\zeta_n \;\middle|\; \sup_{S_m\le t\le S_{m+1}} |\zeta_n(t)-\zeta_\infty(t)| < \frac{\epsilon}{2}\right\}$$
$$\subseteq \left\{\zeta_n \;\middle|\; \sup_{S_m\le t\le S_{m+1}} |\zeta_n(t)-\tilde{\zeta}_\infty(t)| < \epsilon\right\}.$$

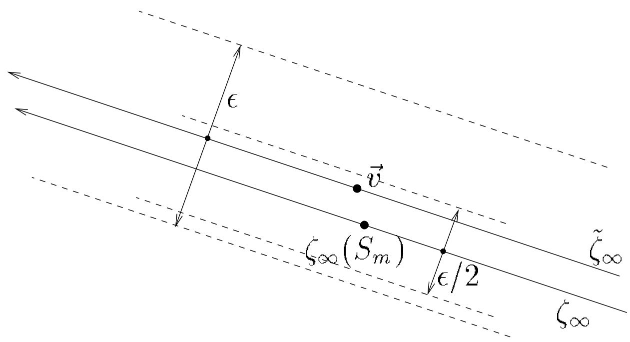


FIGURE 3. ζ_∞ versus $\tilde{\zeta}_\infty$.

Thus,

$$\mathbb{P}\left(\sup_{S_m\le t\le S_{m+1}} |\zeta_n(t)-\zeta_\infty(t)| < \frac{\epsilon}{2} \;\middle|\; \zeta_n(S_m)=\vec{v}\right)$$
$$\le \mathbb{P}\left(\sup_{S_m\le t\le S_{m+1}} |\zeta_n(t)-\tilde{\zeta}_\infty(t)| < \epsilon \;\middle|\; \zeta_n(S_m)=\vec{v}\right),$$

and by shifting the time by $-S_m$ we further obtain

$$\mathbb{P}\left(\sup_{S_m\le t\le S_{m+1}} |\zeta_n(t)-\tilde{\zeta}_\infty(t)| < \epsilon \;\middle|\; \zeta_n(S_m)=\vec{v}\right) = 1-\mathcal{P}(T_m,\epsilon,n).$$

Therefore, it follows from (25) that

$$\mathbb{P}_{\vec{x}}\left(\sup_{t\le S_{m+1}} |\zeta_n(t)-\zeta_\infty(t)| < \frac{\epsilon}{2} \;\cap\; \zeta_n(S_m)=\vec{v}\right)$$
$$\le \big(1-\mathcal{P}(T_m,\epsilon,n)\big)\cdot \mathbb{P}_{\vec{x}}\left(\sup_{t\le S_m} |\zeta_n(t)-\zeta_\infty(t)| < \frac{\epsilon}{2} \;\cap\; \zeta_n(S_m)=\vec{v}\right).$$

Summing over all possible $\vec{v}$ such that $|\vec{v}-\zeta_\infty(S_m)| < \frac{\epsilon}{2}$, we obtain

$$\mathbb{P}_{\vec{x}}\left(\sup_{t\le S_{m+1}} |\zeta_n(t)-\zeta_\infty(t)| < \frac{\epsilon}{2}\right)$$
$$\le \big(1-\mathcal{P}(T_m,\epsilon,n)\big)\cdot \mathbb{P}_{\vec{x}}\left(\sup_{t\le S_m} |\zeta_n(t)-\zeta_\infty(t)| < \frac{\epsilon}{2}\right),$$

and thus

$$\begin{aligned}
1-\mathcal{P}(S_{m+1},\frac{\epsilon}{2},n) &\le \big(1-\mathcal{P}(T_m,\epsilon,n)\big)\big(1-\mathcal{P}(S_m,\frac{\epsilon}{2},n)\big)\\
\mathcal{P}(S_{m+1},\frac{\epsilon}{2},n) &\ge \mathcal{P}(S_m,\frac{\epsilon}{2},n)+\mathcal{P}(T_m,\epsilon,n)\big(1-\mathcal{P}(S_m,\frac{\epsilon}{2},n)\big)\\
\mathcal{P}(S_{m+1},\frac{\epsilon}{2},n) &\ge \mathcal{P}(S_m,\frac{\epsilon}{2},n)+\mathcal{P}(T_m,\epsilon,n)\big(1-\mathcal{P}(S,\frac{\epsilon}{2},n)\big),
\end{aligned}$$

and by the induction assumption (22)

$$\begin{aligned}
\mathcal{P}(S_{m+1},\frac{\epsilon}{2},n) &\ge \big(1-\mathcal{P}(S,\frac{\epsilon}{2},n)\big)\sum_{k=1}^{m}\mathcal{P}(T_k,\epsilon,n)\\
&\qquad + \big(1-\mathcal{P}(S,\frac{\epsilon}{2},n)\big)\mathcal{P}(T_m,\epsilon,n)\\
&\ge \big(1-\mathcal{P}(S,\frac{\epsilon}{2},n)\big)\sum_{k=1}^{m+1}\mathcal{P}(T_k,\epsilon,n),
\end{aligned}$$

and (24) follows.

Now since the base (23) holds, and the assumption (22) implies (24) for any $m\in\mathbb{N}$, the inequality (22) holds for all $m\in\mathbb{N}$. By bringing m to infinity we obtain the desired result (21). □

3.5. **First step.** First we will show that $\vec{z}_n$ stays near $\vec{z}_\infty$ until it arrives to the point Y_1 at the boundary, with sufficient probability.

Let $n\ge 1$, $\delta>0$. Define the stopping time of the arrival to the y-axis:

$$\tau_0=\inf\{t>0:\ z_{n,x}(t)=0\}.$$

Consider the free motion ζ_n starting at $\vec{a}\in\mathbb{D}_0$ (with the third coordinate stripped). ζ_n is coupled with $\vec{z}_n$ (see Remark 4), and its limit process ζ_∞ coincides with $\vec{z}_\infty$ until time T_0:

$$\forall t\le T_0\quad \vec{z}_\infty(t)=\zeta_\infty(t).$$

Let $\widetilde{T_0}=T_0+\frac{\delta}{\mu_0-\lambda_0}$. Then $\zeta_{\infty,x}(\widetilde{T_0})=-\delta$ (see Figure 4).

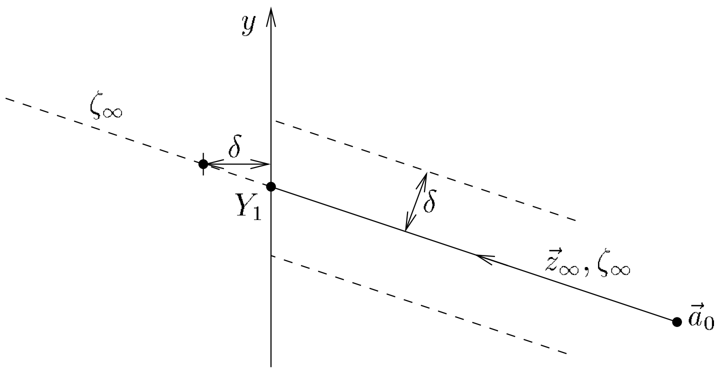


FIGURE 4. First arrival to the boundary.

Denote $\tilde{\tau}_0$ as the stopping time for ζ_n, similarly to τ_0:

$$\tilde{\tau}_0 = \inf\{t > 0 :\ \zeta_{n,x}(t) = 0\}.$$

Let $\sup_{t \le \widetilde{T}_0} |\zeta_n(t) - \zeta_\infty(t)| < \delta$. Then

$$\begin{aligned}
|\zeta_{n,x}(\widetilde{T}_0) - \zeta_{\infty,x}(\widetilde{T}_0)| &< \delta \\
|\zeta_{n,x}(\widetilde{T}_0) + \delta| &< \delta \\
\zeta_{n,x}(\widetilde{T}_0) &< 0 \\
\widetilde{T}_0 &> \tilde{\tau}_0(\zeta_n) \\
\sup_{t \le \widetilde{T}_0} |\zeta_n(t) - \zeta_\infty(t)| &\ge \sup_{t \le \tilde{\tau}_0(\zeta_n)} |\zeta_n(t) - \zeta_\infty(t)|,
\end{aligned}$$

and thus $\sup_{t \le \tilde{\tau}_0(\zeta_n)} |\zeta_n(t) - \zeta_\infty(t)| < \delta$. This implies immediately, that

$$\left\{ \zeta_n \;\middle|\; \sup_{t \le \tilde{\tau}_0} |\zeta_n(t) - \zeta_\infty(t)| < \delta \right\} \supseteq \left\{ \zeta_n \;\middle|\; \sup_{t \le \widetilde{T}_0} |\zeta_n(t) - \zeta_\infty(t)| < \delta \right\}$$

and accordingly,

$$\mathbb{P}_{\vec{a}} \left(\sup_{t \le \tilde{\tau}_0} |\zeta_n(t) - \zeta_\infty(t)| \ge \delta \right) \le \mathbb{P}_{\vec{a}} \left(\sup_{t \le \widetilde{T}_0} |\zeta_n(t) - \zeta_\infty(t)| \ge \delta \right). \tag{26}$$

Therefore

$$\begin{aligned}
\mathcal{P}(\widetilde{T}_0, \delta, n) &= \mathbb{P}_{\vec{a}}\Big(\sup\nolimits_{0 \le t \le \widetilde{T}_0} |\zeta_n(t) - \zeta_\infty(t)| \ge \delta \Big) \\
&\ge \mathbb{P}_{\vec{a}}\Big(\sup\nolimits_{0 \le t \le \tilde{\tau}_0} |\zeta_n(t) - \zeta_\infty(t)| \ge \delta \Big) \quad \text{by (26)} \\
&= \mathbb{P}_{\vec{a}}\Big(\sup\nolimits_{0 \le t \le \tau_0} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) \ge \delta \Big) \quad \text{by coupling.}
\end{aligned} \tag{27}$$

3.6. **Second step.** Now we need to estimate the probability for $\vec{z}_n$ to stay around $\vec{z}_\infty$ during the time interval $[0, T_0 + S_1]$, i.e. along the first two linear pieces of $\vec{z}_\infty$. Using the strong Markov property of $\vec{z}_n$, we can reduce this problem to a problem of estimating the probability for $\vec{z}_n$ to stay around $\vec{z}_\infty$ during the second time interval, provided it did stay around $\vec{z}_\infty$ during the first interval.

Let Δ_0 be the set of points on y-axis which fall inside the δ-neighborhood of ζ_∞ as defined in the first step (see Figure 5).

We denote also α, β as the angles between the y-axis and two adjacent pieces of $\vec{z}_\infty$ respectively, and define $\delta_0 = \delta\left(1 - \frac{\sin\beta}{\sin\alpha}\right)$. The quantities α and β can be readily obtained from

$$\tan\alpha = \frac{\mu_0 - \lambda_0}{\lambda_1}, \quad \tan\beta = \frac{\lambda_0}{\mu_1 - \lambda_1}.$$

It also follows from the stability condition (14) that $\alpha > \beta$, and thus $0 < \delta_0 < \delta$.

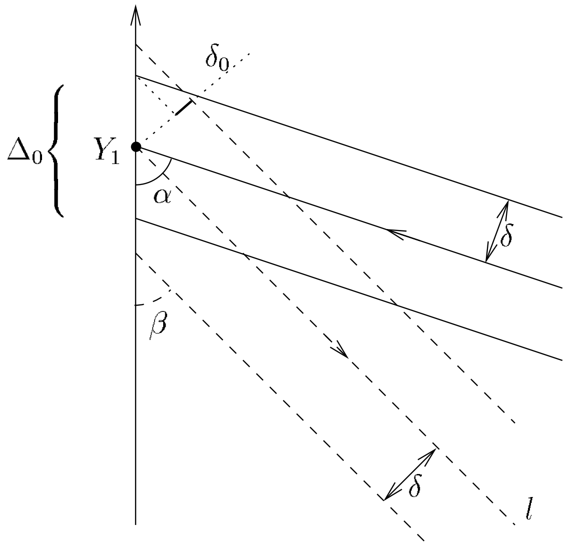


FIGURE 5. Geometry of the second step.

Remark. This reference to stability is crucial for our proof. From now on, we shall extensively use the fact that β is smaller than α in order to deduce that $\vec{z}_\infty$ indeed does arrive to the empty state, and that it stays there.

Note that under these settings $\vec{z}_n(\tau_0) \in \Delta_0$, i.e. located at most $\frac{\delta}{\sin\alpha}$ away from Y_1 along the y-axis. Therefore its initial distance from the straight line l, which contains the second piece of $\vec{z}_\infty$, is at most $\delta \cdot \frac{\sin\beta}{\sin\alpha} = \delta - \delta_0$. We can allow $\vec{z}_n$ to start at any point inside Δ_0, and stay at the distance δ_0 from the line starting at the same point and parallel to l. Thus we will ensure that $\vec{z}_n$ stays at the distance δ from l.

We repeat the reasoning we applied in the first step with regards to the point X_1 located on the x-axis (see Figure 2).

Define the stopping time of the arrival of $\vec{z}_n$ to the x-axis:

$$\sigma_1 = \inf\{t > \tau_0 : z_{n,y}(t) = 0\}.$$

Now we recall the free process ξ coupled with $\vec{z}$ on $\mathbb{D}_1$. In order to formalize our discussion, we denote the process which starts at $(0, p)$ as ξ^p, and its limit process as ξ^p_∞ respectively.

We wish to allocate $\widetilde{S}_1$ large enough, so that any limit process of the kind ξ_∞ originating on Δ_0 would reach at least δ below the x-axis. Clearly, $y_1 = Y_{1,y} + \frac{\delta}{\sin\alpha}$ corresponds to the process $\xi^{y_1}_\infty$ which has the longest way to go. So we can put

$$\widetilde{S}_1 = S_1 + \frac{1}{\mu_1 - \lambda_1}\left(\frac{\delta}{\sin\alpha} + \delta\right).$$

Furthermore, we consider

$$\tilde{\sigma}_1^p = \inf\{t > 0 : \ \xi_{n,y}^p(t) = 0\}.$$

Let $p \in \Delta_0$. Then, similarly to (26),

(28)

$$\mathbb{P}_{(0,p)}\Big(\sup_{t \le \tilde{\sigma}_1^p} |\xi_n^p(t) - \xi_\infty^p(t)| \ge \delta_0\Big) \le \mathbb{P}_{(0,p)}\Big(\sup_{t \le \widetilde{S}_1} |\xi_n^p(t) - \xi_\infty^p(t)| \ge \delta_0\Big).$$

Therefore,

(29)

$$\begin{aligned}
\mathcal{Q}(\widetilde{S}_1, \delta_0, n) &= \mathbb{P}_{(0,p)}\Big(\sup_{t \le \widetilde{S}_1} |\xi_n^p(t) - \xi_\infty^p(t)| \ge \delta_0\Big) \\
&\ge \mathbb{P}_{(0,p)}\Big(\sup_{t \le \tilde{\sigma}_1^p} |\xi_n^p(t) - \xi_\infty^p(t)| \ge \delta_0\Big) && \text{by (28)} \\
&= \mathbb{P}\Big(\sup_{\tau_0 \le t \le \sigma_1} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) \ge \delta_0 \ \Big| \ z_{n,y}(\tau_0) = p\Big) && \text{by cpl.} \\
&\ge \mathbb{P}\Big(\sup_{\tau_0 \le t \le \sigma_1} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) \ge \delta \ \Big| \ z_{n,y}(\tau_0) = p\Big).
\end{aligned}$$

By the strong Markov property [Nor97, Thm 6.5.4],

$$\begin{aligned}
&\mathbb{P}_{\vec{a}}\Big(\sup_{t \le \sigma_1} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) < \delta \ \cap \ \vec{z}_n(\tau_0) = (0,p) \ \cap \ \tau_0 < \widetilde{T}_0\Big) \\
&\quad = \mathbb{P}\Big(\sup_{\tau_0 \le t \le \sigma_1} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) < \delta \ \Big| \ \vec{z}_n(\tau_0) = (0,p)\Big) \\
&\qquad \times \mathbb{P}_{\vec{a}}\Big(\sup_{t \le \tau_0} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) < \delta \ \cap \ \vec{z}_n(\tau_0) = (0,p) \ \cap \ \tau_0 < \widetilde{T}_0\Big).
\end{aligned}$$

We can take the infimum over $p \in \Delta_0$ of the first probability in the righthand side, and sum up the result over $p \in \Delta_0$ again. Thus we will obtain

$$\begin{aligned}
(30) \quad &\mathbb{P}_{\vec{a}}\Big(\sup_{t \le \sigma_1} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) < \delta \ \cap \ \tau_0 < \widetilde{T}_0\Big) \\
&\qquad \ge \inf_{p \in \Delta_0} \mathbb{P}\Big(\sup_{\tau_0 \le t \le \sigma_1} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) < \delta \ \Big| \ \vec{z}_n(\tau_0) = (0,p)\Big) \\
&\qquad\qquad \times \mathbb{P}_{\vec{a}}\Big(\sup_{t \le \tau_0} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) < \delta \ \cap \ \tau_0 < \widetilde{T}_0\Big).
\end{aligned}$$

Recall that in the first step we obtained that

$$\sup_{t \le \widetilde{T}_0} |\zeta_n(t) - \zeta_\infty(t)| < \delta \ \text{ implies } \ \widetilde{T}_0 > \tilde{\tau}_0(\zeta_n).$$

Similarly one can see that

$$\sup_{t\le \widetilde{T}_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \quad \text{implies} \quad \widetilde{T}_0 > \tau_0.$$

Therefore,

$$\begin{aligned}
\mathbb{P}_{\vec{a}}&\bigg(\sup_{t\le\tau_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \ \cap \ \tau_0 < \widetilde{T}_0\bigg)\\
&\qquad\qquad \ge \mathbb{P}_{\vec{a}}\bigg(\sup_{t\le \widetilde{T}_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \ \cap \ \tau_0 < \widetilde{T}_0\bigg)\\
&\qquad\qquad \ge \mathbb{P}_{\vec{a}}\bigg(\sup_{t\le \widetilde{T}_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta\bigg).
\end{aligned}$$

Furthermore, in a similar fashion $\sup_{t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta$ implies again $\widetilde{T}_0 > \tau_0$.

The statement (30) can thus be restated as

$$\begin{aligned}
(31)\quad \mathbb{P}_{\vec{a}}&\bigg(\sup_{t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta\bigg)\\
&\ge \inf_{p\in\Delta_0} \mathbb{P}\bigg(\sup_{\tau_0\le t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \ \bigg|\ \vec{z}_n(\tau_0) = (0,p)\bigg)\\
&\quad \times \mathbb{P}_{\vec{a}}\bigg(\sup_{t\le\tau_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta\bigg).
\end{aligned}$$

Before we proceed, let us denote for simplicity of notation

$$\begin{aligned}
k &= \mathbb{P}_{\vec{a}}\bigg(\sup_{t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) \ge \delta\bigg),\\
l &= \mathbb{P}_{\vec{a}}\bigg(\sup_{t\le\tau_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) \ge \delta\bigg),\\
m_p &= \mathbb{P}\bigg(\sup_{\tau_0\le t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) \ge \delta \ \bigg|\ \vec{z}_n(\tau_0) = (0,p)\bigg).
\end{aligned}$$

It now follows from (31), that

$$\begin{aligned}
1-k &\ge \inf_{p\in\Delta_0} (1-m_p)(1-l)\\
&\ge (1-\sup_{p\in\Delta_0} m_p)(1-l)\\
&\ge 1-l-\sup_{p\in\Delta_0} m_p + l\sup_{p\in\Delta_0} m_p\\
&\ge 1-l-\sup_{p\in\Delta_0} m_p\\
k &\le l+\sup_{p\in\Delta_0} m_p
\end{aligned}$$

and thus

$$
\begin{aligned}
(32)\quad & \mathbb{P}_{\vec{a}}\Big(\sup_{t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)\ge\delta\Big)\\
&\le \mathbb{P}_{\vec{a}}\Big(\sup_{t\le\tau_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)\ge\delta\Big)\\
&\quad+\sup_{p\in\Delta_0}\mathbb{P}\Big(\sup_{\tau_0\le t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)\ge\delta \;\Big|\; \vec{z}_n(\tau_0)=(0,p)\Big)\\
&\overset{(29)}{\le} \mathbb{P}_{\vec{a}}\Big(\sup_{t\le\tau_0} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)\ge\delta\Big)+\mathcal{Q}(\widetilde{S}_1,\delta_0,n)\\
&\overset{(27)}{\le} \mathcal{P}(\widetilde{T}_0,\delta,n)+\mathcal{Q}(\widetilde{S}_1,\delta_0,n).
\end{aligned}
$$

3.7. **Third step and further.** Continuing the direction established earlier, we now wish to estimate the probability of $\vec{z}_n(t)$ to stay around $\vec{z}_\infty(t)$ until the time $T_0+S_1+T_1$, i.e. during the first three linear pieces of $\vec{z}_\infty(t)$. For this purpose, we define the stopping time

$$\tau_1=\inf\{t>\sigma_1:\ z_{n,x}(t)=0\}.$$

Then we can define variables like we did in the second step, with appropriate changes, as the panes $\mathbb{D}_0$ and $\mathbb{D}_1$ switch their roles:

$$
\begin{aligned}
\delta_1&=\delta\left(1-\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\sin\left(\frac{\pi}{2}-\beta\right)}\right)=\delta\left(1-\frac{\cos\alpha}{\cos\beta}\right).\\
\widetilde{T}_1&=T_1+\frac{1}{\mu_0-\lambda_0}\left(\frac{\delta}{\cos\beta}+\delta\right).
\end{aligned}
$$

By the similar approach, as the one we used in the second step, we can now obtain

$$
\begin{aligned}
&\mathbb{P}_{\vec{a}}\Big(\sup_{t\le\tau_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)\ge\delta\Big)\\
&\qquad\le \mathbb{P}_{\vec{a}}\Big(\sup_{t\le\sigma_1} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)\ge\delta\Big)+\mathcal{P}(\widetilde{T}_1,\delta_1,n)\\
&\qquad\overset{(32)}{\le} \mathcal{P}(\widetilde{T}_0,\delta,n)+\mathcal{Q}(\widetilde{S}_1,\delta_0,n)+\mathcal{P}(\widetilde{T}_1,\delta_1,n).
\end{aligned}
$$

We thus continue to advance further towards the origin, until we approach close enough.

Let us elaborate on the notion of "close enough". As far as X_kY_{k+1} doesn't intersect the δ-neighborhood of $(0,0)$, the behavior of $\vec{z}_n(t)$ is quite predictable while it is itself confined to the δ-neighborhood of $\vec{z}_\infty(t)$. Otherwise, $\vec{z}_n(t)$ may change pane several times during a single step of $\vec{z}_\infty(t)$, and so the reasoning we employed until now may not apply anymore.

Therefore we consider X_kY_{k+1} as the last interval in the sequence $\{Y_1X_1, X_1Y_2, Y_2X_2, X_2Y_3, \dots\}$ (see Figure 6), which doesn't intersect the δ-neighborhood of $(0,0)$. (Of course, this last interval could as well be of the kind Y_kX_k, but we can assume the former without loss of generality.)

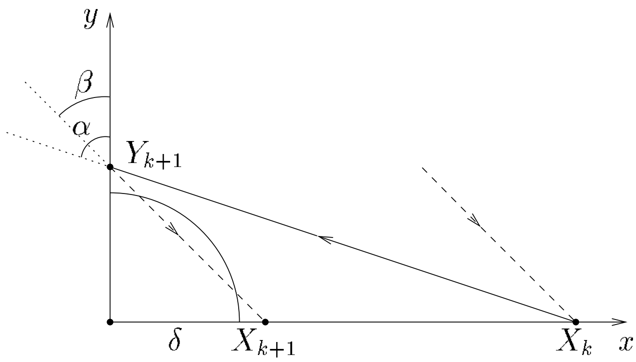


FIGURE 6. Approaching the origin.

A simple geometric calculation yields the following results:

$$X_k > \frac{\delta}{\cos\alpha}, \tag{33}$$

$$Y_{k+1} \le \frac{\delta}{\sin\beta}, \qquad X_{k+1} \le \frac{\delta}{\cos\beta}. \tag{34}$$

We also see that, by further applying the reasons outlined at the beginning of this step, for the appropriately defined stopping time

$$\tau_k = \inf\left\{ t : \begin{array}{c} z_{n,x} = 0 \text{ for the} \\ k+1\text{-th time} \end{array} \right\}$$

one can derive

$$\begin{aligned} \mathbb{P}_{\vec{a}}\Big(\sup_{t \le \tau_k} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) \ge \delta \Big) \\ \le \mathcal{P}(\widetilde{T}_0, \delta, n) + \sum_{i=1}^{k} \Big[\mathcal{P}(\widetilde{T}_i, \delta_1, n) + \mathcal{Q}(\widetilde{S}_i, \delta_0, n) \Big], \end{aligned} \tag{35}$$

where

$$\begin{aligned} \widetilde{T}_i &= T_i + \frac{\delta}{\mu_0 - \lambda_0}\left(1 + \frac{1}{\cos\beta}\right), \\ \widetilde{S}_i &= S_i + \frac{\delta}{\mu_1 - \lambda_1}\left(1 + \frac{1}{\sin\alpha}\right). \end{aligned}$$

We observe that for any $i \in \{1, 2, 3, \dots, k\}$

$$\begin{aligned}
T_i &\overset{(19)}{=} \frac{X_i}{\mu_0 - \lambda_0} \geq \frac{X_k}{\mu_0 - \lambda_0} \\
&\overset{(33)}{>} \frac{\delta}{\cos\alpha(\mu_0 - \lambda_0)} \\
\widetilde{T}_i &= T_i + \frac{\delta}{\mu_0 - \lambda_0}\left(1 + \frac{1}{\cos\beta}\right) \\
&< T_i + \frac{\delta}{\mu_0 - \lambda_0}\left(1 + \frac{1}{\cos\alpha}\right) \\
&< T_i + \frac{\delta}{\mu_0 - \lambda_0} \cdot \frac{2}{\cos\alpha} \\
&< 3T_i.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{k} \mathcal{P}(\widetilde{T}_i, \delta_1, n) \leq \sum_{i=1}^{k} \mathcal{P}(3T_i, \delta_1, n) \leq \sum_{i=1}^{\infty} \mathcal{P}(3T_i, \delta_1, n).$$

Note that $\sum T_i$ is a geometric series, with sum smaller than T. Therefore by Lemma 4,

$$\sum_{i=1}^{k} \mathcal{P}(\widetilde{T}_i, \delta_1, n) \leq \frac{\mathcal{P}(T, \frac{\delta_1}{2}, n)}{1 - \mathcal{P}(T, \frac{\delta_1}{2}, n)}.$$

In the similar fashion,

$$\sum_{i=1}^{k} \mathcal{Q}(\widetilde{S}_i, \delta_0, n) \leq \frac{\mathcal{Q}(T, \frac{\delta_0}{2}, n)}{1 - \mathcal{Q}(T, \frac{\delta_0}{2}, n)}.$$

It now follows from (35), that

$$\begin{aligned}
(36)\quad & \mathbb{P}_{\vec{a}}\left(\sup_{t \leq \tau_k} d\big(\vec{z}_n(t), \vec{z}_\infty(t)\big) \geq \delta\right) \\
&\leq \mathcal{P}(\widetilde{T}_0, \delta, n) + \frac{\mathcal{P}(T, \frac{\delta_1}{2}, n)}{1 - \mathcal{P}(T, \frac{\delta_1}{2}, n)} + \frac{\mathcal{Q}(T, \frac{\delta_0}{2}, n)}{1 - \mathcal{Q}(T, \frac{\delta_0}{2}, n)}.
\end{aligned}$$

Consider two cases:
I. $\mathcal{P}(T, \frac{\delta_1}{2}, n) < \frac{1}{2}$ and $\mathcal{Q}(T, \frac{\delta_0}{2}, n) < \frac{1}{2}$. Then

$$\begin{aligned}
\frac{\mathcal{P}(T, \frac{\delta_1}{2}, n)}{1 - \mathcal{P}(T, \frac{\delta_1}{2}, n)} &< 2\mathcal{P}(T, \frac{\delta_1}{2}, n), \\
\frac{\mathcal{Q}(T, \frac{\delta_0}{2}, n)}{1 - \mathcal{Q}(T, \frac{\delta_0}{2}, n)} &< 2\mathcal{Q}(T, \frac{\delta_0}{2}, n).
\end{aligned}$$

and (36) implies

$$(37)\quad \mathbb{P}_{\vec{a}}\Big(\sup_{t\le\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)\ge\delta\Big)$$
$$\le \mathcal{P}(\widetilde{T}_0,\delta,n)+2\mathcal{P}(T,\frac{\delta_1}{2},n)+2\mathcal{Q}(T,\frac{\delta_0}{2},n).$$

II. If either of $\mathcal{P}(T,\frac{\delta_1}{2},n)$ and $\mathcal{Q}(T,\frac{\delta_0}{2},n)$ is at least $\frac{1}{2}$, then

$$2\mathcal{P}(T,\frac{\delta_1}{2},n)+2\mathcal{Q}(T,\frac{\delta_0}{2},n)\ge 1,$$

and (37) holds trivially again, and thus it is valid in both cases.

3.8. **Staying around the origin.** In this part of the proof we show that if $\vec{z}_n(t)$ starts in a close vicinity of the origin, its chances to travel far away are very small.

Specifically, we consider two neighborhoods of the empty state whose radii are small (order of ϵ), but differ by a significant constant factor. We model the escape by a sample path $\vec{z}_n$ which starts inside the smaller neighborhood, and after some time arrives to the *outside* of the larger neighborhood. We show that for any such sample path there must exist a period of time when $\vec{z}_n(t)$ stays on a single pane (i.e. is supposed to behave like a free motion), but nevertheless it doesn't stay in the vicinity of its most probable path. The most probable path in this case is a solution of (13) with an appropriate initial condition. Second, we show that the probability of the latter event is small.

For the purpose of further discussion we define the term of "γ-escape". We shall say that a sample path $\zeta_n(t)$ (or, similarly, $\xi_n(t)$) starting at $\vec{x}$ performs a γ-*escape*, when its distance from $\zeta_\infty^{\vec{x}}(t)$ exceeds γ without respect to the time (see Figure 7), i.e.

$$\exists t_1,t_2\ge 0\quad |\zeta_n(t_1)-\zeta_\infty^{\vec{x}}(t_2)|>\gamma.$$

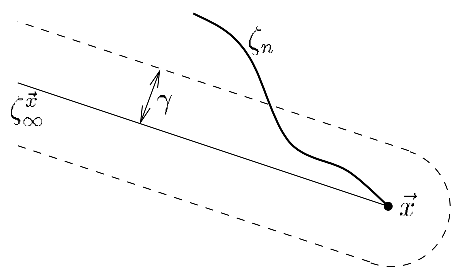


FIGURE 7. A γ-escape.

Furthermore, we shall say that a sample path $\vec{z}_n(t)$ performs a γ-escape starting at t_1, if $\vec{z}_n(t)$ remains on a single pane during some time interval (t_1,t_2), and the sample path of the free motion associated with $\vec{z}_n$ on that pane performs a γ-escape during that time.

We wish to define some auxiliary variables, which are merely multiples of ϵ by some specific constants. Like in many ϵ-δ-style proofs, these definitions look quite artificial, and in fact so they are.

Let

$$\epsilon_1 > 0 \text{ such that } \epsilon_1 < \frac{\epsilon}{2\max\left\{\frac{\sqrt{\mu_0^2+\mu_1^2}}{\mu_0}, \frac{\sqrt{\mu_0^2+\mu_1^2}}{\mu_1}\right\}}. \tag{38}$$

Consider the straight line l which is perpendicular to the vector (μ_1, μ_0) and passes at the distance ϵ_1 from the origin O (see Figure 8).

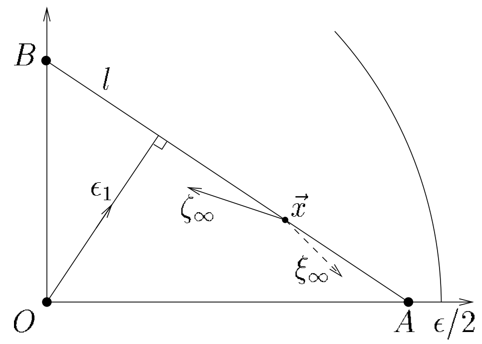


FIGURE 8. The definition of l.

Naturally, the points of intersection of l with the axes are

$$A = \left(\epsilon_1 \frac{\sqrt{\mu_0^2+\mu_1^2}}{\mu_1}, 0\right); \qquad B = \left(0, \epsilon_1 \frac{\sqrt{\mu_0^2+\mu_1^2}}{\mu_0}\right).$$

By choosing ϵ_1 as in (38) we ensured that the segment AB lies entirely inside the $\frac{\epsilon}{2}$-neighborhood of O. The line l also has the nice property, that for any $\vec{x} \in l$, the paths $\zeta_\infty^{\vec{x}}$ and $\xi_\infty^{\vec{x}}$ both lie below l (see Figure 8).

Consider a sample path $\vec{z}_n(t)$, which starts at some $\vec{x}$ and reaches outside the $\frac{\epsilon}{2}$-neighborhood of the origin. Since the segment AB lies inside that neighborhood, there is a moment t_0 of the first crossing of l by $\vec{z}_n$ on either pane.

We intend to prove that any sample path $\vec{z}_n(t)$, which starts inside some ϵ_3-neighborhood of the origin (ϵ_3 is to be defined later), and crosses l, performs an ϵ_2-escape on its way.

We can assume without loss of generality, that $\vec{z}_n(t_0) \in \mathbb{D}_0$, and denote $\vec{z}_n(t_0) = (a, b, 0)$.

Let

$$\epsilon_2 < \min\left\{\frac{\epsilon_1}{2}, \frac{\cos\alpha\cos\beta}{\cos\alpha+\cos\beta}\Big(|OA| - |OB|\frac{\sin\beta}{\cos\beta}\Big)\right\}, \tag{39}$$

$$\epsilon_3 < \min\left\{\frac{\epsilon_1}{2}, \cos\beta\Big(|OA| - \frac{\epsilon_2}{\cos\alpha} - \frac{\epsilon_2}{\cos\beta}\Big)\right\}. \tag{40}$$

On the technical side, we need to consider three different possibilities, depending whether $\vec{z}_n$ escapes without touching the boundary $\partial\mathbb{D}$ (case I), with touching it just once (case II), or more than once (case III). Let us do that now.

I. The sample path $\vec{z}_n(t)$ arrives from $\vec{x}$ to $(a,b,0)$ without touching the boundary $\partial\mathbb{D}$. In his case $\vec{x}$ obviously belongs to $\mathbb{D}_0$. Moreover,

$$d(\vec{x},(0,0)) < \epsilon_3 \overset{(40)}{<} \frac{\epsilon_1}{2}$$

$$d(\vec{x},l) \geq d(l,(0,0)) - d(\vec{x},(0,0)) > \frac{\epsilon_1}{2} \overset{(39)}{>} \epsilon_2.$$

As $\zeta^{\vec{x}}_\infty$ advances, it further recedes from l. Therefore the ϵ_2-neighborhood of $\zeta^{\vec{x}}_\infty$ doesn't reach l, and $\vec{z}_n(t)$ obviously performs an ϵ_2-escape starting at the time 0 (see Figure 9).

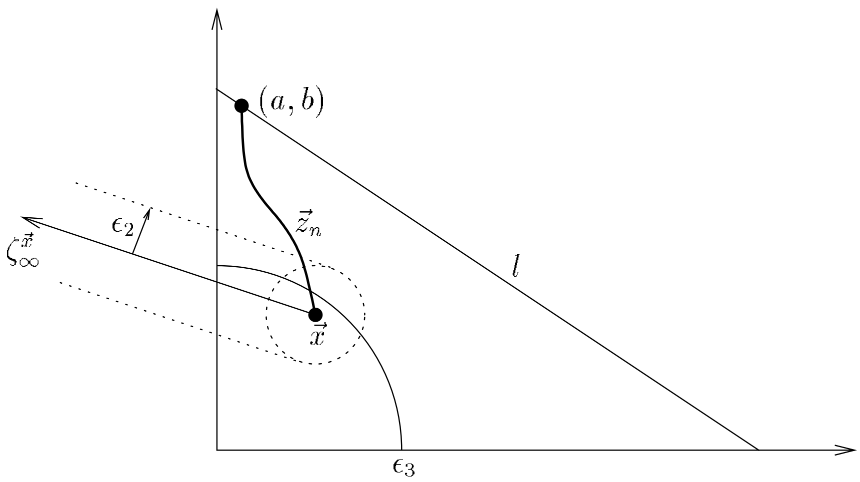


FIGURE 9. Case I: $\vec{z}_n(t)$ goes straight to l.

II. The sample path $\vec{z}_n(t)$ touches $\partial\mathbb{D}$ just once before arriving to $(a,b,0)$. Then the encounter of $\vec{z}_n$ with the boundary must have happened on the x-axis at some moment $t_1 < t_0$, and we know that $\vec{z}_n(t_1)$ lies on OA.

It follows from (40) that

$$\frac{\epsilon_2+\epsilon_3}{\cos\beta} + \frac{\epsilon_2}{\cos\alpha} < |OA|,$$

and we can easily see from Figure 10 that if $\vec{z}_n$ tries to follow the path $\xi^{\vec{x}}_\infty$, it must touch OA within distance ϵ_2 from $\xi^{\vec{x}}_\infty$, so the distance between $\vec{z}_n(t_1)$ and the origin would not exceed $\frac{\epsilon_2+\epsilon_3}{\cos\beta}$.

Furthermore, if $\vec{z}_n$ continues and follows the path $\zeta^{\vec{z}_n(t_1)}_\infty$, it would unavoidably miss the segment AB altogether. Thus, as we know that $\vec{z}_n(t_0) \in AB$, we surely conclude that it performs ϵ_2-escape either starting at the time 0, or starting at t_1.

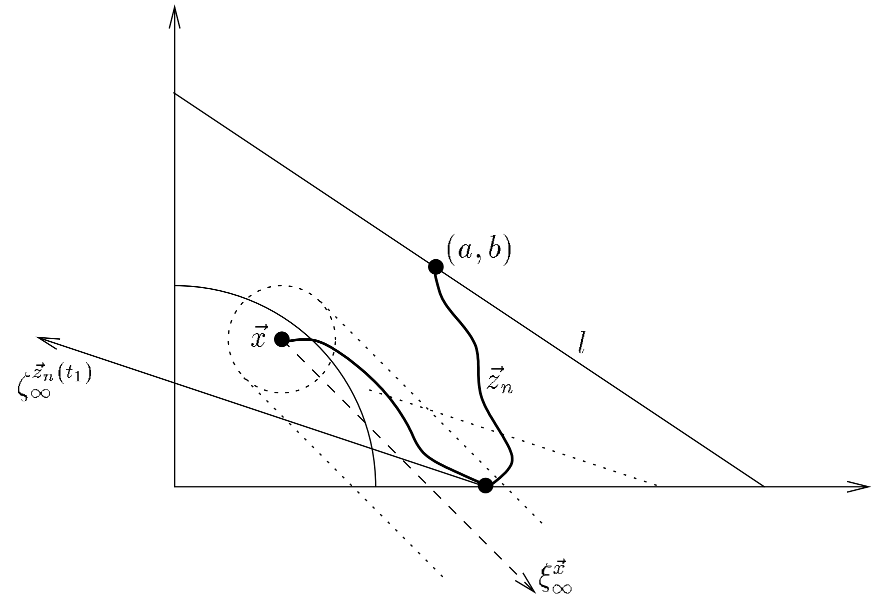


FIGURE 10. Case II: $\vec{z}_n(t)$ touches the boundary once.

III. $\vec{z}_n(t)$ touches $\partial\mathbb{D}$ at least twice before arriving to $(a, b, 0)$. Then we can denote t_1 and t_2 as the moments of $\vec{z}_n$'s last two encounters with the boundary. Obviously, $\vec{z}_n(t_1) \in OB$ and $\vec{z}_n(t_2) \in OA$. We apply reasoning similar to the previous case: if $\vec{z}_n(t)$ follows $\xi_\infty^{\vec{z}_n(t_1)}$ after the moment t_1, it touches OA at the point $\vec{z}_n(t_2)$ which satisfies

$$d(\vec{z}_n(t_2), O) < \frac{|BO| \sin\beta + \epsilon_2}{\cos\beta} \text{ (see Figure 11).}$$

But (39), (40) imply that ϵ_2, ϵ_3 satisfy

$$\frac{|BO| \sin\beta + \epsilon_2}{\cos\beta} + \frac{\epsilon_2}{\cos\alpha} < |OA|,$$

so $\vec{z}_n(t)$ must perform ϵ_2-escape starting from t_2 in order to reach $(a, b) \in l$.

The above discussion was dealing with the case $\vec{z}_n(t_0) \in \mathbb{D}_0$. The case $\vec{z}_n(t_0) \in \mathbb{D}_1$ would lead us to similar results with ϵ_2, ϵ_3 defined appropriately. We conclude therefore, that there exist constants $c_1, c_2 > 0$ which depend only on the system parameters $\lambda_0, \lambda_1, \mu_0, \mu_1$, such that any sample path $\vec{z}_n(t)$ which starts in $c_1\epsilon$-neighborhood of the empty state $(0, 0)$ and escapes outside its $\frac{\epsilon}{2}$-neighborhood, must perform a $c_2\epsilon$-escape in the course of its movement.

Now we wish to find an upper bound for the probability

$$\mathbb{P}\left(\sup_{0\le t\le U} d\big(z_n(t), (0, 0)\big) \ge \frac{\epsilon}{2} \;\middle|\; d\big(z_n(0), (0, 0)\big) < c_1\epsilon\right).$$

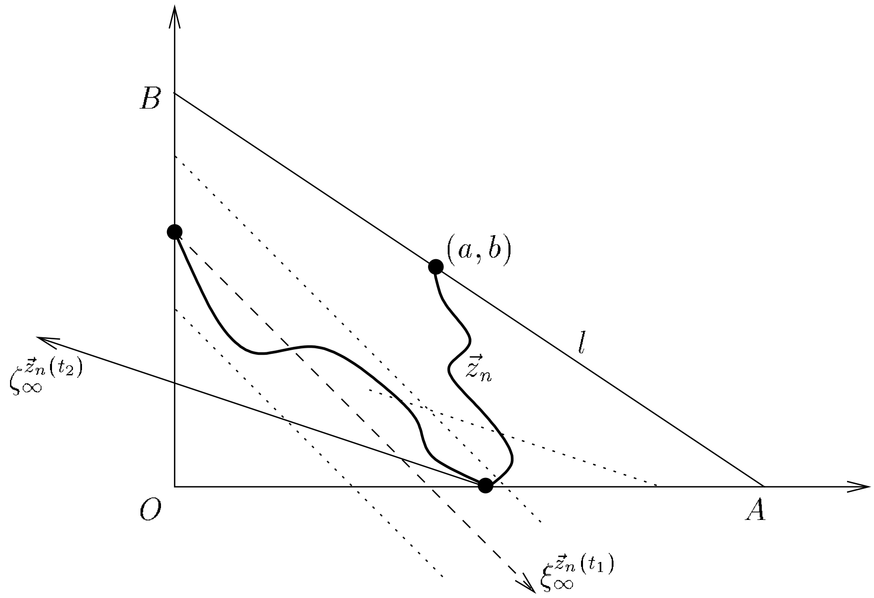


FIGURE 11. Case III: $\vec{z}_n(t)$ touches the boundary twice or more.

Define the following sequence of stopping times:

$$\begin{aligned}
&\rho_0 = 0,\\
&\rho_1 = \inf\{t \geq \rho_0 : \ \vec{z}_n(t) \in \mathbb{D}_0\},\\
&\rho_2 = \inf\{t \geq \rho_1 : \ \vec{z}_n(t) \in \mathbb{D}_1\},\\
&\rho_3 = \inf\{t \geq \rho_2 : \ \vec{z}_n(t) \in \mathbb{D}_0\},\\
&\ \vdots
\end{aligned}$$

Note that $\vec{z}_n(\rho_k)$ lays on the x-axis for odd k, and on the y-axis for even k.

As we saw earlier, for any $\vec{x}$ such that $d(\vec{x}, (0,0)) < c_1\epsilon$

$$\begin{aligned}
(41)\quad &\mathbb{P}_{\vec{x}}\Big(\sup_{0\leq t\leq U} d\big(z_n(t),(0,0)\big) \geq \frac{\epsilon}{2}\Big)\\
&\leq \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape}\Big)\\
&= \sum_{k=0}^{\infty} \mathbb{P}_{\vec{x}}\left(\begin{array}{c} \vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k \\ \cap\ \vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \end{array}\right).
\end{aligned}$$

By the strong Markov property, for any even k

$$\text{(42)}\quad \mathbb{P}_{\vec{x}}\begin{pmatrix} \vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k \\ \cap\ \vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \\ \cap\ \vec{z}_n(\rho_k) = (0,p) \end{pmatrix}$$

$$= \mathbb{P}\Big(\vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \ \Big|\ \vec{z}_n(\rho_k) = (0,p)\Big)$$

$$\times\, \mathbb{P}_{\vec{x}}\begin{pmatrix} \vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k \\ \cap\ \vec{z}_n(\rho_k) = (0,p) \end{pmatrix}.$$

As we defined $c_2\epsilon$-escape, the probability for $\vec{z}_n(t)$ to perform it equals the probability of ξ_n to perform $c_2\epsilon$-escape starting at the time ρ_k from the initial state $\vec{z}_n(\rho_k)$. Also, $\rho_{k+1} - \rho_k$ is obviously smaller than U, and thus

$$\mathbb{P}\Big(\vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \ \Big|\ \vec{z}_n(\rho_k) = (0,p)\Big) \le \mathcal{Q}(U, c_2\epsilon, n).$$

Therefore, (42) now implies

$$\mathbb{P}_{\vec{x}}\begin{pmatrix} \vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k \\ \cap\ \vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \end{pmatrix}$$

$$\le \mathcal{Q}(U, c_2\epsilon, n) \times \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k\Big).$$

In the similar fashion, for odd k

$$\mathbb{P}_{\vec{x}}\begin{pmatrix} \vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k \\ \cap\ \vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \end{pmatrix}$$

$$\le \mathcal{P}(U, c_2\epsilon, n) \times \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k\Big),$$

and we can unify these bounds as

$$\text{(43)}\quad \mathbb{P}_{\vec{x}}\begin{pmatrix} \vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k \\ \cap\ \vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \end{pmatrix}$$

$$\le \big(\mathcal{P}(U, c_2\epsilon, n) + \mathcal{Q}(U, c_2\epsilon, n)\big)$$

$$\times\, \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k\Big), \qquad \forall k \ge 0.$$

We notice a simple fact, that $\vec{z}_n(t)$ must perform at least one jump in the direction $(1,0)$ or $(0,1)$ between two consecutive stopping times

ρ_k and ρ_{k+1}. Therefore

$$
\begin{aligned}
(44)\quad \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k\Big) \\
&\leq \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ changes pane at least } k \text{ times until time } U\Big) \\
&\leq \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ jumps at least } k \text{ times until time } U\Big) \\
&\leq \mathbb{P}\big(N(U) \geq k\big),
\end{aligned}
$$

where $N(t)$ is a Poisson process with intensity $\max\{\lambda_0, \lambda_1\}$.

It now follows that

$$
\begin{aligned}
&\mathbb{P}_{\vec{x}}\left(\sup_{0\leq t\leq U} d\big(z_n(t),(0,0)\big) \geq \frac{\epsilon}{2}\right) \\
&\quad \overset{(41)}{\leq} \sum_{k=0}^{\infty} \mathbb{P}_{\vec{x}}\left(\begin{array}{c} \vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k \\ \cap\ \vec{z}_n(t) \text{ performs } c_2\epsilon\text{-escape starting at } \rho_k \end{array}\right) \\
&\quad \overset{(43)}{\leq} \big(\mathcal{P}(U, c_2\epsilon, n) + \mathcal{Q}(U, c_2\epsilon, n)\big) \\
&\qquad \times \sum_{k=0}^{\infty} \mathbb{P}_{\vec{x}}\Big(\vec{z}_n(t) \text{ doesn't perform } c_2\epsilon\text{-escape until } \rho_k\Big) \\
&\quad \overset{(44)}{\leq} \big(\mathcal{P}(U, c_2\epsilon, n) + \mathcal{Q}(U, c_2\epsilon, n)\big) \\
&\qquad \times \sum_{k=0}^{\infty} \mathbb{P}\big(N(U) \geq k\big).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{P}\big(N(U) \geq k\big) &= \sum_{k=0}^{\infty} (k+1)\mathbb{P}\big(N(U) = k\big) \\
&= 1 + \sum_{k=0}^{\infty} k\mathbb{P}\big(N(U) = k\big) \\
&= 1 + EN(U) \\
&= 1 + U\max\{\lambda_0, \lambda_1\},
\end{aligned}
$$

and thus for any $\vec{x}$ satisfying $d\big(\vec{x},(0,0)\big) < c_1\epsilon$

$$
\begin{aligned}
(45)\quad &\mathbb{P}_{\vec{x}}\left(\sup_{0\leq t\leq U} d\big(z_n(t),(0,0)\big) \geq \frac{\epsilon}{2}\right) \\
&\qquad \leq \big(\mathcal{P}(U, c_2\epsilon, n) + \mathcal{Q}(U, c_2\epsilon, n)\big)\big(1 + U\max\{\lambda_0, \lambda_1\}\big).
\end{aligned}
$$

$$
\begin{aligned}
(46)\quad &\mathbb{P}\left(\sup_{0\leq t\leq U} d\big(z_n(t),(0,0)\big) \geq \frac{\epsilon}{2} \;\middle|\; d\big(z_n(0),(0,0)\big) < c_1\epsilon\right) \\
&\qquad \leq \big(\mathcal{P}(U, c_2\epsilon, n) + \mathcal{Q}(U, c_2\epsilon, n)\big)\big(1 + U\max\{\lambda_0, \lambda_1\}\big).
\end{aligned}
$$

3.9. **Finalizing the proof.** Recall that we have shown by now that $\vec{z}_n(t)$ stays sufficiently close to $\vec{z}_\infty(t)$ until it arrives to a neighborhood of zero, and that $\vec{z}_n(t)$ stays sufficiently close to zero once it reaches it.

Now we can turn to our ultimate goal of providing an upper bound for

$$\mathbb{P}_{\vec{a}}\left(\sup_{t\le T} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \epsilon\right).$$

We shall perform some preparatory work. Recall that we can choose c_1 in (45) as small as we wish, so we require

$$c_1 < \frac{1}{4}.$$

Let also

$$\delta = c_1\epsilon\min\left\{\frac{\sin\beta}{2},\frac{\cos\alpha}{2}\right\}.$$

The choice of δ was made in such a manner that the δ-neighborhoods of Y_{k+1} and X_{k+1} (see Figure 6 for reference) lay entirely inside the $c_1\epsilon$-neighborhood of the origin, namely satisfying

$$\begin{aligned}\big\{\vec{x}\in\mathbb{D}:\ d(\vec{x},X_{k+1})<\delta\big\}\cup\big\{\vec{x}\in\mathbb{D}:\ d(\vec{x},Y_{k+1})<\delta\big\}&\\ \subseteq\big\{\vec{x}\in\mathbb{D}:\ d(\vec{x},(0,0))<c_1\epsilon\big\}&.\end{aligned}$$

That says, if $\vec{z}_n(t)$ follows the most probable path $\vec{z}_\infty(t)$ closely enough, and ultimately arrives to the vicinity of the point Y_{k+1} (or X_{k+1}), then it should find itself by than resting sufficiently close to the origin. For the purpose of this discussion, "sufficiently close" means the ability to apply the results of paragraph 3.8.

Recall the definition of τ_k in paragraph 3.7. We assert that

$$\begin{aligned}\left\{\vec{z}_n:\ \sup_{t\le\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)<\delta,\ \sup_{\tau_k\le t\le T} d\big(\vec{z}_n(t),(0,0)\big)<\frac{\epsilon}{2}\right\}&\\ \subseteq\left\{\vec{z}_n:\ \sup_{t\le T} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)<\epsilon\right\}&.\end{aligned}\tag{47}$$

Indeed, we know that $\delta<\epsilon$ and thus

$$\forall t\le\tau_k\quad \sup_{t\le\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big)<\delta<\epsilon.$$

On the other hand, note that for any sample path $\vec{z}_n$

$$\begin{aligned}&d\big(\vec{z}_\infty(\tau_k(\vec{z}_n)),(0,0)\big)\\ &\qquad\le d\big((0,0),Y_{k+1}\big)+d\big(Y_{k+1},\vec{z}_n(\tau_k)\big)+d\big(\vec{z}_n(\tau_k),\vec{z}_\infty(\tau_k(\vec{z}_n))\big)\\ &\qquad< c_1\epsilon+\delta+\delta<\frac{\epsilon}{2}.\end{aligned}$$

Therefore for any $t \geq \tau_k$

$$\begin{aligned} d\big(\vec{z}_\infty(t),(0,0)\big) &< \frac{\epsilon}{2} \qquad \text{(see Figure 6)} \\ d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) &\leq d\big(\vec{z}_n(t),(0,0)\big) + d\big((0,0),\vec{z}_\infty(t)\big) \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon. \end{aligned}$$

and this proves the assertion (47).

The most immediate consequence of (47) is the following relation:

$$\begin{aligned} (48)\quad & \mathbb{P}_{\vec{a}}\bigg(\sup_{t\leq T} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \epsilon\bigg) \\ &\geq \mathbb{P}_{\vec{a}}\bigg(\sup_{t\leq \tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \ \cap \sup_{\tau_k\leq t\leq T} d\big(\vec{z}_n(t),(0,0)\big) < \frac{\epsilon}{2}\bigg). \end{aligned}$$

For any $\vec{x}$ satisfying $d\big(\vec{x},(0,0)\big) < c_1\epsilon$ we can state using the strong Markov property:

$$\begin{aligned} (49)\quad & \mathbb{P}_{\vec{a}}\left(\begin{array}{c} \sup_{t\leq\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \\ \cap \sup_{\tau_k\leq t\leq T} d\big(\vec{z}_n(t),(0,0)\big) < \frac{\epsilon}{2} \ \cap \ \vec{z}_n(\tau_k)=\vec{x} \end{array}\right) \\ &= \mathbb{P}\bigg(\sup_{\tau_k\leq t\leq T} d\big(\vec{z}_n(t),(0,0)\big) < \frac{\epsilon}{2} \;\bigg|\; \vec{z}_n(\tau_k)=\vec{x}\bigg) \\ &\quad\times \mathbb{P}_{\vec{a}}\bigg(\sup_{t\leq\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \ \cap \ \vec{z}_n(\tau_k)=\vec{x}\bigg). \end{aligned}$$

Denote $\tilde{z}_n(t) = \vec{z}_n(\tau_k+t)$. Obviously, $\tilde{z}_n$ is a jump Markov process with the same generator as $\vec{z}_n$, and thus

$$\begin{aligned} & \mathbb{P}\bigg(\sup_{\tau_k\leq t\leq T} d\big(\vec{z}_n(t),(0,0)\big) < \frac{\epsilon}{2} \;\bigg|\; \vec{z}_n(\tau_k)=\vec{x}\bigg) \\ &\geq \mathbb{P}\bigg(\sup_{\tau_k\leq t\leq T+\tau_k} d\big(\vec{z}_n(t),(0,0)\big) < \frac{\epsilon}{2} \;\bigg|\; \vec{z}_n(\tau_k)=\vec{x}\bigg) \\ &= \mathbb{P}_{\vec{x}}\bigg(\sup_{t\leq T} d\big(\tilde{z}_n(t),(0,0)\big) < \frac{\epsilon}{2}\bigg) \\ &\overset{(45)}{\geq} 1-\big(\mathcal{P}(T,c_2\epsilon,n)+\mathcal{Q}(T,c_2\epsilon,n)\big)\big(1+T\max\{\lambda_0,\lambda_1\}\big). \end{aligned}$$

By summing (49) over $\vec{x}$ we further obtain

$$
\begin{aligned}
(50)\quad & \mathbb{P}_{\vec{a}}\left(\sup_{t\le\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta \,\cap \sup_{\tau_k\le t\le T} d\big(\vec{z}_n(t),(0,0)\big) < \frac{\epsilon}{2}\right)\\
&\ge \mathbb{P}_{\vec{a}}\left(\sup_{t\le\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta\right)\\
&\quad \times \Big(1-\big(\mathcal{P}(T,c_2\epsilon,n)+\mathcal{Q}(T,c_2\epsilon,n)\big)\big(1+T\max\{\lambda_0,\lambda_1\}\big)\Big).
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\mathbb{P}_{\vec{a}}\left(\sup_{t\le T} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \epsilon\right)\\
&\overset{(48),(50)}{\ge} \Big(1-\big(\mathcal{P}(T,c_2\epsilon,n)+\mathcal{Q}(T,c_2\epsilon,n)\big)\big(1+T\max\{\lambda_0,\lambda_1\}\big)\Big)\\
&\qquad \times \mathbb{P}_{\vec{a}}\left(\sup_{t\le\tau_k} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) < \delta\right)\\
&\overset{(37)}{\ge} \Big(1-\big(\mathcal{P}(T,c_2\epsilon,n)+\mathcal{Q}(T,c_2\epsilon,n)\big)\big(1+T\max\{\lambda_0,\lambda_1\}\big)\Big)\\
&\qquad \times \Big(1-\mathcal{P}(\widetilde{T}_0,\delta,n)-2\mathcal{P}(T,\frac{\delta_1}{2},n)-2\mathcal{Q}(T,\frac{\delta_0}{2},n)\Big).
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}_{\vec{a}}\left(\sup_{t\le T} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) \ge \epsilon\right)\\
&\qquad\qquad \le \big(\mathcal{P}(T,c_2\epsilon,n)+\mathcal{Q}(T,c_2\epsilon,n)\big)\big(1+T\max\{\lambda_0,\lambda_1\}\big)\\
&\qquad\qquad\quad +\mathcal{P}(T,\delta,n)+2\mathcal{P}(T,\frac{\delta_1}{2},n)+2\mathcal{Q}(T,\frac{\delta_0}{2},n).
\end{aligned}
$$

The right-hand side of the last expression is a finite sum of probabilities of the form (20). As ζ and ξ are free motions with constant jump rates in each direction, these probabilities can be bounded in the terms of Kurtz theorem:

$$
\begin{aligned}
\mathcal{P}(T,\epsilon,n) &\le A_1 e^{-nA_2(\epsilon)},\\
\mathcal{Q}(T,\epsilon,n) &\le B_1 e^{-nB_2(\epsilon)}
\end{aligned}
$$

for some constants A_1, B_1 and functions $A_2(\epsilon)$, $B_2(\epsilon)$ satisfying (15). Therefore Lemma 27 ensures now that

$$
\mathbb{P}_{\vec{a}}\left(\sup_{t\le T} d\big(\vec{z}_n(t),\vec{z}_\infty(t)\big) \ge \epsilon\right) \le C_1 e^{-nC_2(\epsilon)}
$$

for some C_1, $C_2(\epsilon)$ as required in the statement of the theorem.

4. The rate function

It was stressed in Chapter 2, that our model exhibits, at large, two major modes of behavior, which correspond to the motion $\vec{z}$ being on each of the two panes. In this chapter we are going to address the question of cost for a given path $\vec{r}$ over $\mathbb{D}$. This requires us to build a *rate function* which would assign such cost to any path $\vec{r}$. We know, that at the intuitive level, the rate function somehow reflects the probability of a scaled motion $\vec{z}_n$ to stay near $\vec{r}$. Thus, the purpose of this chapter would be to introduce the proper rate function, which will later be shown to satisfy the Large Deviations Principle.

Judging on the general theory, we guess that the property of cost may bear some local meaning, depending whether $\vec{r}$ is located on either of the panes, or on the boundary at any given moment.

For the purpose of further discussion we wish to introduce the notation for the set of paths over $\mathbb{D}$.

Definition 6. *Consider the set of all paths $\vec{r}(t)$ which travel on $\mathbb{D}$ during the time $t \in [0, T]$. This set, equipped with the* sup *metrics, forms a metric space, and will be denoted as $\mathcal{D}^2[0, T]$.*

Definition 7. *For any fixed $\vec{x} \in \mathbb{D}$, the subset of $\mathcal{D}^2[0, T]$ which consists of paths beginning at $\vec{x}$ will be denoted as $\mathcal{D}^2_{\vec{x}}[0, T]$.*

Remark 8. The reader surely noticed, that we equipped $\mathcal{D}^2[0, T]$ with *sup* metrics, contrary to the usual practice of using the Skorohod metrics in this sort of models. The explanation, why it doesn't matter, appears later in Remark 11. Throughout this chapter the topologic properties of $\mathcal{D}^2[0, T]$ will be used only in Proposition 13, and even that result holds equivalently for both *sup* and Skorohod metrics.

4.1. Structure of a path on $\mathbb{D}$.

In this section we wish to discuss the structure of a given path $\vec{r}$. This requires an insight into the way $\vec{r}$ travels over $\mathbb{D}$, with emphasis on how it moves from pane to pane, how it visits the boundary and the empty state, and so forth.

When discussing various paths over $\mathbb{D}$, it is essential to distinguish between two core situations. We ask for each path $\vec{r}$, whether some random motion can follow it with positive probability. As one may guess, the paths for which the answer is negative, can be ignored as they are largely irrelevant to the probabilistic discussion.

In Definition 9 below we attempt to introduce the notion of *feasibility* which aims to address the question of such relevance, but let us first present some examples which will hopefully clarify our intentions.

As a first example, consider a path which starts at a point $(1, 1, 0)$ and moves straight towards the origin. If we consider a motion which keeps close to this path, we ultimately come to the conclusion that as the time passes by, its value decreases over both coordinates x and y, thus indicating that the service occurs in both queues, without having our motion touch the boundary. This situation is, of course, forbidden by the terms of our model, and therefore the aforementioned path is not feasible.

Another situation involves a point of discontinuity of $\vec{r}$. Whenever $\vec{r}$ is not continuous, we can find a positive ϵ small enough, such that the ϵ-neighbourhood of $\vec{r}$ would also contain a gap in the sense that it contains no other continuous paths on the same domain of time as $\vec{r}$. In this case a desire to keep some n-scaled motion $\vec{z}_n$ in an ϵ-neighbourhood of $\vec{r}$ will force us to choose a relatively large n. But then each single step of $\vec{z}_n$ would be just $\frac{1}{n}$ long, and in order to stay just ϵ away from $\vec{r}$, $\vec{z}_n$ would have to perform a larger step right at the moment of discontinuity of $\vec{r}$. This means that $\vec{z}_n$ incorporates two or more arrival/service events which occur simultaneously, and thus the probability of $\vec{z}_n$ must be zero. Once again we see that such a path must not be feasible. The conclusion of this paragraph is formalized in Proposition 5.

On the contrary, the most probable path discussed in Chapter 3, has the property that for a scaled motion $\vec{z}_n$ staying along it is quite a likely event. This property speaks in favour of its alleged feasibility.

Having seen all these examples, we now proceed to the rigorous definition of feasibility.

Definition 9. *A path $\vec{r} \in \mathcal{D}^2[0,T]$ is feasible, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$*

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} d\big(\vec{z}_n(t), \vec{r}(t)\big) < \epsilon \right) > 0. \tag{51}$$

Proposition 5. *A feasible path is continuous.*

Proof. Consider a feasible path $\vec{r}$. Assume to the contrary, that $\vec{r}$ is not continuous, i.e. there exist $s \in [0,T]$ and a sequence $\{s_m\}_{m=1}^{\infty}$ converging to s, such that

$$\lim_{m \to \infty} \vec{r}(s_m) = \vec{x} \neq \vec{r}(s).$$

Choose some positive ϵ such that

$$\epsilon < \frac{1}{4} d\big(\vec{r}(s), \vec{x}\big).$$

Let $M \in \mathbb{N}$ such that for any $m > M$

$$d\big(\vec{r}(s_m), \vec{x}\big) < \epsilon.$$

Let also $n \in \mathbb{N}$, and consider a sample path $\vec{z}_n$ which satisfies

$$\sup_{0 \leq t \leq T} d\big(\vec{z}_n(t), \vec{r}(t)\big) < \epsilon.$$

Then for any $m > M$

$$\begin{aligned} d\big(\vec{z}_n(s_m), \vec{z}_n(s)\big) &\geq d\big(\vec{x}, \vec{r}(s)\big) - \Big(d\big(\vec{z}_n(s), \vec{r}(s)\big) \\ &\quad + d\big(\vec{z}_n(s_m), \vec{r}(s_m)\big) + d\big(\vec{r}(s_m), \vec{x}\big)\Big) \\ &\geq d\big(\vec{x}, \vec{r}(s)\big) - (\epsilon + \epsilon + \epsilon) \\ &\geq \frac{1}{4} d\big(\vec{x}, \vec{r}(s)\big) > 0. \end{aligned}$$

Since $\{s_m\}$ converges to s, we unavoidably conclude, that $\vec{z}_n$ is not continuous in s. Therefore, $\vec{z}_n$ performs a jump at the moment s. But since $\vec{z}_n$ is comprised of Poisson processes, its probability to jump at a specific moment is zero. Thus for any n,

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} d\big(\vec{z}_n(t), \vec{r}(t)\big) < \epsilon\right) \leq \mathbb{P}(\vec{z}_n \text{ jumps at the moment } s) = 0,$$

contrary to (51). This contradiction shows that the initial assumption was wrong, and thus $\vec{r}$ must be continuous. $\square$

Now we turn to determine the structure of a feasible path running over $\mathbb{D}$. This would allow us to better understand the dynamics of a random walk which runs near that path.

Recall, that we have defined our space $\mathbb{D}$ roughly as a set of two quadrant layers with glued boundaries, i.e. $\mathbb{D} \approx \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z}_2$, where the boundary points $(x, 0, 0)$ and $(x, 0, 1)$ are viewed as the same (and similarly for the y-boundary).

As a matter of convenience, we shall identify several distinct areas of $\mathbb{D}$. Denote

$$\begin{aligned} D_0 &= \{(x, y, 0) \in \mathbb{D} \mid x, y > 0\}, \\ D_1 &= \{(x, y, 1) \in \mathbb{D} \mid x, y > 0\}, \\ \partial_x D &= \{(x, 0) \in \mathbb{D} \mid x > 0\} \\ \partial_y D &= \{(0, y) \in \mathbb{D} \mid y > 0\} \\ \mathbf{0} &= \{(0, 0)\}. \end{aligned}$$

Note that $\partial_x D$ and $\partial_y D$ are basically the x-axis and the y-axis, without the origin which deserves a set of its own.

Consider a feasible path $\vec{r}$. At the moment, we don't require $\vec{r}$ to be absolutely continuous. Its continuity, which follows from the feasibility, will satisfy us for the purpose of the coming discussion.

Our next duty will be the classification of various time intervals within $[0, T]$, which are characterized by different modes of behavior of $\vec{r}$.

More precisely, we consider a scaled motion $\vec{z}_n$ for some n, which lies close to $\vec{r}$, and its behavior is the criterion we actually apply.

First let us consider the time that $\vec{r}$ spends on D_0. We denote it as follows:

$$\tilde{A}_0(\vec{r}) = \vec{r}^{\,-1}(D_0) \cap (0, T). \tag{52a}$$

Note that since $\vec{r}$ is continuous, and D_0 is an open set, $\tilde{A}_0(\vec{r})$ is an open set too (see Remark 10 below). This domain has the nice property that if we take an interval $[s_1, s_2] \subset \tilde{A}_0(\vec{r})$, any $\vec{z}_n$ which is close enough to $\vec{r}$, would behave on $[s_1, s_2]$ just like a free motion.

Remark 10. While the points $\vec{r}(0)$ and $\vec{r}(T)$ may reside on D_0, we explicitly remove them from consideration when building $\tilde{A}_0(\vec{r})$ (and later, $A_0(\vec{r})$). That ensures that $\tilde{A}_0(\vec{r})$ is an open set in $\mathbb{R}$, with all the nice properties stemming from this.

We further consider the situation when $\vec{r}$ walks over D_0, visits the boundary $\partial_y D$ and leaves back to D_0 (see Figure 12). Now if we take an interval $[s_1, s_2]$ on which it all happened, it should be clear that a scaled motion $\vec{z}_n$ near $\vec{r}$ must not touch the boundary $\partial_y D$, because then by definition of the model, it would have to move to D_1.

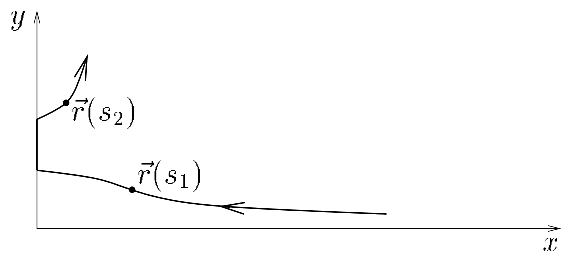


FIGURE 12. The rationale behind $A_0(\vec{r})$

For this reason we can state that such $\vec{z}_n$ can be coupled with the free motion ζ_n (see Remark 4).

Another special case arises when $\vec{r}$ starts at the boundary $\partial_x D$ (i.e. $\vec{r}(0) \in \partial_x D$), and stays on it for some positive amount of time. This situation corresponds to the scenario which begins with an empty second queue, and has no arrivals occur in that queue for some period of time starting with $t = 0$. It is clear that in this situation $\vec{z}_n$ would also behave like ζ_n, and we can further expand $\tilde{A}_0(\vec{r})$ with some interval starting at 0 and ending as soon as $\vec{r}$ first touches $\partial_y D$ or $\mathbf{0}$. Formally speaking, there is an interval

$$(0, t_0), \text{ s.t. } t_0 = \inf\bigl\{t \in [0, T] \mid r_x(t) = 0\bigr\}.$$

Therefore we define

$$A_0(\vec{r}) = \tilde{A}_0(\vec{r}) \cup \left\{ t \in (0, T) \;\middle|\; \begin{array}{c} t \text{ lies inside some } [s_1, s_2] \\ \text{s.t. } \vec{r}(s_1), \vec{r}(s_2) \in D_0 \text{ and} \\ \vec{r}([s_1, s_2]) \subseteq D_0 \cup \partial_y D \end{array} \right\} \tag{52b}$$
$$\Bigl[\cup (0, t_0), \text{ when } \vec{r}(0) \in \partial_x D\Bigr]$$

and we expect that $\vec{z}_n$ would behave like ζ_n over the time domain $A_0(\vec{r})$, conditioned on both not touching the boundary $\partial_y D$ in-between.

It is easy to check that $A_0(\vec{r})$ is an open set, provided the set $\tilde{A}_0(\vec{r})$ is open.

In a similar fashion we define

$$\tilde{A}_1(\vec{r}) = \vec{r}^{\,-1}(D_1) \cap (0,T),$$

$$A_1(\vec{r}) = \tilde{A}_1(\vec{r}) \cup \left\{ t \in (0,T) \;\middle|\; \begin{array}{c} t \text{ lies inside some } [s_1, s_2] \\ \text{s. t. } \vec{r}(s_1), \vec{r}(s_2) \in D_1 \text{ and} \\ \vec{r}([s_1,s_2]) \subseteq D_1 \cup \partial_x D \end{array} \right\}$$

$$\left[\begin{array}{ll} \cup\ (0,t_1), & \text{when } \vec{r}(0) \in \partial_y D, \\ & t_1 = \inf\{t \in [0,T] \mid r_y(t) = 0\} \end{array} \right].$$

Now we turn to discuss yet another mode of behavior of $\vec{r}$. Consider the situation where $\vec{r}$ arrives from D_0 to $\partial_y D$, stays on this boundary for some time, and then leaves to D_1 (or, possibly, to $\mathbf{0}$). Our desire to deal with this setting separately emerges from the following observation. Whenever $\vec{r}$ behaves as described above, and $\vec{z}_n$ is some scaled random walk near it, $\vec{z}_n$ obviously arrives to a neighborhood of $\partial_y D$ together with $\vec{r}$, and also leaves together. But as long as $\vec{r}$ stays on $\partial_y D$, $\vec{z}_n$ can first spend some time at D_0, then touch the boundary at some moment, and spend some more time still near $\partial_y D$ and $\vec{r}$, but on D_1. Needless to say that in such case it would be difficult to know in advance at what moment $\vec{z}_n$ is most likely to cross the boundary, and respectively, its probability to stay around $\vec{r}$.

Therefore, we define the collection of sojouring times of $\vec{r}$ on $\partial_y D$ as the complete set of its visits to $\partial_y D$, without intervals already included in $A_0(\vec{r})$ and without momentary visits which will prove to be insignificant for us. Denote

$$A_2(\vec{r}) = \left(\vec{r}^{\,-1}(\partial_y D) \setminus A_0(\vec{r})\right)^o,$$

i. e. the interior of $\vec{r}^{\,-1}(\partial_y D) \setminus A_0(\vec{r})$.

In a similar fashion we denote

$$A_3(\vec{r}) = \left(\vec{r}^{\,-1}(\partial_x D) \setminus A_1(\vec{r})\right)^o.$$

Note that by construction, all the sets $A_0(\vec{r})$, $A_1(\vec{r})$, $A_2(\vec{r})$ and $A_3(\vec{r})$ are disjoint open sets. By the characteristic property of open sets on $\mathbb{R}$, each $A_i(\vec{r})$ can be represented as a countable collection of disjoint open intervals:

$$A_i(\vec{r}) = \bigcup_{k=1}^{\infty} (s_k^i, t_k^i).$$

For our convenience we denote

$$\mathcal{A}_i(\vec{r}) = \{(s_k^i, t_k^i)\}_{k=1}^{\infty},$$

$$\mathcal{A}(\vec{r}) = \mathcal{A}_0(\vec{r}) \cup \mathcal{A}_1(\vec{r}) \cup \mathcal{A}_2(\vec{r}) \cup \mathcal{A}_3(\vec{r}). \tag{53}$$

Now in order to achieve a complete classification of points in $[0,T]$, it remains to define the last two sets.

Let $A_4(\vec{r})$ be the collection of endpoints of the intervals making up either of the sets $\mathcal{A}_0(\vec{r}), \mathcal{A}_1(\vec{r}), \mathcal{A}_2(\vec{r}), \mathcal{A}_3(\vec{r})$, together with $\{0,T\}$; and for whatever remains:

$$A_5(\vec{r}) = [0,T] \setminus \bigcup_{i=0}^{4} A_i(\vec{r}).$$

As a final note on the structure analysis, we bring here a statement regarding $A_5(\vec{r})$. It turns out that whatever moments of time belong to $A_5(\vec{r})$, $\vec{r}$ spends them in the origin.

Proposition 6. *Let $\vec{r} \in \mathcal{D}^2[0,T]$ be a feasible path. Then*

$$\forall t \in A_5(\vec{r}) \quad \vec{r}(t) = (0,0).$$

Moreover, if $\vec{r}$ is absolutely continuous, then

$$\vec{r}\,'(t) = (0,0)$$

almost surely in the sense of the Lebesgue measure in $A_5(\vec{r})$.

Proof: Let $t \in A_5(\vec{r})$. Clearly, $\vec{r}(t) \notin D_0 \cup D_1$, as otherwise t would belong to either $A_0(\vec{r})$ or $A_1(\vec{r})$. Assume to the contrary that $\vec{r}(t) \neq (0,0)$. Then we can assume without loss of generality that $\vec{r}(t) \in \partial_x D$, i. e. $\vec{r}(t) = (x,0)$ for some $x > 0$.

Since $\vec{r}$ is continuous and t is not an endpoint of $[0,T]$, we can choose some $\epsilon > 0$ such that

$$\forall s \in (t-\epsilon, t+\epsilon) \subset [0,T] \quad \vec{r}(s) \notin \partial_y D \cup \mathbf{0}.$$

$\vec{r}((t-\epsilon,t))$ must also not lie entirely in $D_1 \cup \partial_x D$, or otherwise t would belong either to $A_1(\vec{r})$ or to $A_4(\vec{r})$. Thus there exists $t_1 \in (t-\epsilon, t)$ such that $\vec{r}(t_1) \in D_0$. But now it comes out that during the time $[t_1, t]$ $\vec{r}$ travelled from some point in D_0 to $(x,0)$ without touching the boundary $\partial_y D$, and such a path can't be feasible.

From the contradiction we conclude that $\vec{r}(t) = (0,0)$.

Furthermore, by [Jon93, p. 550], the absolutely continuous path $\vec{r}$ is almost surely differentiable. Assume that $\vec{r}\,'(t)$ exists for some $t \in A_5(\vec{r})$. Then there exists the limit from above

$$r_x'(t) = \lim_{\epsilon \downarrow 0} \frac{r_x(t+\epsilon) - r_x(t)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{r_x(t+\epsilon)}{\epsilon}.$$

Since $r_x(t+\epsilon) \geq 0$ for any $\epsilon > 0$, we conclude that $r_x'(t) \geq 0$.

But on the other hand, there also exists a limit from below:

$$r_x'(t) = \lim_{\epsilon \uparrow 0} \frac{r_x(t+\epsilon) - r_x(t)}{\epsilon} = \lim_{\epsilon \uparrow 0} \frac{r_x(t+\epsilon)}{\epsilon}.$$

Since $r_x(t+\epsilon) \geq 0$ for any $\epsilon < 0$, we conclude that $r_x'(t) \leq 0$ and therefore $r_x'(t) = 0$.

Similarly, $r_y'(t) = 0$ and $\vec{r}\,'(t) = (0,0)$, as required. ■

We now prove one more simple, yet useful statement about $\mathcal{A}(\vec{r})$.

Proposition 7. *Let $\vec{r} \in \mathcal{D}^2[0,T]$ be a feasible path. Let $(u,v) \subseteq [0,T]$ be such that for any $t \in (u,v)$, $\vec{r}(t) \in D_0 \cup \partial_y D$. Then (u,v) is covered by at most three members of $\mathcal{A}(\vec{r})$, except for a finite number of points.*

Proof: We split the proof into two cases.

I. If $\vec{r}((u,v)) \subseteq \partial_y D$, then we can consider some arbitrary point $t_0 \in (u,v)$. If there exist $s_1 < t_0 < s_2$ such that $\vec{r}(s_1), \vec{r}(s_2) \in D_0$ and $\vec{r}([s_1,s_2]) \subseteq D_0 \cup \partial_y D$, then obviously $t_0 \in A_0(\vec{r})$, and so are all the other points in (u,v). Thus, $(u,v) \subseteq A_0(\vec{r})$ and it is covered by a single member of $\mathcal{A}_0(\vec{r})$.

Otherwise, $t_0 \in A_2(\vec{r})$ and again, so are the other points in (u,v), and (u,v) is covered by a single member of $\mathcal{A}_2(\vec{r})$.

II. If $\vec{r}((u,v)) \not\subseteq \partial_y D$, then the set $\{t \in (u,v) \mid \vec{r}(t) \in D_0\}$ is not empty, and we denote

$$
\begin{aligned}
u_1 &= \inf\{t \in (u,v) \mid \vec{r}(t) \in D_0\},\\
u_2 &= \sup\{t \in (u,v) \mid \vec{r}(t) \in D_0\}.
\end{aligned}
$$

Then there exist points located arbitrarily close to u_1, u_2 with images under $\vec{r}$ lying in D_0, and therefore

$$(u_1,u_2) \subseteq A_0(\vec{r}).$$

Thus, (u_1,u_2) is covered by a member of $\mathcal{A}_0(\vec{r})$.

Moreover, obviously

$$
\begin{aligned}
\vec{r}((u,u_1)) &\subseteq \partial_y D,\\
\vec{r}((u_2,v)) &\subseteq \partial_y D,
\end{aligned}
$$

and, as we have already seen, each of (u,u_1) and (u_2,v) is covered by a single interval from $\mathcal{A}_0(\vec{r})$ or $\mathcal{A}_2(\vec{r})$. Therefore, the entire interval (u,v) is covered by at most three members of $\mathcal{A}(\vec{r})$ except for, possibly, the points u_1 and u_2. ■

Lemma 8. *Let $\vec{r} \in \mathcal{D}^2[0,T]$ be a fixed feasible absolutely continuous path, and let $\delta > 0$. Define $\mathcal{B}_\delta$ as the collection of all intervals in $\mathcal{A}(\vec{r}) = \mathcal{A}_0(\vec{r}) \cup \mathcal{A}_1(\vec{r}) \cup \mathcal{A}_2(\vec{r}) \cup \mathcal{A}_3(\vec{r})$, on which $\vec{r}$ travels at least δ away from the empty state:*

$$\mathcal{B}_\delta = \{(u,v) \in \mathcal{A}(\vec{r}) \mid \vec{r}((u,v)) \not\subseteq B_\delta(\mathbf{0})\}\,. \tag{54}$$

Then $\mathcal{B}_\delta$ is finite. Moreover, $m_\delta = |\mathcal{B}_\delta|$ satisfies

$$\lim_{\delta\to 0} m_\delta \cdot \delta = 0. \tag{55}$$

Proof: Let $\delta > 0$. Note that

$$\mathcal{B}_\delta = (\mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_2(\vec{r}))) \cup (\mathcal{B}_\delta \cap (\mathcal{A}_1(\vec{r}) \cup \mathcal{A}_3(\vec{r}))),$$

and so we can assume without loss of generality, that $|\mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_2(\vec{r}))| \geq \frac{m_\delta}{2}$. At this point, we allow m_δ to be infinity, and if this happens, the last statement would simply mean that $\mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_2(\vec{r}))$ is infinite.

Consider four intervals located inside $[0, T]$ in this order:

$$(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4),$$

where each interval belongs to $\mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_2(\vec{r}))$, and $u_1 \neq 0$. From now on we shall neglect the interval $(0, u) \in \mathcal{A}(\vec{r})$, whenever it exists at all. Obviously, such neglection doesn't affect the asymptotic result (55).

If $\vec{r}((u_1, v_4)) \subseteq D_0 \cup \partial_y D$, then by Proposition 7, (u_1, v_4) is covered by at most three members of $\mathcal{A}(\vec{r})$, and this leads to contradiction.

Therefore there exists $t_0 \in (u_1, v_4)$ such that $\vec{r}(t_0) \in D_1 \cup \partial_x D \cup \mathbf{0}$.

If $\vec{r}(t_0) \in D_1$, then there must exist $t_0 < t_1 < v_4$ such that $\vec{r}(t_1) \in \partial_x D \cup \mathbf{0}$, or otherwise $\vec{r}$ would be unable to travel from D_1 to $\partial_y D$ and retain feasibility. Obviously, $t_1 \in [v_1, u_4]$.

Denote

$$t_2 = \inf\{t \in [v_1, u_4] \mid \vec{r}(t) \in \partial_x D \cup \mathbf{0}\}.$$

Since $\partial_x D \cup \mathbf{0}$ is a closed set, $\vec{r}(t_2) \in \partial_x D \cup \mathbf{0}$.

Now we have the following setting:

$$\begin{gathered} \vec{r}(u_1) \in \partial_y D \cup \mathbf{0}, \\ \exists s \in (u_1, v_1) \quad \vec{r}(s) \notin B_\delta(\mathbf{0}), \\ \vec{r}(t_2) \in \partial_x D \cup \mathbf{0}, \end{gathered}$$

and $u_1 < s < t_2$.

By a simple geometric reasoning we obtain

$$d(\vec{r}(u_1), \vec{r}(s)) + d(\vec{r}(s), \vec{r}(t_2)) > \delta. \tag{56}$$

Recall that by Proposition 6, $\vec{r}(A_5(\vec{r})) = (0, 0)$, and therefore by defining t_2 as an infimum, we ensured that (u_1, s) and (s, t_2) are covered by the members of $\mathcal{A}(\vec{r})$ almost surely in the sense of the standard euclidean measure.

Assume to the contrary, that $m_\delta = \infty$. Then we can choose arbitrarily large finite subset

$$\mathcal{B}' \subseteq \mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_2(\vec{r})).$$

We can divide $\mathcal{B}'$ into quadruples of consecutive intervals and obtain the same result (56) for each of the quadruples. Summing up. we obtain that for some finite collection of non-overlapping intervals $\{[s_i, t_i]\}_{i \in I}$,

$$\sum_{i \in I} d(\vec{r}(t_i), \vec{r}(s_i)) > \left[\frac{|\mathcal{B}'|}{4}\right] \cdot \delta \geq \left(\frac{|\mathcal{B}'|}{4} - 1\right) \cdot \delta. \tag{57}$$

The right term of (57) (and therefore also the left) can be arbitrarily large, thus contradicting the absolute continuity of $\vec{r}$.

Now when we know that $m_\delta < \infty$, by substituting $\mathcal{B}' = \mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_2(\vec{r}))$, (57) transforms into

$$\sum_{i \in I} d(\vec{r}(t_i), \vec{r}(s_i)) > \left(\frac{m_\delta}{8} - 1\right) \cdot \delta \tag{58}$$

With this result in mind, we proceed to finalize the proof.

Let $\alpha > 0$. As $\vec{r}$ is absolutely continuous, we can choose $\epsilon > 0$ such that any finite collection $\{(s_i, t_i)\}_{i \in I}$ of non-overlapping intervals with total length below ϵ satisfies

$$\sum_{i \in I} d(\vec{r}(t_i), \vec{r}(s_i)) < \alpha.$$

Denote

$$L = \sum_{(u,v) \in \mathcal{A}(\vec{r})} |v - u|.$$

For any $(u, v) \in \mathcal{A}(\vec{r})$ there exists $\delta > 0$ such that $(u, v) \in \mathcal{B}_\delta$. Therefore we can choose δ_0 in such a manner that

$$\sum_{(u,v) \in \mathcal{B}_{\delta_0}} |v - u| > L - \epsilon.$$

Consider $\delta_0 > \delta > 0$. Obviously, $\mathcal{B}_{\delta_0} \subseteq \mathcal{B}_\delta$.

We enumerate the members of $\mathcal{B}_{\delta_0}$ according to their order on the real line:

$$\mathcal{B}_{\delta_0} = \{(u_i, v_i)\}_{i=1}^{m_{\delta_0}}.$$

Between each two consecutive members (u_i, v_i) and (u_{i+1}, v_{i+1}) of $\mathcal{B}_{\delta_0}$ there are m_δ^i members of $\mathcal{B}_\delta \setminus \mathcal{B}_{\delta_0}$. In addition, there are m_δ^0 members of $\mathcal{B}_\delta \setminus \mathcal{B}_{\delta_0}$ between 0 and u_1, and $m_\delta^{m_{\delta_0}}$ between the last interval of $\mathcal{B}_{\delta_0}$ and T. According to (58), for each i there is some finite collection of non-overlapping intervals $\{[s_j, t_j]\}_{j \in J_i}$, satisfying

$$\sum_{j \in J_i} d(\vec{r}(t_j), \vec{r}(s_j)) > \left(\frac{m_\delta^i}{8} - 1\right) \cdot \delta.$$

Recall, that we chose the intervals (s_j, t_j) in such a manner that they are almost surely covered by $\mathcal{A}(\vec{r})$. Therefore their total length doesn't exceed ϵ.

Taking sum over $i = 0, \ldots, m_{\delta_0}$, we obtain

$$\epsilon > \sum d(\vec{r}(t_j), \vec{r}(s_j)) > \sum_{i=0}^{m_{\delta_0}} \left(\frac{m_\delta^i}{8} - 1\right) \cdot \delta$$

or equivalently,

$$\begin{aligned}\epsilon &> \frac{\delta}{8}\cdot\sum_{i=0}^{m_{\delta_0}} m_\delta^i - \delta(m_{\delta_0}+1)\\ &= \frac{\delta}{8}(m_\delta - m_{\delta_0}) - \delta(m_{\delta_0}+1)\\ &= \frac{\delta\cdot m_\delta}{8} - \delta(\frac{9}{8}m_{\delta_0}+1)\end{aligned}$$

Bringing δ to zero, we obtain

$$\lim_{\delta\to 0}\frac{\delta\cdot m_\delta}{8} < \epsilon$$

for all $\epsilon > 0$. Therefore,

$$\lim_{\delta\to 0}\delta\cdot m_\delta = 0$$

■

4.2. **Definition of the rate function.** Recall that in Section 2.3 we defined the two free motions which comprise the base of our model, and denoted them as ζ and ξ. To remind, ζ described the service mode on D_0 (first queue served), and ξ described the mode on D_1 (second queue served).

The behavior of a free motion has been extensively studied, and appropriate estimations derived. It will be our goal to describe the behavior of the exhaustive service model in terms of ζ and ξ.

In Section 4.1 we discussed the structure of a feasible path on $\mathbb{D}$ and divided any such path $\vec{r}$ into pieces of different nature. Now we need to define the rate function $\mathcal{I}_0^T(\vec{r})$ in a manner that would reflect correctly the bounds set by the Large Deviations Principle. We understand intuitively that the total value of the rate function of $\vec{r}$ over the interval $[0, T]$ should be the sum of partial rate functions over smaller intervals $[u, v]$ which together comprise $[0, T]$ (up to some degree of neglection as we will further see). We rely on our dissection of $[0, T]$ into $\mathcal{A}_i(\vec{r})$'s to provide such collection of smaller intervals. On each of them, $\vec{r}$ has a structure simple enough to allow for an easy definition of $\mathcal{I}$ on it.

Remark. While in Section 4.1 we were primarilly dealing with general feasible paths, in this section we wish to consider only paths which are also absolutely continuous. As we noted earlier, absolutely continuous paths are differentiable Lebesgue–a.e. [Jon93, p. 550].

We now turn to describe each set $\mathcal{A}_i(\vec{r})$ of intervals in terms of the good rate functions I and J related to ζ and ξ respectively. The latter reflect in turn the probability of a scaled random walk $\vec{z}_n$ to stay around $\vec{r}$ during the given interval of time.

Take for example, an interval $(u, v) \in \mathcal{A}_0(\vec{r})$. As far as we are concerned with (u, v), $\vec{r}$ travels on D_0, possibly spends some time on $\partial_y D$, and returns back to D_0 by the time v. One can observe that a random walk $\vec{z}_n$ which stays near $\vec{r}$ during the time $[u, v]$, must not cross $\partial_y D$, or otherwise the setting of the model would force it to run away to D_1. Therefore $\vec{z}_n$ actually behaves like a scaled free motion ζ_n, and it would be natural to define

$$\mathcal{I}_u^v(\vec{r}) = I_u^v(\vec{r}) = \int_u^v l_\zeta(\vec{r}\,'(t))dt, \tag{59a}$$

where the integral definition is valid due to the a.e. differentiability of $\vec{r}$. Similarly, for $(u, v) \in \mathcal{A}_1(\vec{r})$ we define

$$\mathcal{I}_u^v(\vec{r}) = J_u^v(\vec{r}) = \int_u^v l_\xi(\vec{r}\,'(t))dt. \tag{59b}$$

Now we consider an interval of another kind: $(u, v) \in \mathcal{A}_2(\vec{r})$. During the time $[u, v]$, $\vec{r}$ stays on $\partial_y D$, and it can travel back and forth on it. This time we observe that a random walk $\vec{z}_n$ which stays near $\vec{r}$ during the time $[u, v]$, may originate either on D_0 or D_1. In the simpler case of $\vec{z}_n(u) \in D_1$, we observe that $\vec{z}_n$ must stay on D_1, since the definition of the model wouldn't allow it to move to D_0. In this case, the cost of moving on D_1 along $\vec{r}$ is $J_u^v(\vec{r})$. On the other hand, if $\vec{z}_n$ originated on D_0, it has just one chance to cross the boundary $\partial_y D$ during the time $[u, v]$, and then it must continue moving on D_1. If we force the moment of crossing $\partial_y D$ to be some $s \in [u, v]$, the cost of such behavior for $\vec{z}_n$ would be

$$I_u^s(\vec{r}) + J_s^v(\vec{r}).$$

But since the moment s is not forced, and the highest probability for $\vec{z}_n$ to stay around $\vec{r}$ is reflected by the *lowest* cost, we need to alter the above expression and define

$$\begin{aligned} \mathcal{I}_u^v(\vec{r}) &= \min_{s\in[u,v]} \{I_u^s(\vec{r}) + J_s^v(\vec{r})\} \\ &= \min_{s\in[u,v]} \left\{ \int_u^s l_\zeta(\vec{r}\,'(t))dt + \int_s^v l_\xi(\vec{r}\,'(t))dt \right\}. \end{aligned} \tag{59c}$$

Remark. $\mathcal{I}_u^t(\vec{r})$ is *not local*, in the sense that a change in $\vec{r}$ at $t_1 < t$ such that $\mathcal{I}_{t_1}^t(\vec{r})$ remains *unchanged*, may still change $\mathcal{I}_u^t(\vec{r})$. See (60b) and (61) below.

Note that whenever $I_u^v(\vec{r})$ and $J_u^v(\vec{r})$ are finite, the minimum is attained because the integral is a continuous operator and s varies over compact set $[u, v]$. It turns out that in fact the minimum of (59c) is attained regardless of the finiteness of I and J, as we show below.

Proposition 9. *Let $(u, v) \in \mathcal{A}_2(\vec{r})$. Then there exists $s_0 \in [u, v]$ such that*

$$\min_{s\in[u,v]} \{I_u^s(\vec{r}) + J_s^v(\vec{r})\} = I_u^{s_0}(\vec{r}) + J_{s_0}^v(\vec{r}).$$

Proof. Let

$$\tilde{u} = \sup\{s \in [u,v] \mid I_u^s(\vec{r}) < \infty\},$$
$$\tilde{v} = \inf\{s \in [u,v] \mid J_s^v(\vec{r}) < \infty\}.$$

If $\tilde{u} < \tilde{v}$, then any $s \in [u,v]$ satisfies either $s < \tilde{v}$ or $s > \tilde{u}$, and therefore at least one of the values $I_u^s(\vec{r})$, $J_s^v(\vec{r})$ is infinite, and by (59c),

$$\mathcal{I}_u^v(\vec{r}) = \infty.$$

The minimum of (59c) is thus attained at any point $s \in [u,v]$.

Otherwise, the expression $I_u^s(\vec{r}) + J_s^v(\vec{r})$ is finite for any $s \in [\tilde{v}, \tilde{u}]$, and is infinite for any $s \in [u,v] \setminus [\tilde{v}, \tilde{u}]$. Bt in this case the expression $I_u^s(\vec{r}) + J_s^v(\vec{r})$ is a finite integral operator over the range $s \in [\tilde{v}, \tilde{u}]$. Therefore it is continuous and attains a minimum over the compact set $[\tilde{v}, \tilde{u}]$, as required. □

Despite its apparent naïvety, the definition (59c) successfully addresses all complications that may emerge. For example, if $\vec{r}$ moves forth, back and forth as described in Figure 13, any $\vec{z}_n$ that stays around $\vec{r}$ with positive probability, must travel during $[s_1, s_2]$ over D_1, because during that time the service occurs in the second queue. This effectively restricts the crossing time s to $[u, s_1]$. But for any $s > s_1$, it can be easily seen that $I_u^s(\vec{r}) = \infty$, and the minimum operator in (59b) effectively removes all such times from consideration.

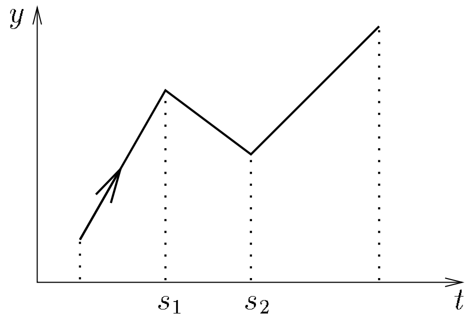


FIGURE 13. Graph of $\vec{r}$ changing direction several times while on $\partial_y D$.

Back to the definition, we now similarly treat the collection $\mathcal{A}_3(\vec{r})$ and define for $(u,v) \in \mathcal{A}_3(\vec{r})$

$$\begin{aligned} \mathcal{I}_u^v(\vec{r}) &= \min_{s \in [u,v]} \{J_u^s(\vec{r}) + I_s^v(\vec{r})\} = \\ &= \min_{s \in [u,v]} \left\{ \int_u^s l_\xi(\vec{r}\,'(t))dt + \int_s^v l_\zeta(\vec{r}\,'(t))dt \right\}. \end{aligned} \tag{59d}$$

The sets $A_4(\vec{r})$ and $A_5(\vec{r})$ are discarded, because $A_4(\vec{r})$ is countable, and $A_5(\vec{r})$ denotes the sojouring time of $\mathbf{0}$, which comes free of charge as it is the most probable behavior.

Remark. While we don't make any rigorous statements at this moment, it is clear that the path $\vec{r}$ can stay at the empty state "free of charge" only on condition of stability. The zero cost of $A_5(\vec{r})$ becomes evident, once we recall the probability of the scaled motion $\vec{z}_n$ to stay near the origin, as stated in Theorem 3.

As promised, we now sum up the pieces of (59) to obtain

$$\mathcal{I}_0^T(\vec{r}) = \sum_{(u,v)\in\mathcal{A}(\vec{r})} \mathcal{I}_u^v(\vec{r}). \tag{60a}$$

For all $\vec{r}$ which are either not absolutely continuous or not feasible, we define

$$\mathcal{I}_0^T(\vec{r}) = \infty. \tag{60b}$$

Remark. It follows immediately from the definition, that any path with finite rate function value must be feasible and absolutely continuous, and thus a.e. differentiable.

In order to simplify some further calculations, we also define the "local" rate function $l_{\vec{r}}$. Obviously, $l_{\vec{r}}$ is not a local rate function in the true sense of the word, because it doesn't depend solely on $\vec{r}(t)$ and $\vec{r}\,'(t)$. Any attempt to define a true local rate function for $\mathcal{I}$ would surely fail, as it loses the required information about the past and the future of $\vec{r}$, which must be retained.

Therefore, we construct the following definition:

$$l_{\vec{r}}(t) = \begin{cases} l_\varsigma(\vec{r}\,'(t)), & t \in A_0(\vec{r}) \\ l_\xi(\vec{r}\,'(t)), & t \in A_1(\vec{r}) \\ l_\varsigma(\vec{r}\,'(t)), & t \in (u,v) \in \mathcal{A}_2(\vec{r}) \text{ and } u < t < s, \\ & \qquad \text{where } s \text{ is the minimum at (59b)} \\ l_\xi(\vec{r}\,'(t)), & t \in (u,v) \in \mathcal{A}_2(\vec{r}) \text{ and } s < t < v, \\ & \qquad \text{where } s \text{ is the minimum at (59b)} \\ l_\xi(\vec{r}\,'(t)), & t \in (u,v) \in \mathcal{A}_3(\vec{r}) \text{ and } u < t < s, \\ & \qquad \text{where } s \text{ is the minimum at (59c)} \\ l_\varsigma(\vec{r}\,'(t)), & t \in (u,v) \in \mathcal{A}_3(\vec{r}) \text{ and } s < t < v, \\ & \qquad \text{where } s \text{ is the minimum at (59c)} \\ 0, & t \in A_4(\vec{r}) \cup A_5(\vec{r}). \end{cases} \tag{61}$$

The most important property of $l_{\vec{r}}$ is that

$$\forall (u,v) \in \mathcal{A}(\vec{r}) \quad \mathcal{I}_u^v(\vec{r}) = \int_u^v l_{\vec{r}}(t)dt$$

$$\mathcal{I}_0^T(\vec{r}) = \int_0^T l_{\vec{r}}(t)dt.$$

This allows us to extend the definition of $\mathcal{I}$ to any $u, v \in [0,T]$:

$$\mathcal{I}_u^v(\vec{r}) = \int_u^v l_{\vec{r}}(t)dt. \tag{62}$$

4.3. **Properties of the rate function.** In this section we shall show that the function $\mathcal{I}_0^T$ defined in Section 4.2 is indeed a rate function on the metric space $\mathbb{D}$, in the sense of definition in [DZ93, p. 4]. Moreover, we shall establish some basic properties of $\mathcal{I}_0^T$, which will serve us later.

The first property says that the level sets of $\mathcal{I}_0^T$ are uniformly absolutely continuous collections of paths. As a point of interest, its proof demonstrates an idea of path cost comparison using an arbitrarily built free motion. A simple lower bound (64) for $\mathcal{I}_0^T(\vec{r})$, expressed in terms of that motion, is provided as a corollary.

Proposition 10. (uniform absolute continuity)

Let $\mathcal{I}_0^T(\vec{r}) \le K$, and fix $\epsilon > 0$. Then there exists $\delta > 0$, uniform for all $\vec{r}$, such that for any collection of non-overlapping intervals in $[0, T]$

$$\{[s_n, t_n],\ n = 1, \dots, N\} \textit{ with } \sum_{n=1}^{N} t_n - s_n < \delta$$

it holds

$$\sum_{n=1}^{N} d(\vec{r}(s_n), \vec{r}(t_n)) < \epsilon.$$

Proof. Consider the free motion ω with the following jump directions and rates:

$$\begin{array}{ll} (1,0) & \text{with rate } \lambda_0 \\ (-1,0) & \text{with rate } \mu_0 \\ (0,1) & \text{with rate } \lambda_1 \\ (0,-1) & \text{with rate } \mu_1 \end{array} \tag{63}$$

We attempt to estimate the cost of a given absolutely continuous path $\vec{r}$ in the terms of the motions ω and ζ.

Define

$$g_\omega(\vec{a}, \vec{\theta}) = g_\omega(\vec{\theta}) = \lambda_0(e^{\theta_x} - 1) + \mu_0(e^{-\theta_x} - 1) + \lambda_1(e^{\theta_y} - 1) + \mu_1(e^{-\theta_y} - 1)$$

and l_ω, ${}_\omega I_0^T$ accordingly. Since for any $\vec{\theta}$

$$\mu_1(e^{-\theta_y} - 1) = \mu_1 e^{-\theta_y} - \mu_1 > -\mu_1,$$

we conclude that

$$\begin{aligned} g_\omega(\vec{\theta}) &> g_\zeta(\vec{\theta}) - \mu_1 \\ l_\zeta(\vec{b}) &= \sup_{\vec{\theta}} \{\langle \vec{\theta}, \vec{b} \rangle - g_\zeta(\vec{\theta})\} \\ &\ge \sup_{\vec{\theta}} \{\langle \vec{\theta}, \vec{b} \rangle - g_\omega(\vec{\theta}) - \mu_1\} \\ &= \sup_{\vec{\theta}} \{\langle \vec{\theta}, \vec{b} \rangle - g_\omega(\vec{\theta})\} - \mu_1 \\ &= l_\omega(\vec{b}) - \mu_1 \end{aligned}$$

Similarly, $l_\xi(\vec{b}) \geq l_\omega(\vec{b}) - \mu_0$ and

$$\min\{l_\zeta(\vec{b}), l_\xi(\vec{b})\} \geq l_\omega(\vec{b}) - \max\{\mu_0, \mu_1\}.$$

It follows from (61) that for any $t \in \bigcup_{i=0}^{3} A_i(\vec{r})$

$$l_{\vec{r}}(t) \geq \min\{l_\zeta(\vec{r}\,'(t)), l_\xi(\vec{r}\,'(t))\} \geq l_\omega(\vec{r}\,'(t)) - \max\{\mu_0, \mu_1\}.$$

Also, if $t \in A_5(\vec{r})$, then by Proposition 6,

$$l_{\vec{r}}(t) = 0 = l_\omega(\vec{r}\,'(t)) - l_\omega(\vec{0})$$

almost surely. Since $A_4(\vec{r})$ is a zero set, we obtain that

$$l_{\vec{r}}(t) \geq l_\omega(\vec{r}\,'(t)) - l_\omega(\vec{0}) - \max\{\mu_0, \mu_1\} \text{ a.s.}$$

$$\begin{aligned} \mathcal{I}_0^T(\vec{r}) &\geq \int_0^T l_\omega(\vec{r}\,'(t))dt - T\left(\max\{\mu_0, \mu_1\} + l_\omega(\vec{0})\right) \\ &= {}_\omega I_0^T(\vec{r}) - T\left(\max\{\mu_0, \mu_1\} + l_\omega(\vec{0})\right) \end{aligned}$$

and therefore

$$(64) \qquad {}_\omega I_0^T(\vec{r}) \leq \mathcal{I}_0^T(\vec{r}) + T\left(\max\{\mu_0, \mu_1\} + l_\omega(\vec{0})\right).$$

Now since $\mathcal{I}_0^T(\vec{r}) \leq K$, we have

$${}_\omega I_0^T(\vec{r}) \leq K + T\left(\max\{\mu_0, \mu_1\} + l_\omega(\vec{0})\right)$$

and by Lemma 5.18 [SW95] for the free motion ω, there exists δ which satisfies the requirements of the lemma for the bound $K + T(\max\{\mu_0, \mu_1\} + l_\omega(\vec{0}))$.

Choose a collection $\mathcal{N}$ of non-overlapping intervals with total length below δ. For each $[s,t] \in \mathcal{N}$ such that $\vec{r}(s)$ and $\vec{r}(t)$ reside on different layers of $\mathbb{D}$, choose u such that $\vec{r}(u) \in \partial D$ and replace $[s,t]$ with $[s,u]$ and $[u,t]$. Thus we obtain a new collection $\widetilde{\mathcal{N}}$ of non-overlapping intervals with total length below δ. By the triangle inequality for metric d and the mentioned lemma,

$$\sum_{[s,t]\in\mathcal{N}} d(\vec{r}(s), \vec{r}(t)) \leq \sum_{[s,t]\in\widetilde{\mathcal{N}}} d(\vec{r}(s), \vec{r}(t)) = \sum_{[s,t]\in\widetilde{\mathcal{N}}} |\vec{r}(t) - \vec{r}(s)| < \epsilon$$

□

Corollary 11. *The rate function* ${}_\omega I_0^T$ *of the free motion* ω *defined in* (63) *satisfies*

$$\mathcal{I}_0^T(\vec{r}) \geq {}_\omega I_0^T(\vec{r}) - T\left(\max\{\mu_0, \mu_1\} + l_\omega(\vec{0})\right)$$

for any path $\vec{r} : [0,T] \to \mathbb{D}$.

Proof. For the case $\mathcal{I}_0^T(\vec{r}) < \infty$ the objective is proved in the preceding proposition in form of equation (64). On the other hand, for the case $\mathcal{I}_0^T(\vec{r}) = \infty$ the objective holds trivially. □

The next property is a rather arbitrary statement, which can be viewed as a sort of lower bound for the rate function over small distances and time intervals. While this property will only be used in Chapter 5, with regards to small movenets of $\vec{r}$ over small time intervals, we still chose to present it here due to the generality of its statement.

Proposition 12. *There exist positive constants C_1, C_2 and C_3, such that for any path $\vec{r} : [0,T] \to \mathbb{D}$*

(65)

$$|\vec{r}(T) - \vec{r}(0)| \ \log \frac{T}{|\vec{r}(T) - \vec{r}(0)|} > -C_1|\vec{r}(T) - \vec{r}(0)| - C_2 T - C_3 \mathcal{I}_0^T(\vec{r}).$$

Remark. The distance applied in this proposition is not the distance on $\mathbb{D}$, but rather the Euclidean distance between two points, without regards to their respective panes.

Proof. For any $\vec{r}$ such that $\mathcal{I}_0^T(\vec{r}) = \infty$ the estimate (65) holds trivially with any $C_3 > 0$. Therefore we can restrict our discussion to paths with finite rate function value. Obviously, such paths are feasible and absolutely continuous.

In the course of the proof of Proposition 10 we showed that for a free motion ω defined in (63), a relation (64) holds.

Moreover, Lemma 5.17 [SW95] asserts that there exist constants C and B such that

$$\forall |\vec{y}| \geq B \quad l_\omega(\vec{y}) \geq C|\vec{y}|\log|\vec{y}|. \tag{66}$$

Although not stated explicitly, the proof of this lemma implies that we can choose C to be positive. We can also choose B to be larger than 1, simply because narrowing the range of $|\vec{y}|$ will not impair the conclusion of the lemma.

Yet another result, Lemma 5.16 [SW95], states that for an absolutely continuous $\vec{r}$

$${}_\omega I_0^T(\vec{r}) \geq T \cdot l_\omega\left(\frac{\vec{r}(T) - \vec{r}(0)}{T}\right). \tag{67}$$

Let us first assume, that $\left|\frac{\vec{r}(T)-\vec{r}(0)}{T}\right| \geq B$. Bringing together (64), (66) and (67), we obtain

$$\begin{aligned}
\mathcal{I}_0^T(\vec{r}) &\geq {}_\omega I_0^T(\vec{r}) - T\max\{\mu_0, \mu_1\} \\
&\geq T \cdot l_\omega\left(\frac{\vec{r}(T) - \vec{r}(0)}{T}\right) - T\max\{\mu_0, \mu_1\} \\
&\geq TC\left|\frac{\vec{r}(T) - \vec{r}(0)}{T}\right| \log\left|\frac{\vec{r}(T) - \vec{r}(0)}{T}\right| - T\max\{\mu_0, \mu_1\} \\
&= -C|\vec{r}(T) - \vec{r}(0)| \ \log\frac{T}{|\vec{r}(T) - \vec{r}(0)|} - T\max\{\mu_0, \mu_1\}.
\end{aligned}$$

Thus,

$$|\vec{r}(T)-\vec{r}(0)|\ \log\frac{T}{|\vec{r}(T)-\vec{r}(0)|}\geq -\frac{\mathcal{I}_0^T(\vec{r})}{C}-T\cdot\frac{\max\{\mu_0,\mu_1\}}{C}.$$

In the other case, namely when $\left|\frac{\vec{r}(T)-\vec{r}(0)}{T}\right|<B$,

$$\begin{aligned}|\vec{r}(T)-\vec{r}(0)|\ \log\frac{T}{|\vec{r}(T)-\vec{r}(0)|}&\geq |\vec{r}(T)-\vec{r}(0)|\ \log\frac{1}{B}\\&=-|\vec{r}(T)-\vec{r}(0)|\log B,\end{aligned}$$

and the righthand side of the last statement is negative, because $B>1$.

Note also that the constants B and C only depend on the parameters of the free motion ω, and so do their derived constants C_1, C_2, C_3. Therefore, the objective now follows immediately from the two discussed cases. □

The last three propositions show together that $\mathcal{I}_0^T(\vec{r})$ is a rate function by proving its lower semicontinuity.

Proposition 13. *Let $\vec{x}$ be some fixed point in $\mathbb{D}$, and let $\vec{r}\in\mathcal{D}_{\vec{x}}^2[0,T]$ be a feasible absolutely continuous path. Then for any sequence $\{\vec{r}_n\}\subseteq\mathcal{D}_{\vec{x}}^2[0,T]$ with limit $\vec{r}$ under the Skorohod metric,*

$$\liminf_{n\to\infty}\mathcal{I}_0^T(\vec{r}_n)\geq\mathcal{I}_0^T(\vec{r}).\tag{68}$$

Proof. Let $\{\vec{r}_n\}$ be a sequence of paths in $\mathcal{D}_{\vec{x}}^2[0,T]$, converging to $\vec{r}$. Naturally, it would suffice to consider only sequences of continuous paths, as the non-continuous paths are not feasible, and their costs are always infinite. Thus we can assume that $\{\vec{r}_n\}$ converges to $\vec{r}$ under the *sup* metric, which is equivalent to the Skorohod metric for continuous functions.

Consider the splitting $\mathcal{A}(\vec{r})$ of $[0,T]$, as defined in (53). It follows from (60a), that the value of $\mathcal{I}_0^T(\vec{r})$ is determined as a sum of $\mathcal{I}_u^v(\vec{r})$ over $(u,v)\in\mathcal{A}(\vec{r})$, a countable collection of intervals.

The notion of convergence to $\mathcal{I}_0^T(\vec{r})$ differs with respect to the finiteness of $\mathcal{I}_0^T(\vec{r})$. If $\mathcal{I}_0^T(\vec{r})<\infty$, then we consider $\epsilon>0$ and choose a finite subset $\mathcal{C}\subseteq\mathcal{A}(\vec{r})$ such that

$$\sum_{(u,v)\in\mathcal{C}}\mathcal{I}_u^v(\vec{r})\geq\mathcal{I}_0^T(\vec{r})-\epsilon.\tag{69a}$$

On the other hand, for $\mathcal{I}_0^T(\vec{r})=\infty$ we consider some $M>0$ and choose again a finite subset $\mathcal{C}\subseteq\mathcal{A}(\vec{r})$ such that

$$\sum_{(u,v)\in\mathcal{C}}\mathcal{I}_u^v(\vec{r})\geq M.\tag{69b}$$

Recall, that the rate function $\mathcal{I}_u^v(\vec{q})$ can be defined for any feasible path $\vec{q}$ as an integral of some function $l_{\vec{q}}$ over the interval (u,v) (see formulae (61)–(62)). Therefore,

$$(70)\qquad \liminf_{n\to\infty} \mathcal{I}_0^T(\vec{r}_n) \geq \liminf_{n\to\infty} \sum_{(u,v)\in\mathcal{C}} \mathcal{I}_u^v(\vec{r}_n) \geq \sum_{(u,v)\in\mathcal{C}} \liminf_{n\to\infty} \mathcal{I}_u^v(\vec{r}_n).$$

Thus it would be sufficient to prove that for each $(u,v) \in \mathcal{A}(\vec{r})$

$$(71)\qquad \liminf_{n\to\infty} \mathcal{I}_u^v(\vec{r}_n) \geq \mathcal{I}_u^v(\vec{r}),$$

and the proof of the proposition will be completed.

Let $(u,v) \in \mathcal{A}(\vec{r})$. We proceed separately with each of the cases where (u,v) belongs to either $\mathcal{A}_0(\vec{r})$, $\mathcal{A}_1(\vec{r})$, $\mathcal{A}_2(\vec{r})$ or $\mathcal{A}_3(\vec{r})$.

I. Let $(u,v) \in \mathcal{A}_0(\vec{r})$. When $u \neq 0$ or $\vec{r}(u) \notin \partial_x D$, i.e. the special case mentioned at the end of (52b) doesn't hold, we can choose a strictly increasing sequence of points $\{u_m\}_{m\in\mathbb{Z}}$ such that

$$\lim_{m\to-\infty} u_m = u$$
$$\lim_{m\to\infty} u_m = v$$
$$\forall m \in \mathbb{Z} \quad u_m \in \tilde{A}_0(\vec{r}).$$

It is easy to build such a sequence by induction, as we shall now see.

Choose $v_0 \in (u,v)$. By the definition (52b) of $A_0(\vec{r})$, there exist points $u_0 < v_0 < u_1$ such that $u_0, u_1 \in \tilde{A}_0(\vec{r})$, and $[u_0, u_1]$ lies inside (u,v).

From this base we can build an increasing sequence $\{u_m\}$ as follows: given $u_m \in \tilde{A}_0(\vec{r})$, we can set $v_m = \frac{u_m+v}{2}$, and select $u_{m+1} \in \tilde{A}_0(\vec{r})$ in such a manner that $u_m < v_m < u_{m+1}$, and $[u_m, u_{m+1}]$ lies inside (u,v). Note that the distance of u_{m+1} from v is less than hald of the distance between u_m and v, so $\{u_m\}$ indeed converges to v, as m goes to infinity. Similarly, we can build the sequence $\{u_m\}$ downwards, as m decreases towards $-\infty$.

For any $m \in \mathbb{Z}$,

$$\forall t \in [u_m, u_{m+1}] \quad \vec{r}(t) \in D_0 \cup \partial_y D \subseteq D_0 \cup D_1 \cup \partial_y D.$$

Since the latter is an open set, and $\vec{r}$ is a continuous function, it follows that there exists $\delta > 0$ such that

$$\forall t \in [u_m, u_{m+1}] \quad d(\vec{r}(t), \partial_x D \cup \mathbf{0}) > \delta.$$

As both $\vec{r}(u_m)$ and $\vec{r}(u_{m+1})$ lie in the open set D_0, we can safely choose δ such that it also satisfies

$$d(\vec{r}(u_m), \partial_y D) > \delta,$$
$$d(\vec{r}(u_{m+1}), \partial_y D) > \delta.$$

Moreover, as $\vec{r}_n$ converges in *sup* to $\vec{r}$, we can find N such that

$$\forall n > N,\ \forall 0 \leq t \leq T \quad d(\vec{r}_n(t), \vec{r}(t)) < \delta.$$

Note that for any $n > N$, $\vec{r}_n$ can't visit D_1 during the time interval $[u_m, u_{m+1}]$, because $\vec{r}_n(u_m), \vec{r}_n(u_{m+1}) \in D_0$, and $\vec{r}_n$ doesn't visit

$\partial_x D \cup \mathbf{0}$, so visiting D_1 would make $\vec{r}_n$ a non-feasible path. Therefore $\vec{r}_n([u_m, u_{m+1}]) \subseteq D_0 \cup \partial_y D$ and $[u_m, u_{m+1}] \subseteq A_0(\vec{r}_n)$.

In the case $u = 0$ and $\vec{r}(u) \in \partial_x D$, it follows immediately that $\vec{r}_n(u) = \vec{r}_n(0) \in \partial_x D$. Using the same technique as before, we obtain the very same result, namely that for any $m \in \mathbb{Z}$ there exists N such that

$$\forall n > N \qquad [u_m, u_{m+1}] \subseteq A_0(\vec{r}_n).$$

The main conclusion of the above discussion is that starting from that N, $\vec{r}_n$'s belong to the domain of the free motion ζ on $D_0 \cup \partial_y D$. We conclude therefore that the general theory applies, and by Lemma 5.42 [SW95] the rate function of ζ is lower-semicontinuous, which allows us to state:

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{I}_{u_m}^{u_{m+1}}(\vec{r}_n) &= \liminf_{n\to\infty} \int_{u_m}^{u_{m+1}} l_\zeta(\vec{r}_n{}'(t))dt \\
&\geq \int_{u_m}^{u_{m+1}} l_\zeta(\vec{r}\,'(t))dt \\
&= \mathcal{I}_{u_m}^{u_{m+1}}(\vec{r}).
\end{aligned} \tag{72}$$

Furthermore,

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{I}_u^v(\vec{r}_n) &= \liminf_{n\to\infty} \sum_{m=-\infty}^{\infty} \mathcal{I}_{u_m}^{u_{m+1}}(\vec{r}_n) \\
&\geq \sum_{m=-\infty}^{\infty} \liminf_{n\to\infty} \mathcal{I}_{u_m}^{u_{m+1}}(\vec{r}_n) \\
&\overset{(72)}{\geq} \sum_{m=-\infty}^{\infty} \mathcal{I}_{u_m}^{u_{m+1}}(\vec{r}) \\
&= \mathcal{I}_u^v(\vec{r}),
\end{aligned}$$

as required in (71). The case of $(u, v) \in \mathcal{A}_1(\vec{r})$ is treated in the same fashion.

II. Let $(u, v) \in \mathcal{A}_2(\vec{r})$. Consider some subinterval $[\tilde{u}, \tilde{v}] \subset (u, v)$. As $\tilde{u} > u$ and $\tilde{v} < v$, it follows that for any $t \in [\tilde{u}, \tilde{v}]$, $\vec{r}(t) \in \partial_y D$, and due to the compactness of $[\tilde{u}, \tilde{v}]$, the distance of $\vec{r}$ from $\partial_x D \cup \mathbf{0}$ attains a minimum value $\gamma > 0$ over $[\tilde{u}, \tilde{v}]$.

Thus there exists $N \in \mathbb{N}$ such that for all $n > N$, $\vec{r}_n$ doesn't visit $\partial_x D \cup \mathbf{0}$ during the time $[\tilde{u}, \tilde{v}]$, and therefore it may cross the boundary $\partial_y D$ at most once during that time.

As $\vec{r}_n$ is feasible, we can denote

$$\begin{aligned}
s_{n,0} &= \sup\{t \in [\tilde{u}, \tilde{v}] \mid \vec{r}_n(t) \in D_0\}, \\
s_{n,1} &= \inf\{t \in [\tilde{u}, \tilde{v}] \mid \vec{r}_n(t) \in D_1\},
\end{aligned}$$

and they satisfy

$$s_{n,0} \leq s_{n,1}.$$

Then it follows from the definition (59) that

$$\begin{aligned}\mathcal{I}_{\tilde u}^{\tilde v}(\vec r_n) &= \int_{\tilde u}^{s_n} l_\zeta(\vec r_n{}'(t))dt + \int_{s_n}^{\tilde v} l_\xi(\vec r_n{}'(t))dt\\ &= I_{\tilde u}^{s_n}(\vec r_n) + J_{s_n}^{\tilde v}(\vec r_n)\end{aligned}$$

for some $s_n \in [s_{n,0}, s_{n,1}]$.

Choose a subsequence $\{\vec r_{n_k}\}_{k=1}^{\infty}$ of $\{\vec r_n\}$, such that $n_1 > N$, and

$$\liminf_{n\to\infty}\big(\mathcal{I}_{\tilde u}^{\tilde v}(\vec r_n)\big) = \lim_{k\to\infty}\big(\mathcal{I}_{\tilde u}^{\tilde v}(\vec r_{n_k})\big). \tag{73}$$

Since for each $\vec r_{n_k}$ the value of s_{n_k} is determined and it belongs to a compact range $[\tilde u, \tilde v]$, we can further choose a subsequence of $\{s_{n_k}\}$ that converges to some $\tilde s$. In order to avoid a mess of subindices, we assume that we made the choice of $\{\vec r_{n_k}\}$ at the first hand in such a manner that $\{s_{n_k}\}$ converges to $\tilde s$.

Then for any $\delta > 0$

$$\begin{aligned}\lim_{k\to\infty}\big(\mathcal{I}_{\tilde u}^{\tilde v}(\vec r_{n_k})\big) &= \lim_{k\to\infty}\big(I_{\tilde u}^{s_{n_k}}(\vec r_{n_k}) + J_{s_{n_k}}^{\tilde v}(\vec r_{n_k})\big)\\ &\geq \lim_{k\to\infty}\big(I_{\tilde u}^{\tilde s-\delta}(\vec r_{n_k}) + J_{\tilde s+\delta}^{\tilde v}(\vec r_{n_k})\big)\\ &\geq \liminf_{k\to\infty} I_{\tilde u}^{\tilde s-\delta}(\vec r_{n_k}) + \liminf_{k\to\infty} J_{\tilde s+\delta}^{\tilde v}(\vec r_{n_k})\\ &\geq I_{\tilde u}^{\tilde s-\delta}(\vec r) + J_{\tilde s+\delta}^{\tilde v}(\vec r),\end{aligned}$$

where the last inequality follows from the lower-semicontinuity of I and J (see [SW95, Lemma 5.42]).

We can now take δ to zero and obtain

$$\lim_{k\to\infty}\big(\mathcal{I}_{\tilde u}^{\tilde v}(\vec r_{n_k})\big) \geq I_{\tilde u}^{\tilde s}(\vec r) + J_{\tilde s}^{\tilde v}(\vec r).$$

Using (73), we further obtain

$$\liminf_{n\to\infty}\big(\mathcal{I}_u^v(\vec r_n)\big) \geq \liminf_{n\to\infty}\big(\mathcal{I}_{\tilde u}^{\tilde v}(\vec r_n)\big) \overset{(73)}{\geq} I_{\tilde u}^{\tilde s}(\vec r) + J_{\tilde s}^{\tilde v}(\vec r). \tag{74}$$

Let us elaborate a little bit on this important intermediate result. The lefthand site of (74) is equal to the lefthand side of (71). The righthand side involves the interval $[\tilde u, \tilde v]$ which can be taken arbitrarily inside (u, v), and the value $\tilde s \in [\tilde u, \tilde v]$ which depends solely on $\tilde u$, $\tilde v$.

Now we proceed as follows: choose sequences $\{\tilde u_k\} \to u$, $\{\tilde v_k\} \to v$. Since for any $[\tilde u_k, \tilde v_k]$, its appropriate value $\tilde s_k$ is confined to the compact range $[u, v]$, we can choose from $\{\tilde s_k\}$ a converging subsequence. Again, we assume for the simplicity of notation, that $\{\tilde u_k\}$, $\{\tilde v_k\}$ were chosen at the first hand in such a manner that $\{\tilde s_k\}$ converges to some $s_0 \in [u, v]$.

But then for any $\delta > 0$, for any k large enough

$$\liminf_{n\to\infty}\big(\mathcal{I}_u^v(\vec r_n)\big) \geq I_{\tilde u_k}^{\tilde s_k}(\vec r) + J_{\tilde s_k}^{\tilde v_k}(\vec r) \geq I_{\tilde u_k}^{s_0-\delta}(\vec r) + J_{s_0+\delta}^{\tilde v_k}(\vec r).$$

Now we take k to infinity and obtain

$$\liminf_{n\to\infty}\big(\mathcal{I}_u^v(\vec r_n)\big) \geq I_u^{s_0-\delta}(\vec r) + J_{s_0+\delta}^{v}(\vec r),$$

and, as we already said, this holds for any $\delta > 0$. By taking in turn δ to zero we further obtain

$$\begin{aligned}\liminf_{n\to\infty}\big(\mathcal{I}_u^v(\vec{r}_n)\big) &\geq I_u^{s_0}(\vec{r}) + J_{s_0}^v(\vec{r}) \\ &\geq \min_{s\in[u,v]}\big\{I_u^s(\vec{r}) + J_s^v(\vec{r})\big\} \\ &= \mathcal{I}_u^v(\vec{r}),\end{aligned}$$

and (71) holds. The case of $(u, v) \in \mathcal{A}_3(\vec{r})$ is treated in the same fashion.

We have shown that (71) holds for any $(u, v) \in \mathcal{A}(\vec{r})$, and thus it now follows that

$$\liminf_{n\to\infty}\big(\mathcal{I}_u^v(\vec{r}_n)\big) \overset{(70)}{\geq} \sum_{(u,v)\in\mathcal{C}} \liminf_{n\to\infty} \mathcal{I}_u^v(\vec{r}_n) \geq \sum_{(u,v)\in\mathcal{C}} \mathcal{I}_u^v(\vec{r})$$

for any finite subset $\mathcal{C}$ of $\mathcal{A}(\vec{r})$. By bringing ϵ to zero, or M to infinity in (69) we reach the required conclusion (68). $\square$

Unlike in the continuous model, we can't establish lower semicontinuity of $\mathcal{I}$ without fixing $\vec{r}(0)$ first. Indeed, one can perhaps find a setting of λ_0, λ_1, μ_0, μ_1 and a special path

$$\vec{r} : [0, T] \to \partial_x D,$$

such that

$$J_0^s(\vec{r}) + I_s^T(\vec{r}) < I_0^T(\vec{r}) = \mathcal{I}_0^T(\vec{r}).$$

Then it could be possible to choose a sequence of paths $\vec{r}_n$ starting on D_1 near $\vec{r}(0)$ and converging to $\vec{r}$, such that

$$\mathcal{I}_0^T(\vec{r}_n) \approx J_0^s(\vec{r}) + I_s^T(\vec{r}),$$

and the lower semicontinuity would not hold.

In the scope of this paper we don't demonstrate any specific case, but the reader must be aware of such possibility.

Proposition 14. *Let $\vec{r} \in \mathcal{D}^2[0, T]$ be a feasible and not absolutely continuous path. Then for any sequence $\{\vec{r}_n\} \subseteq \mathcal{D}^2[0, T]$ with limit $\vec{r}$ under the Skorohod metric,*

$$\lim_{n\to\infty} \mathcal{I}_0^T(\vec{r}_n) = \infty = \mathcal{I}_0^T(\vec{r}). \tag{75}$$

Proof. Assume to the contrary that there exists a sequence $\{\vec{r}_n\}$ such that (75) is not satisfied. Then it has a subsequence $\{\vec{r}_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \mathcal{I}_0^T(\vec{r}_{n_k}) = K < \infty.$$

Obviously, the set $\{\mathcal{I}_0^T(\vec{r}_{n_k})\}_{k=1}^{\infty}$ is bounded by some finite constant K_1, and therefore by Proposition 10, the sequence $\{\vec{r}_{n_k}\}$ is uniformly absolutely continuous.

Since $\vec{r}_{n_k}$ and $\vec{r}$ are continuous, their Skorohod distance equals to their *sup* distance, and thus $\{\vec{r}_{n_k}\}$ converges to $\vec{r}$ in *sup* metric. Therefore, by Proposition 28, $\vec{r}$ is absolutely continuous, contrary to the conditions. This contradiction completes the proof of the proposition. $\square$

Proposition 15. *The rate function $\mathcal{I}_0^T(\vec{r})$ is lower-semicontinuous.*

Proof. We need to show that for any $\vec{r} \in \mathcal{D}^2[0,T]$, and a sequence $\{\vec{r}_m\} \subseteq \mathcal{D}^2[0,T]$ which converges to $\vec{r}$ under the Skorohod metric,

$$\liminf_{m\to\infty} \mathcal{I}_0^T(\vec{r}_m) \geq \mathcal{I}_0^T(\vec{r}). \tag{76}$$

Note that this result has already been achieved for all feasible paths $\vec{r}$ in Propositions 13 and 14. Thus we only need to address the non-feasible paths in $\mathcal{D}^2[0,T]$.

Let $\vec{r}$ be a non-feasible path. Then by (60b),

$$\mathcal{I}_0^T(\vec{r}) = \infty.$$

Moreover, by Definition 9, there exist $\epsilon > 0$ and $N \in \mathbb{N}$ such that for any $n \geq N$

$$\mathbb{P}\left(\sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}(t)\big) < \epsilon\right) = 0. \tag{77}$$

As $\{\vec{r}_n\}$ converges to $\vec{r}$, we can choose $M \in \mathbb{N}$ such that for any $m \geq M$

$$\sup_{0\leq t\leq T} d\big(\vec{r}_m(t), \vec{r}(t)\big) < \frac{\epsilon}{2}.$$

Consider a sample path $\vec{z}_n$ such that

$$\sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}_m(t)\big) < \frac{\epsilon}{2}.$$

By the triangle inequality,

$$\begin{aligned}
\sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}(t)\big) &\leq \sup_{0\leq t\leq T} \Big\{ d\big(\vec{z}_n(t), \vec{r}_m(t)\big) + d\big(\vec{r}_m(t), \vec{r}(t)\big) \Big\} \\
&\leq \sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}_m(t)\big) + \sup_{0\leq t\leq T} d\big(\vec{r}_m(t), \vec{r}(t)\big) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

and therefore

$$\left\{ \vec{z}_n \;\middle|\; \sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}_m(t)\big) < \frac{\epsilon}{2} \right\} \subseteq \left\{ \vec{z}_n \;\middle|\; \sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}(t)\big) < \epsilon \right\}.$$

By using (77) we now conclude that for any $n \geq N$ and $m \geq M$

$$\mathbb{P}\left(\sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}_m(t)\big) < \frac{\epsilon}{2}\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq T} d\big(\vec{z}_n(t), \vec{r}(t)\big) < \epsilon\right) = 0,$$

and thus for any $m \geq M$, $\vec{r}_m$ is not feasible, and it satisfies

$$\mathcal{I}_0^T(\vec{r}_m) = \infty.$$

It is now obvious that

$$\liminf_{m\to\infty} \mathcal{I}_0^T(\vec{r}_m) = \infty = \mathcal{I}_0^T(\vec{r}),$$

and (76) follows. $\square$

5. The lower bound

In this chapter we shall establish one part of the Large Deviations Principle, namely the lower bound. Our ultimate goal therefore will be

Theorem 16. *For every open set* $G \subseteq \mathcal{D}^2[0,T]$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}(\vec{z}_n \in G) \geq -\inf\left\{\mathcal{I}_0^T(\vec{r}) \mid \vec{r} \in G, \vec{r}(0) = \vec{x}\right\}. \tag{78}$$

When comparing (78) to the general Large Deviations Principle [SW95, 5.1(ii)], one can notice that our formulation lacks the statement about the uniformity for $\vec{x}$ in compact sets. A corresponding statement on our part would assert that

$$\lim_{\vec{x}_m \to \vec{x}} \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}_m}(\vec{z}_n \in G) \geq -\inf\left\{\mathcal{I}_0^T(\vec{r}) \mid \vec{r} \in G, \vec{r}(0) = \vec{x}\right\}. \tag{79}$$

The uniformity holds in our model only when $\vec{x} \notin \partial\mathbb{D}$, but we shall not provide a rigorous proof in this paper. Paragraph 5.3 provides a more in-depth discussion on this matter.

To establish Theorem 16, we shall first prove a seemingly weaker local statement. Nevertheless, as we will see at the end of this section, the theorem is actually its trivial conclusion.

Consider a path $\vec{r}$ and a small neighborhood $B_\epsilon(\vec{r})$ of radius ϵ. We wish to establish the following "local" lower bound statement for this neighborhood:

Proposition 17. *For any* $\vec{r} \in \mathcal{D}^2[0,T]$ *and* $\epsilon > 0$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}(\vec{z}_n \in B_\epsilon(\vec{r})) \geq -\mathcal{I}_0^T(\vec{r}), \tag{80}$$

where $\vec{x} = \vec{r}(0)$.

Remark 11. Recall that the set of paths $\mathcal{D}^2[0,T]$ is equipped with *sup* metric (see Definition 6 at the beginning of Chapter 4), and thus both Theorem 16 and Proposition 17 are formulated in terms of the topology induced by that metric. This contrasts with the usual formulation of the LDP which comes in terms of Skorohod metric with all its useful properties such as completeness and separability.

We point out that the LDP holds equivalently for Skorohod and for *sup* metric. The reason for this lies in the fact that any sample path $\vec{z}_n$ can be approximated by a piecewise-linear continuous path $\tilde{z}_n$, which lies at most $\frac{1}{n}$ away from $\tilde{z}_n$. The LDPs for $\vec{z}_n$ and $\tilde{z}_n$ are therefore equivalent due to the exponential equivalence of the appropriate distributions, and once we employ continuous sample paths, their Skorohod distance coincides with their *sup* distance.

Let us elaborate on the idea behind the proof of Proposition 17. We employ the splitting of $\vec{r}$ into pieces according to $\mathcal{A}(\vec{r})$, that was introduced in Section 4.1.

First, we choose a finite subset of $\mathcal{A}(\vec{r})$, which includes only those intervals which are most significant in terms of their length and contribution to $\mathcal{I}_0^T(\vec{r})$. These subintervals cover $\mathcal{A}(\vec{r})$ almost completely. Thus the entire interval $[0,T]$ becomes splitted into subintervals which nearly coincide with $\mathcal{A}(\vec{r})$, and the transitions between them. It will be our goal to show that the "significant" subintervals supply the bulk of the cost of path $\vec{r}$, while the transitional intervals bear negligible cost. For this purpose we will split intervals belonging to either $\mathcal{A}_2(\vec{r})$ or $\mathcal{A}_3(\vec{r})$ in two pieces, separated by another transitional interval.

The task of estimating the cost of a significant interval is rather straightforward. During any such interval $\vec{z}_n$ can be restricted to a single pane, and thus its probability to stay near $\vec{r}$ would be easily estimated by comparison with an appropriate free motion.

On the other hand, it would be necessary to show that the total cost of all transitional intervals is negligible. Two issues complicate this task. First, as we refine the splitting of $[0,T]$, more and more transitional intervals may emerge. Second, the sojouring time of $\mathbf{0}$ is not covered by $\mathcal{A}(\vec{r})$, and thus it is always contained in the scope of transitional intervals. Both these issues are addressed in the course of the proof of the proposition.

The rest of this chapter is mostly dedicated to the proof of the local lower bound, as stated in Proposition 17. It is divided into sections as follows:

5.1 *Small movements.* The results obtained here are mostly concerned with the mentioned transitional intervals. It is common for auxiliary statements to be formulated in a very cumbersome way, which tailors them specially to the needs of a specific proof, and the results of this section are not exception to this rule. The reader may choose to skip this section and proceeds directly to Section 5.2. Upon finishing the latter, the reader should return back to complete the missing pieces.

5.2 *The local lower bound.* This section is mostly dedicated to the proof of Proposition 17, according to the structure we outlined above. At the end, the lower bound is proved in its canonical form, thus completing the discussion.

5.1. **Estimating small movements.** In this section we will be dealing with the transitional intervals, during which $\vec{r}$ changes its behavior. Due to the momentary nature of such change, these intervals generally correspond to some small movements of $\vec{r}$ around the point of change. Before we proceed, let us elaborate on the meaning of a "small movement".

Let $\epsilon > 0$. We take some $\delta > 0$, which is sufficiently smaller than ϵ. Since $\vec{r}$ is absolutely continuous, we can find some $\tau > 0$ such that

$$|u - v| < \tau \quad \text{implies} \quad d(\vec{r}(u), \vec{r}(v)) < \delta.$$

If we take some particular $u, v \in [0, T]$ which satisfy $|u - v| < \tau$, it follows that the entire path $\vec{r}|_u^v$ lays inside a δ-neighborhood of $\vec{r}(u)$.

Now consider the expression

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\vec{z}_n \in B_\epsilon(\vec{r}|_u^v) \mid d(\vec{z}_n(u), \vec{r}(u)) < \delta\right). \tag{81}$$

Since ϵ is quite large relatively to δ, any sample path $\vec{z}_n$ that starts sufficiently close to $\vec{r}(u)$, ends sufficiently close to $\vec{r}(v)$ and doesn't walk too far away, would stay inside the ϵ-neighborhood of $\vec{r}|_u^v$ (see Figure 14).

This allows us to restrict the event in (81) to some smaller set, which probably would be easier to estimate.

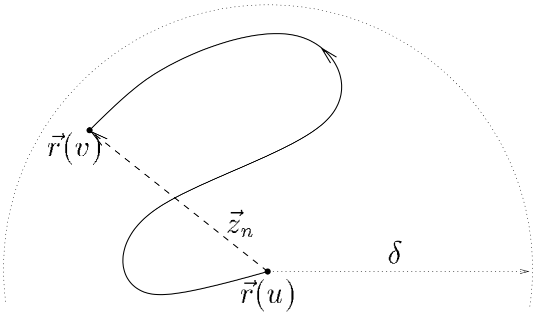


FIGURE 14. A small movement of $\vec{r}$

The figure shows that under certain conditions on δ, a sample path $\vec{z}_n$ may look very differently from $\vec{r}$, and still stay in its proximity.

To begin with, we study a random walk $\vec{z}_n$ of a very specific nature. Specifically, consider the event where $\vec{z}_n$ advances a distance of ϵ in a time τ solely by means of clients arrival to the first queue. This event woud occur if we observe roughly $n\epsilon$ arrivals to the first queue, while the second queue and the server are stalled. In the following lemma we treat this situation in more detail and in a more abstract manner.

Lemma 18. *Let $N_1(t), N_2(t), \ldots, N_m(t)$ be m independent Poisson processes with intensities $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively. Let $k \in \mathbb{N}$ and $\tau, \epsilon > 0$. Consider the following event expressed in terms of scaled processes:*

$$\Theta_n = \left\{ \frac{N_1(n\tau)}{n} = \frac{[n\epsilon]}{n}, \frac{N_2(n\tau)}{n} = \ldots = \frac{N_m(n\tau)}{n} = 0 \right\}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\Theta_n) = -\tau \sum_{i=1}^{m} \lambda_i + \epsilon \left(1 + \log \frac{\lambda_1 \tau}{\epsilon}\right) \tag{82}$$

Proof. Since the processes $\{N_i\}$ are independent, it follows that

$$\begin{aligned}\mathbb{P}(\Theta_n) &= \mathbb{P}\big(N_1(n\tau) = [n\epsilon]\big) \cdot \prod_{i=2}^{m} \mathbb{P}\big(N_i(n\tau) = 0\big) \\ &= e^{-\lambda_1 n\tau} \cdot \frac{(\lambda_1 n\tau)^{[n\epsilon]}}{[n\epsilon]!} \cdot \prod_{i=2}^{m} e^{-\lambda_i n\tau}\end{aligned}$$

$$\frac{1}{n}\log \mathbb{P}(\Theta_n) = -\tau \sum_{i=1}^{m} \lambda_i + \frac{1}{n}[n\epsilon]\log(\lambda_1 n\tau) - \frac{1}{n}\sum_{i=1}^{[n\epsilon]} \log i.$$

The sum of logarithms can be estimated integrally:

$$\int_1^{n\epsilon-1} \log x dx \le \sum_{i=1}^{[n\epsilon]} \log i \le \int_1^{n\epsilon+2} \log x dx$$

$$\Big(\epsilon - \frac{1}{n}\Big)(\log(n\epsilon - 1) - 1) \le \frac{1}{n}\sum_{i=1}^{[n\epsilon]} \log i \le \Big(\epsilon + \frac{2}{n}\Big)(\log(n\epsilon + 2) - 1)$$

$$\frac{1}{n}\sum_{i=1}^{[n\epsilon]} \log i = \epsilon(\log(n\epsilon) - 1) + O\Big(\frac{\log n}{n}\Big).$$

Therefore,

$$\begin{aligned}\frac{1}{n}\log \mathbb{P}(\Theta_n) &= -\tau \sum_{i=1}^{m} \lambda_i + \epsilon\log(\lambda_1 n\tau) + O\Big(\frac{\log n}{n}\Big) + \epsilon \\ &\quad - \epsilon\log(n\epsilon) + O\Big(\frac{\log n}{n}\Big) \\ &= -\tau \sum_{i=1}^{m} \lambda_i + \epsilon\Big(1 + \log\frac{\lambda_1\tau}{\epsilon}\Big) + O\Big(\frac{\log n}{n}\Big),\end{aligned}$$

and the conclusion follows. □

Let us now interpret the result we just obtained, as it applies to our model. Consider a path $\vec{r}$ which starts at the moment 0 at some point $\vec{r}(0) \in D_0$, and goes at constant velocity to some other point $\vec{r}(\tau) = \vec{r}(0) + (\epsilon, 0)$.

Let $m = 3$ and let the Poisson processes $N_1(t)$, $N_2(t)$, $N_3(t)$ have the intensities λ_0, λ_1 and μ_0 respectively. Then the event Θ_n from Lemma 18 corresponds to the situation where $n\epsilon$ clients arrive to the first queue, while no clients arrive to the second, and no service occurs. Figure 15 shows the x-coordinate of the path $\vec{r}$ together with several sample paths $\vec{z}_n$ which belong to Θ_n. One can notice that the sample paths begin at $\vec{r}(0)$, end at $\vec{r}(\tau)$, and remain in the proximity of $\vec{r}$ during the time $[0, \tau]$.

The expression (82) provides an estimate for the probability $\mathbb{P}(\Theta_n)$ in terms of time τ and displacement ϵ. Furthermore, the definition

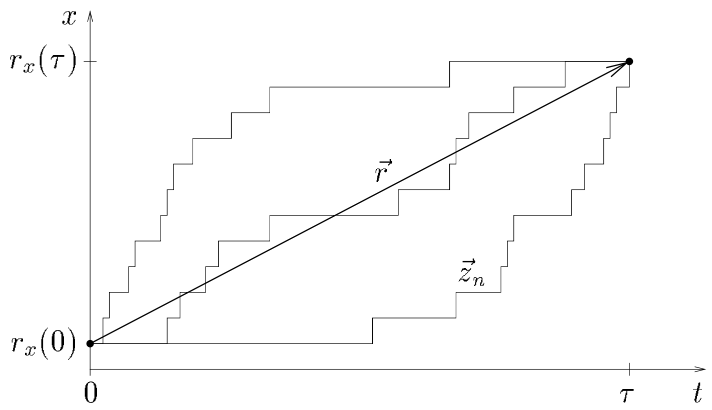


FIGURE 15. A short linear path $\vec{r}$

(59a) allows us to compute easily the cost of $\vec{r}$ as follows:

$$\mathcal{I}_0^\tau(\vec{r}) = I_0^\tau(\vec{r}) = \tau l_\zeta\left(\frac{\epsilon}{\tau}\right) = C\epsilon \log \frac{\epsilon}{\tau},$$

and therefore (82) can be restated as

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\Theta_n) = C_1\epsilon + C_2\tau + C_3\mathcal{I}_0^\tau(\vec{r}),$$

where the constants C_1, C_2, C_3 depend only on λ_0, λ_1, μ_0. Now we wish to establish an estimate of a similar form for the general case of small movement.

Lemma 19. *Let $\vec{r} : [0,T] \to \mathbb{D}$ be a feasible absolutely continuous path. Then there exist positive constants C_1, C_2 and C_3 such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{r}(u)+\vec{q}_1} \begin{pmatrix} \vec{z}_n \in B_\epsilon(\vec{r}|_u^{u+\tau}), \\ \vec{z}_n(u+\tau) = \vec{r}(u+\tau) + \vec{q}_2 \end{pmatrix} \tag{83}$$
$$> -C_1\tau - C_2\delta - C_3\mathcal{I}_u^{u+\tau}(\vec{r}).$$

provided the following conditions are met:

1. $|\vec{q}_1| < 2\tau$.
2. $|\vec{q}_2| < 2\tau$.
3. *there exists a feasible path from $\vec{r}(u) + \vec{q}_1$ to $\vec{r}(u+\tau) + \vec{q}_2$ which belongs also to $B_\epsilon(\vec{r}|_u^{u+\tau})$.*

There exists a positive $\delta < \frac{\epsilon}{10}$ such that

4. $|\vec{q}_1|, |\vec{q}_2| < \delta$,
5. *For any $t_1, t_2 \in (u, u+\tau)$, $\quad d(\vec{r}(t_1), \vec{r}(t_2)) < \delta$.*

Proof. We shall employ the following strategy. Instead of allowing $\vec{z}_n$ to move freely under the given constraints, we tightly restrict it to move piecewise linearly, in the fashion described by Lemma 18. More specifically, we consider the start point $\vec{r}(u) + \vec{q}_1$, and the end point $\vec{r}(u+\tau) + \vec{q}_2$ of the motion $\vec{z}_n$, and construct a piecewise linear path from $\vec{r}(u) + \vec{q}_1$ to $\vec{r}(u+\tau) + \vec{q}_2$ with each piece parallel to some axis. Then we allow $\vec{z}_n$ to move strictly along each piece of the path in turn by freezing all arrival/service processes except one each time. With this setting, we would be able to estimate the probability for $\vec{z}_n$ to follow our scheme using (82).

Our model allows for a great deal of different initial conditions. Specifically, $\vec{r}(u)$ and $\vec{r}(u+\tau)$ may be located on the same pane, or on two different panes, or perhaps even on the boundary. We will not discuss every single possibility, but it is worth mentioning that even in the worst-case scenario the piecewise linear path will consist of just four pieces.

To illustrate this approach, consider the following situation: $\vec{r}(u) \in D_0$ and $\vec{r}(u+\tau) \in D_1$. By the rules of vector addition on $\mathbb{D}$, $\vec{r}(u) + \vec{q}_1$ belongs to the same pane, as $\vec{r}(u)$, and the same holds for $\vec{r}(u+\tau) + \vec{q}_2$. Then the feasibility of $\vec{r}$ implies that there is some $\sigma \in [u, u+\tau]$ which satisfies

$$\vec{r}(\sigma) = \vec{p} \in \partial_y D.$$

Now we construct a piecewise linear path as follows (see Figure 16):

$$
\begin{aligned}
&t \in [u, u + \frac{\tau}{4}] : \\
&\qquad \vec{z}_n(t) \text{ moves from } \vec{r}(u) + \vec{q}_1 \text{ to } (r_x(u) + q_{1x}, p_y, 0); \\
&t \in [u + \frac{\tau}{4}, u + \frac{\tau}{2}] : \\
&\qquad \vec{z}_n(t) \text{ moves from } (r_x(u) + q_{1x}, p_y, 0) \text{ to } \vec{p}; \\
&t \in [u + \frac{\tau}{2}, u + \frac{3\tau}{4}] : \\
&\qquad \vec{z}_n(t) \text{ moves from } \vec{p} \text{ to } (r_x(u+\tau) + q_{2x}, p_y, 1); \\
&t \in [u + \frac{3\tau}{4}, u + \tau] : \\
&\qquad \vec{z}_n(t) \text{ moves from } (r_x(u+\tau) + q_{2x}, p_y, 1) \text{ to } \vec{r}(u+\tau) + \vec{q}_2.
\end{aligned}
\tag{84}
$$

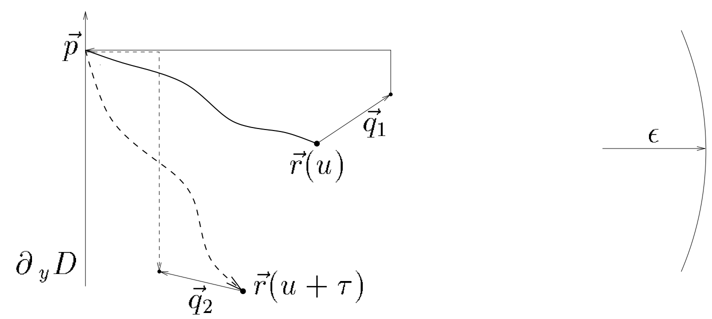


FIGURE 16. Approximation of a small movement of $\vec{r}$ by a linear path.

It's easy to see that this path is indeed located inside $B_\epsilon(\vec{r}|_u^{u+\tau})$, because the length of each single piece is at most 2δ, and correspondingly, the length of the entire path $\vec{z}_n$ makes is at most 8δ. Therefore the distance between any point of this path and $\vec{r}(t)$ doesn't exceed 10δ for any $u < t < u + \tau$.

An event of having $\vec{z}_n$ move along a particular piece is precisely the one described in Lemma 18, with appropriate substitutions.

As an example, we shall treat just the first interval among the four described in (84), to show how we deal with it. Others can be treated similarly. For convenience, we denote the displacement of $\vec{z}_n$ as

$$\Delta_y = p_y - (r_y(u) + q_{1y}).$$

Indeed, note that the event

$$\Theta_n = \left\{ \begin{array}{c} \vec{z}_n(t) \text{ moves from } \vec{r}(u) + \vec{q}_1 \text{ to } (r_x(u) + q_{1x}, p_y, 0), \\ \text{as } t \text{ advances from } u \text{ to } u + \frac{\tau}{4} \end{array} \right\}$$

can be equivalently described as:

$$\left\{ \frac{N_1\left(n\frac{\tau}{4}\right)}{n} = \frac{[n \cdot |\Delta_y|]}{n}, \frac{N_2\left(n\frac{\tau}{4}\right)}{n} = \frac{N_3\left(n\frac{\tau}{4}\right)}{n} = 0 \right\},$$

where

$$\begin{aligned} N_1 &\sim \left\{ \begin{array}{ll} \text{Pois}(\lambda_1), & \Delta_y > 0 \\ \text{Pois}(\mu_1), & \Delta_y \leq 0 \end{array} \right. \\ N_2 &\sim \left\{ \begin{array}{ll} \text{Pois}(\mu_1), & \Delta_y > 0 \\ \text{Pois}(\lambda_1), & \Delta_y \leq 0 \end{array} \right. \\ N_3 &\sim \text{Pois}(\lambda_0). \end{aligned}$$

By Lemma 18,

$$\begin{aligned} (85) \quad & \lim_{n\to\infty} \frac{1}{n} \mathbb{P}(\Theta_n) \\ & = \begin{cases} -\frac{\tau}{4}(\lambda_0 + \lambda_1 + \mu_1) + |\Delta_y| \left(1 + \log \frac{\lambda_1 \tau}{4|\Delta_y|}\right), & \Delta_y > 0 \\ -\frac{\tau}{4}(\lambda_0 + \lambda_1 + \mu_1) + |\Delta_y| \left(1 + \log \frac{\mu_1 \tau}{4|\Delta_y|}\right), & \Delta_y \leq 0 \end{cases} \end{aligned}$$

and by consolidating the branches, we further obtain

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{P}(\Theta_n) \geq -\frac{\tau}{4}(\lambda_0 + \lambda_1 + \mu_1) + |\Delta_y| \left(1 + \log \frac{\min\{\lambda_1, \mu_1\}\tau}{4|\Delta_y|}\right).$$

Before proceeding further, we shall perform a rough estimation of the movement of $\vec{z}_n$. Specifically,

$$\begin{aligned} |\Delta_y| &= |p_y - (r_y(u) + q_{1y})| \\ &\leq d(\vec{r}(u) + \vec{q}_1, \vec{p}) \\ &\leq d(\vec{r}(u) + \vec{q}_1, \vec{r}(u)) + d(\vec{r}(u), \vec{p}) \\ &\leq |\vec{q}_1| + d(\vec{r}(u), \vec{p}) \\ &\leq 2\max\{2\tau, d(\vec{r}(u), \vec{p})\}. \end{aligned}$$

If $1+\log\frac{\min\{\lambda_1,\mu_1\}\tau}{4|\Delta_y|}>0$, we can further obtain

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{P}(\Theta_n)\geq -\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1).$$

Otherwise, if $2\tau\geq d(\vec{r}(u),\vec{p})$ then $|\Delta_y|\leq 4\tau$ and

$$\begin{aligned}
&-\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1)+|\Delta_y|\left(1+\log\frac{\min\{\lambda_1,\mu_1\}\tau}{4|\Delta_y|}\right)\\
&\qquad\qquad\geq -\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1)+4\tau\left(1+\log\frac{\min\{\lambda_1,\mu_1\}\tau}{16\tau}\right)\\
&\qquad\qquad= -\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1)+4\tau\left(1+\log\frac{\min\{\lambda_1,\mu_1\}}{16}\right).
\end{aligned}$$

And finally, if $2\tau<d(\vec{r}(u),\vec{p})$ then $|\Delta_y|\leq 2d(\vec{r}(u),\vec{p})$ and

$$\begin{aligned}
&-\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1)+|\Delta_y|\left(1+\log\frac{\min\{\lambda_1,\mu_1\}\tau}{4|\Delta_y|}\right)\\
&\geq -\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1)+2d(\vec{r}(u),\vec{r}(\sigma))\left(1+\log\frac{\min\{\lambda_1,\mu_1\}\tau}{8d(\vec{r}(u),\vec{r}(\sigma))}\right)\\
&\geq -\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1)+2d(\vec{r}(u),\vec{r}(\sigma))\left(1+\log\frac{\min\{\lambda_1,\mu_1\}(\sigma-u)}{8d(\vec{r}(u),\vec{r}(\sigma))}\right)\\
&= -\frac{\tau}{4}(\lambda_0+\lambda_1+\mu_1)+2d(\vec{r}(u),\vec{r}(\sigma))\left(1+\log\frac{\min\{\lambda_1,\mu_1\}}{8}\right)\\
&\quad+2d(\vec{r}(u),\vec{r}(\sigma))\log\frac{\sigma-u}{d(\vec{r}(u),\vec{r}(\sigma))}.
\end{aligned}$$

Note, that since $\vec{r}(\sigma)$ is located on the boundary, the notions of $d(\vec{r}(u),\vec{r}(\sigma))$ and $|\vec{r}(\sigma)-\vec{r}(u)|$ coincide. Therefore we can use Proposition 12 to conclude that

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{P}(\Theta_n)\geq -C_1'\tau-C_2'd(\vec{r}(u),\vec{r}(\sigma))-C_3'\mathcal{I}_u^\sigma(\vec{r}). \tag{86}$$

for some constants C_1', C_2' and C_3', uniformly in $\vec{r}$.

Now we recall the construction we made in the beginning of the proof. Since the requirement for $\vec{z}_n$ to move along the given piecewise linear path was a restriction of the original event, we may state, using the Markov property of $\vec{z}_n$, that

$$\begin{aligned}
\mathbb{P}_{\vec{r}(u)+\vec{q}_1}&\begin{pmatrix} \vec{z}_n \in B_\epsilon(\vec{r}|_u^{u+\tau}), \\ \vec{z}_n(u+\tau) = \vec{r}(u+\tau)+\vec{q}_2 \end{pmatrix} \\
&\geq \mathbb{P}_{\vec{r}(u)+\vec{q}_1}\Big(\vec{z}_n \text{ goes directly to } (r_x(u)+q_{1x}, p_y, 0)\Big) \\
&\quad \times\ \mathbb{P}_{(r_x(u)+q_{1x},p_y,0)}\Big(\vec{z}_n \text{ goes directly to } \vec{p}\Big) \\
&\quad \times\ \mathbb{P}_{\vec{p}}\Big(\vec{z}_n \text{ goes directly to } (r_x(u+\tau)+q_{2x}, p_y, 1)\Big) \\
&\quad \times\ \mathbb{P}_{(r_x(u+\tau)+q_{2x},p_y,1)}\Big(\vec{z}_n \text{ goes directly to } \vec{r}(u+\tau)+\vec{q}_2\Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{r}(u)+\vec{q}_1}&\begin{pmatrix} \vec{z}_n \in B_\epsilon(\vec{r}|_u^{u+\tau}), \\ \vec{z}_n(u+\tau) = \vec{r}(u+\tau)+\vec{q}_2 \end{pmatrix} \\
&\geq \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{r}(u)+\vec{q}_1}\Big(\vec{z}_n \text{ goes to } (r_x(u)+q_{1x}, p_y, 0)\Big) \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{(r_x(u)+q_{1x},p_y,0)}\Big(\vec{z}_n \text{ goes to } \vec{p}\Big) \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{p}}\Big(\vec{z}_n \text{ goes to } (r_x(u+\tau)+q_{2x}, p_y, 1)\Big) \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{(r_x(u+\tau)+q_{2x},p_y,1)}\Big(\vec{z}_n \text{ goes to } \vec{r}(u+\tau)+\vec{q}_2\Big).
\end{aligned}$$

Using the estimate of the form (86) for each of the terms of the righthand side, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{r}(u)+\vec{q}_1}&\begin{pmatrix} \vec{z}_n \in B_\epsilon(\vec{r}|_u^{u+\tau}), \\ \vec{z}_n(u+\tau) = \vec{r}(u+\tau)+\vec{q}_2 \end{pmatrix} \\
&\geq -C_1\tau - C_2 d(\vec{r}(u), \vec{r}(\sigma)) - C_3 \mathcal{I}_u^{u+\tau}(\vec{r}) \\
&> -C_1\tau - C_2\delta - C_3 \mathcal{I}_u^{u+\tau}(\vec{r}).
\end{aligned}$$

Note that the righthand estimate is uniform with respect to $\vec{q}_1, \vec{q}_2$. □

Corollary 20. *Under the assumptions of Lemma 19, the following conclusion holds:*

$$\begin{aligned}
(87)\quad \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}&\left(\begin{matrix} \vec{z}_n \in B_\epsilon(\vec{r}|_u^{u+\tau}), \\ \vec{z}_n(u+\tau) = \vec{r}(u+\tau)+\vec{q}_2 \end{matrix}\ \middle|\ \vec{z}_n(u) \in A\right) \\
&> -C_1\tau - C_2\delta - C_3 \mathcal{I}_u^{u+\tau}(\vec{r}),
\end{aligned}$$

where A is any subset of the open ball $B_{\min\{\delta,\tau\}}\big(\vec{r}(u)\big)$.

5.2. **Proof of the lower bound.** Now it is the time to turn to the proof of the main result of this chapter, namely the local version of the large deviations principle, which was stated in Proposition 17.

Let us return once again to the overview of the proof, this time with more technicalities. We have shown that for each $\vec{r}$, the time interval $[0, T]$ is basically composed of intervals of various nature, namely $\mathcal{A}(\vec{r})$. We attempt to single out the most significant members of $\mathcal{A}(\vec{r})$, whether by their length, or by the length of the walk $\vec{r}$ makes during them. We assert, that these intervals bear the significant part of $\mathcal{I}_0^T(\vec{r})$, and the rest is negligible. For this purpose we need to evaluate three different situations.

First, when $(u, v) \in \mathcal{A}(\vec{r})$ is a significant interval, we need to verify that the behavior of $\vec{r}$ on (u, v) is simple enough for its cost to be estimated using the general theory. Second, when (u, v) is a period of time during which $\vec{r}$ stays near the origin, we must apply Theorem 3 to show that $\vec{z}_n$ can stay near $\vec{r}$ at a negligible cost during that time. Third, we need to estimate the total cost of all transitions between the above two states, and show once again that it is negligible.

Now let us proceed with the proof.

Proof of Proposition 17. Note that the claim (80) becomes trivial in the case $\mathcal{I}_0^T(\vec{r}) = \infty$. Thus we can safely consider only the paths $\vec{r}$ which bear finite cost, and are thus feasible and absolutely continuous.

Our first goal is to define a finite subset of $\mathcal{A}(\vec{r})$ which will contain the most significant part of it. For this purpose we consider the total length of the intervals in $\mathcal{A}(\vec{r})$:

$$L = \sum_{(u,v) \in \mathcal{A}(\vec{r})} |v - u| \leq T.$$

Choose $\alpha > 0$ and let $\mathcal{B}_1 \subseteq \mathcal{A}(\vec{r})$ be a finite collection of intervals with total length exceeding $L - \alpha$. Obviously, such a collection exists, as L is a countable sum. Furthermore, for any interval $(u, v) \in \mathcal{B}_1$, $\vec{r}$ doesn't visit the origin during it. Therefore it is possible to choose a positive δ small enough such that $\vec{r}$ wanders away from $B_\delta(0)$ during any interval in $\mathcal{B}_1$. Fix ϵ and choose a positive $\delta < \epsilon/10$ such that

$$\forall (u, v) \in \mathcal{B}_1 \quad \exists t \in (u, v) \quad d(\vec{r}(t), \vec{0}) > \delta.$$

Then $\mathcal{B}_1 \subseteq \mathcal{B}_\delta$, where $\mathcal{B}_\delta$ is defined in (54).

Let us elaborate on the above discussion. We have chosen $\mathcal{B}_1$ in such a manner that it contains almost all the contents of $\mathcal{A}(\vec{r})$ in terms of the total length. Then we expanded it with intervals which walk far away from the origin, and received the set $\mathcal{B}_\delta$, which is more comfortable to work with due to some of its known properties.

As we know from Proposition 8, $\mathcal{B}_\delta$ is finite and contains non-overlapping intervals. Therefore its members have a natural order on the real line. Consider the collection of all endpoints of $\mathcal{B}_\delta$ together with $\{0, T\}$. This collection is finite, and it can be enumerated as follows:

$$0 = u_1 < u_2 < \ldots < u_m < u_{m+1} = T.$$

Each interval (u_i, u_{i+1}) either belongs to $\mathcal{B}_\delta$, or constitutes a gap between two members (u_{i-1}, u_i) and (u_{i+1}, u_{i+2}) of $\mathcal{B}_\delta$. Of course, the leftmost interval (u_1, u_2) and the rightmost interval (u_m, u_{m+1}) may stay out of $\mathcal{B}_\delta$ and have just one neighbor in $\mathcal{B}_\delta$, but this exception is not critical for the further discussion.

Note also that since for all $i = 2, \dots, m$ u_i is an endpoint of some member of $\mathcal{B}_\delta$, one can immediately see that

$$\begin{aligned} m - 1 &\le 2|\mathcal{B}_\delta| \\ m &\le 2m_\delta + 1, \end{aligned} \tag{88}$$

where m_δ is defined in Lemma 8.

Our next step is a bit complicated. We decompose the interval $[0, T]$ into pieces by cutting it at the points $u_i \pm \tau$, where τ is some small increment which accomodates the transition state (see Figure 17). At each point $u_i + \tau$ we force $\vec{z}_n(u_i + \tau)$ into a position $\vec{q}_i$ which is close enough to $\vec{r}(u_i + \tau)$. This setting leaves us with intervals of the form $(u_{i-1} + \tau, u_i + \tau)$, where any such interval incorporates the range $(u_{i-1} + \tau, u_i - \tau)$ of homogeneous behavior and the range $(u_i - \tau, u_i + \tau)$ of transitional behavior. Our choice of τ and $\vec{q}_i$ is quite artificial, and it is dictated purely by technical reasons.

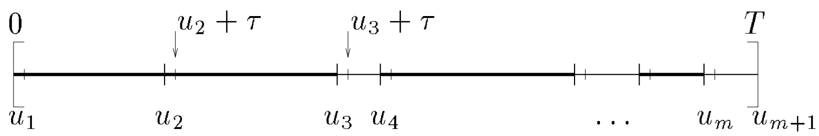


FIGURE 17. The decomposition of $[0, T]$ by $\mathcal{B}_\delta$.

Referring back to the explanation we provided right after the formulation of the proposition, the reader can already see the splitting of $[0, T]$ into pieces. Indeed, the time axis is almost entirely covered by intervals of $\mathcal{B}_\delta$, with small "gaps" remaining between them. One can guess from Figure 17, that there are two kinds of transition intervals: 2τ-long ones, like near u_2, and larger intervals like (u_3, u_4). As τ and δ vanish, the set $\mathcal{B}_\delta$ covers larger and larger parts of $\mathcal{A}(\vec{r})$, and the transitional intervals of both kinds become smaller. Consequently, as we seen in Corollary 20, the cost of staying in ϵ-neighborhood of $\vec{r}$ during a single transitional interval become smaller and eventually vanishes. With some extra work, we shall see at the end of the proof that the total cost of all such transitions vanishes too, despite their increasing number.

Recall that due to the absolute continuity of $\vec{r}$, we can find $\tau' > 0$ such that for any $u, v \in [0, T]$

$$|u - v| < \tau' \quad \text{implies} \quad d(\vec{r}(u), \vec{r}(v)) < \delta. \tag{89}$$

For each $i = 1, \dots, m$ let

$$\tau = \min\left\{\frac{\delta}{4}, \frac{\tau'}{2}, \frac{\alpha}{m}, \frac{u_2 - u_1}{6}, \frac{u_3 - u_2}{6}, \dots, \frac{u_{m+1} - u_m}{6}\right\}. \tag{90}$$

Define also

$$\vec{q}_i = \begin{cases} \vec{r}(u_i + \tau) + (\tau, \tau, 0), & (u_i, u_{i+1}) \in \mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_2(\vec{r})) \\ \vec{r}(u_i + \tau) + (\tau, \tau, 1), & (u_i, u_{i+1}) \in \mathcal{B}_\delta \cap (\mathcal{A}_1(\vec{r}) \cup \mathcal{A}_3(\vec{r})) \\ \vec{r}(u_i + \tau), & (u_i, u_{i+1}) \notin \mathcal{B}_\delta. \end{cases}$$

Now the probability in (80) can be estimated as follows:

$$\begin{aligned}
&\mathbb{P}_{\vec{x}}\Big(\vec{z}_n \in B_\epsilon(\vec{r})\Big) \\
&\qquad \geq \mathbb{P}_{\vec{r}(0)}\begin{pmatrix} \vec{z}_n \in B_\epsilon(\vec{r}) \text{ and} \\ \vec{z}_n(u_i + \tau) = \vec{q}_i \text{ for all } i = 1, \dots, m \end{pmatrix} \\
&\qquad = \mathbb{P}_{\vec{r}(0)}\begin{pmatrix} \vec{z}_n \in B_\epsilon(\vec{r}|_0^{u_1+\tau}) \text{ and } \vec{z}_n(u_1 + \tau) = \vec{q}_1, \\ \vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i + \tau) = \vec{q}_i \\ \text{for all } i = 1, \dots, m \ , \\ \vec{z}_n \in B_\epsilon(\vec{r}|_{u_m+\tau}^{T}) \end{pmatrix} \\
&\qquad = \mathbb{P}_{\vec{r}(0)}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_0^{u_1+\tau}) \text{ and } \vec{z}_n(u_1 + \tau) = \vec{q}_1\Big) \\
&\qquad\quad \times \prod_{i=2}^{m} \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i + \tau) = \vec{q}_i\Big) \\
&\qquad\quad \times \mathbb{P}_{\vec{q}_m}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_m+\tau}^{T})\Big)
\end{aligned}$$

We shall denote the terms of the righthand side as $\mathcal{P}_1^n$, $\mathcal{P}_2^n$, $\dots$, $\mathcal{P}_m^n$, $\mathcal{P}_{m+1}^n$ respectively, and using this notation, the last equation takes the form

$$\mathbb{P}_{\vec{x}}\Big(\vec{z}_n \in B_\epsilon(\vec{r})\Big) \geq \mathcal{P}_1^n \cdot \mathcal{P}_2^n \cdot \dots \cdot \mathcal{P}_{m+1}^n.$$

Furthermore,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}\Big(\vec{z}_n \in B_\epsilon(\vec{r})\Big) \geq \sum_{i=1}^{m+1} \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_i^n. \tag{91}$$

Now it is our goal to estimate the expression

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_i^n$$

for each $i = 1, 2, \dots, m + 1$.

We shall first consider the regular cases, i.e. $i = 2, \dots, m$, and then the two extreme cases $i = 1, m + 1$. For this purpose we first consider three different cases, depending on the nature of an interval (u_{i-1}, u_i) corresponding to $\mathcal{P}_i^n$.

I. Let $(u_{i-1}, u_i) \in \mathcal{B}_\delta \cap (\mathcal{A}_0(\vec{r}) \cup \mathcal{A}_1(\vec{r}))$. Obviously, $\mathcal{A}_0(\vec{r})$ and $\mathcal{A}_1(\vec{r})$ are treated in a similar way, so without loss of generality we may consider only the possibility $(u_{i-1}, u_i) \in \mathcal{B}_\delta \cap \mathcal{A}_0(\vec{r})$.

$\mathcal{P}_i^n$ is defined over the interval $(u_{i-1}+\tau, u_i+\tau)$. Consider the subinterval $(u_{i-1}+\tau, u_i-\tau)$. On this interval $\vec{r}(t)$ is generally located on $D_0 \cap \partial_y D$, while in the case of $\vec{x} \in \partial_x D$, as addressed in (52b), it can also reside on $\partial_x D$ at the beginning. If we translate $\vec{r}$ by $(\tau, \tau, 0)$, the resulting path would be located inside the ϵ-neighborhood of $\vec{r}|_{u_{i-1}+\tau}^{u_i-\tau}$ (see Figure 18). Furthermore, the τ-neighborhood of the translated path would be located entirely in D_0, and so the probability of $\vec{z}_n$ being inside it could be estimated by the general theory.

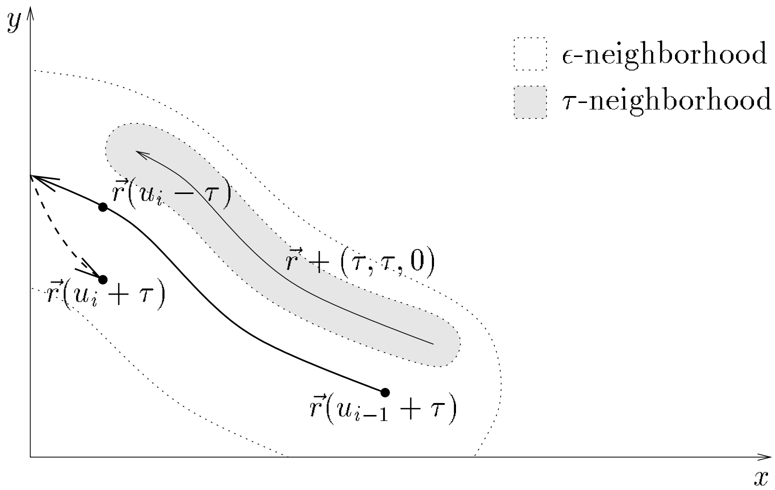


FIGURE 18. The case $(u_{i-1}, u_i) \in \mathcal{A}_0(\vec{r})$.

More specifically,

$$
\begin{aligned}
\mathcal{P}_i^n &= \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i+\tau) = \vec{q}_i\Big) \\
&\geq \mathbb{P}_{\vec{q}_{i-1}}\begin{pmatrix} \vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{u_i-\tau} + (\tau,\tau,0)), \\ \vec{z}_n \in B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i+\tau) = \vec{q}_i \end{pmatrix} \\
&= \mathbb{P}_{\vec{r}(u_{i-1}+\tau)+(\tau,\tau,0)}\Big(\vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{u_i-\tau} + (\tau,\tau,0))\Big) \\
&\quad \times \mathbb{P}\left(\begin{matrix} \vec{z}_n \in B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau}), \\ \vec{z}_n(u_i+\tau) = \vec{q}_i \end{matrix} \;\middle|\; d\begin{pmatrix} \vec{z}_n(u_i-\tau), \\ \vec{r}(u_i-\tau)+(\tau,\tau,0) \end{pmatrix} < \tau \right).
\end{aligned}
$$

Consequently,

$$\begin{aligned}
(92)\quad & \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_i^n \\
& \geq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{r}(u_{i-1}+\tau)+(\tau,\tau,0)}\Big(\vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{u_i-\tau} + (\tau,\tau,0))\Big) \\
& \quad + \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau}), \\ \vec{z}_n(u_i+\tau) = \vec{q}_i \end{array} \right| \\
& \qquad\qquad\qquad\qquad d\left(\begin{array}{c} \vec{z}_n(u_i-\tau), \\ \vec{r}(u_i-\tau)+(\tau,\tau,0) \end{array} \right) < \tau \Bigg).
\end{aligned}$$

Since any $\vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{u_i-\tau} + (\tau,\tau,0))$ is located on D_0, $\vec{z}_n$ can be coupled with the free motion ζ_n, and the first term of the righthand side of (92) can be estimated using the general Large Deviations Principle [SW95, 5.1(ii)]:

$$\begin{aligned}
(93)\quad & \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{r}(u_{i-1}+\tau)+(\tau,\tau,0)}\Big(\vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{u_i-\tau} + (\tau,\tau,0))\Big) \\
& \qquad\qquad \geq -I_{u_{i-1}+\tau}^{u_i-\tau}(\vec{r} + (\tau,\tau,0)) \\
& \qquad\qquad = -I_{u_{i-1}+\tau}^{u_i-\tau}(\vec{r}) = -\mathcal{I}_{u_{i-1}+\tau}^{u_i-\tau}(\vec{r}).
\end{aligned}$$

Note, that the first equality is due to the fact that λ_0, λ_1 and μ_0 are fixed over D_0, and thus a translation of a path doesn't change its cost.

The second term of the righthand side of (92) deals with the behavior of $\vec{z}_n$ during the transition state $(u_i-\tau, u_i+\tau)$. Therefore we will have to estimate it by means of Corollary 20 of Lemma 19. Let us verify the conditions of Lemma 19.

1. The vector $\vec{q}_1$, as referred in Lemma 19, denotes the displacement between $\vec{r}(u_i-\tau)$ and $\vec{z}_n(u_i-\tau)$, for any sample path $\vec{z}_n$ which satisfies the conditional part of the probability expression. Thus,
$$\begin{aligned}
|\vec{q}_1| &= d\big(\vec{r}(u_i-\tau), \vec{z}_n(u_i-\tau)\big) \\
&\leq d\big(\vec{r}(u_i-\tau), \vec{r}(u_i-\tau)+(\tau,\tau,0)\big) \\
&\quad + d\big(\vec{r}(u_i-\tau)+(\tau,\tau,0), \vec{z}_n(u_i-\tau)\big) \\
&< \tau\sqrt{2} + \tau < 4\tau.
\end{aligned}$$
2. The vector $\vec{q}_2$ denotes the displacement between $\vec{r}(u_i+\tau)$ and $\vec{z}_n(u_i+\tau)$ accordingly, so
$$|\vec{q}_2| = d\big(\vec{r}(u_i+\tau), \vec{z}_n(u_i+\tau)\big) = |(\tau,\tau,0)| = \tau\sqrt{2} < 4\tau.$$
3. The existence of the appropriate feasible path is illustrated by Figure 19. Obviously, only one among several possible situations is shown, but the other possibilities can be treated similarly. We leave it as an exercise to the reader to verify that the shown feasible path does belong to $B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau})$.

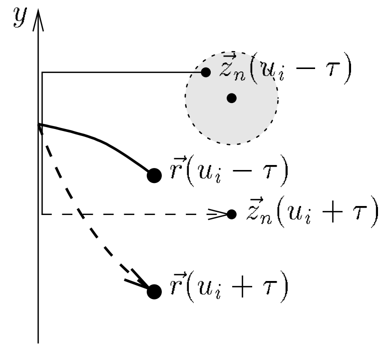


FIGURE 19. The transitional interval $(u_i-\tau, u_i+\tau)$.

4. As $\tau < \frac{\delta}{4}$, and $|\vec{q}_1|, |\vec{q}_2| < 4\tau$, this condition holds immediately.
5. Any $t_1, t_2 \in (u_i-\tau, u_i+\tau)$ satisfy $|t_1-t_2| < 2\tau \le \tau'$, and thus $d\big(\vec{r}(t_1), \vec{r}(t_2)\big) < \delta$ by (89).

The desired estimate now follows, and it takes the form

(94)
$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau}), \\ \vec{z}_n(u_i+\tau) = \vec{q}_i \end{array} \middle| d\left(\begin{array}{c} \vec{z}_n(u_i-\tau), \\ \vec{r}(u_i-\tau) + (\tau,\tau,0) \end{array} \right) < \tau \right)$$
$$\ge -C_1 \cdot 2\tau - C_2\delta - C_3 \mathcal{I}_{u_i-\tau}^{u_i+\tau}(\vec{r}).$$

Therefore, (92) together with (93), (94) implies

(95)
$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_i^n &\ge -\mathcal{I}_{u_{i-1}+\tau}^{u_i-\tau}(\vec{r}) - C_1 \cdot 2\tau - C_2\delta - C_3 \mathcal{I}_{u_i-\tau}^{u_i+\tau}(\vec{r}) \\ &= -\mathcal{I}_{u_{i-1}+\tau}^{u_i+\tau}(\vec{r}) - 2C_1\tau - C_2\delta - (C_3-1)\mathcal{I}_{u_i-\tau}^{u_i+\tau}(\vec{r}). \end{aligned}$$

II. Let $(u_{i-1}, u_i) \in \mathcal{B}_\delta \cap (\mathcal{A}_2(\vec{r}) \cup \mathcal{A}_3(\vec{r}))$. Without loss of generality we assume this time that $(u_{i-1}, u_i) \in \mathcal{B}_\delta \cap \mathcal{A}_3(\vec{r})$.

Recall the definition of $\mathcal{I}_{u_{i-1}}^{u_i}(\vec{r})$ in (59d) and let $s \in [u_{i-1}, u_i]$ be some point at which the expression in (59d) attains its minimum, with respect to the interval (u_{i-1}, u_i).

While discussing the definition, we already noticed that during the time (u_{i-1}, u_i) a sample path $\vec{z}_n$ which stays near $\vec{r}$, may move once from D_1 to D_0 by crossing $\partial_x D$. We wish to select s_i in order to define a transition state around it, but we don't want it to interfere with the transition states at the beginning and the end of (u_{i-1}, u_i), in case s is too close to one of them.

Therefore we define (see Figure 20)

$$s_i = \begin{cases} u_{i-1} + 3\tau, & s \in [u_{i-1}, u_{i-1}+3\tau) \\ s, & s \in [u_{i-1}+3\tau, u_i - 3\tau] \\ u_i - 3\tau, & s \in (u_i - 3\tau, u_i]. \end{cases}$$

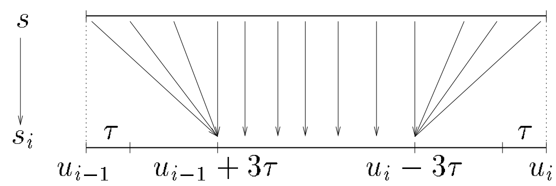


FIGURE 20. The definition of s_i.

Like we did in part I, we restrict the "habitat" of allowed sample paths $\vec{z}_n$ to force them to stay in the interior of some pane most of time, by restricting the transitional intervals. Specifically, we define

$$\vec{p} = \vec{r}(s_i + \tau) = (\tau, \tau, 0),$$

and state

(96)

$$\begin{aligned}
\mathcal{P}_i^n &= \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i+\tau) = \vec{q}_i\Big) \\
&\geq \mathbb{P}_{\vec{q}_{i-1}}\begin{pmatrix} \vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{s_i-\tau} + (\tau,\tau,1)), \\ \vec{z}_n \in B_\epsilon(\vec{r}|_{s_i-\tau}^{s_i+\tau}) \text{ and } \vec{z}_n(s_i+\tau) = \vec{p}, \\ \vec{z}_n \in B_\tau(\vec{r}|_{s_i+\tau}^{u_i-\tau} + (\tau,\tau,0)), \\ \vec{z}_n \in B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i+\tau) = \vec{q}_i \end{pmatrix} \\
&= \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{s_i-\tau} + (\tau,\tau,1))\Big) \\
&\quad \times \mathbb{P}\left(\begin{matrix} \vec{z}_n \in B_\epsilon(\vec{r}|_{s_i-\tau}^{s_i+\tau}), \\ \vec{z}_n(s_i+\tau) = \vec{p} \end{matrix} \;\middle|\; d\begin{pmatrix} \vec{z}_n(s_i-\tau), \\ \vec{r}(s_i-\tau) + (\tau,\tau,1) \end{pmatrix} < \tau\right) \\
&\quad \times \mathbb{P}_{\vec{p}}\Big(\vec{z}_n \in B_\tau(\vec{r}|_{s_i+\tau}^{u_i-\tau} + (\tau,\tau,0))\Big) \\
&\quad \times \mathbb{P}\left(\begin{matrix} \vec{z}_n \in B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau}), \\ \vec{z}_n(u_i+\tau) = \vec{q}_i \end{matrix} \;\middle|\; d\begin{pmatrix} \vec{z}_n(u_i-\tau), \\ \vec{r}(u_i-\tau) + (\tau,\tau,0) \end{pmatrix} < \tau\right)
\end{aligned}$$

As one can see from (96), we want $\vec{z}_n$ to stay near $\vec{r}+(\tau,\tau,1)$ during the time $(u_{i-1}+\tau, s_i-\tau)$, and this also implies staying on D_1 during that time. The next interval $(s_i-\tau, s_i+\tau)$ is a transitional interval, when $\vec{z}_n$ moves from D_1 to D_0, and then it stays on D_0 near $\vec{r}+(\tau,\tau,0)$. The last transitional interval is $(u_i-\tau, u_i+\tau)$. This approach is illustrated by Figure 21.

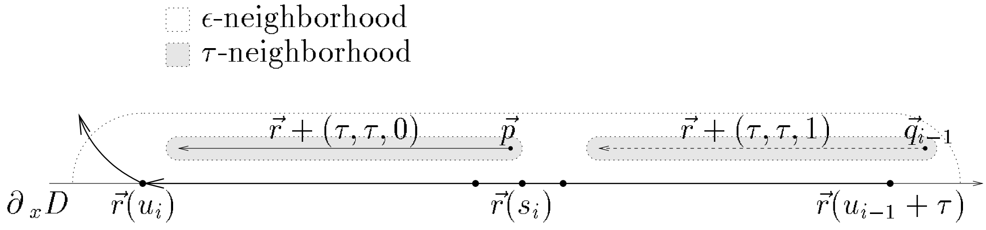


FIGURE 21. The case $(u_{i-1}, u_i) \in \mathcal{A}_3(\vec{r})$.

Now, the first and the third probability expressions in the righthand side of (96) can be bounded as follows:

$$\begin{aligned}(97)\quad \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\tau(\vec{r}|_{u_{i-1}+\tau}^{s_i-\tau} + (\tau,\tau,1))\Big) \\ \geq -J_{u_{i-1}+\tau}^{s_i-\tau}(\vec{r}+(\tau,\tau,1)) = -J_{u_{i-1}+\tau}^{s_i-\tau}(\vec{r}), \\ \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{p}}\Big(\vec{z}_n \in B_\tau(\vec{r}|_{s_i+\tau}^{u_i-\tau} + (\tau,\tau,0))\Big) \\ \geq -I_{s_i+\tau}^{u_i-\tau}(\vec{r}+(\tau,\tau,0)) = -I_{s_i+\tau}^{u_i-\tau}(\vec{r}).\end{aligned}$$

The second and the forth expressions in the righthand side of (96) can be bounded using Lemma 19, like we did in part I. The conditions of the Lemma are verified in a similar fashion, and we obtain

(98)

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\begin{matrix} \vec{z}_n \in B_\epsilon(\vec{r}|_{s_i-\tau}^{s_i+\tau}), \\ \vec{z}_n(s_i+\tau) = \vec{p} \end{matrix} \;\middle|\; d\begin{pmatrix} \vec{z}_n(s_i-\tau), \\ \vec{r}(s_i-\tau)+(\tau,\tau,1) \end{pmatrix} < \tau \right) \\ \geq -C_1\cdot 2\tau - C_2\delta - C_3\mathcal{I}_{s_i-\tau}^{s_i+\tau}(\vec{r}), \\ \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\begin{matrix} \vec{z}_n \in B_\epsilon(\vec{r}|_{u_i-\tau}^{u_i+\tau}), \\ \vec{z}_n(u_i+\tau) = \vec{q}_i \end{matrix} \;\middle|\; d\begin{pmatrix} \vec{z}_n(u_i-\tau), \\ \vec{r}(u_i-\tau)+(\tau,\tau,0) \end{pmatrix} < \tau \right) \\ \geq -C_1\cdot 2\tau - C_2\delta - C_3\mathcal{I}_{u_i-\tau}^{u_i+\tau}(\vec{r}).\end{aligned}$$

Putting (96), (97) and (98) together, we further obtain

$$(99)\quad \begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_i^n &\geq -J_{u_{i-1}+\tau}^{s_i-\tau}(\vec{r}) - C_1\cdot 2\tau - C_2\delta - C_3\mathcal{I}_{s_i-\tau}^{s_i+\tau}(\vec{r}) \\ &\quad - I_{s_i+\tau}^{u_i-\tau}(\vec{r}) - C_1\cdot 2\tau - C_2\delta - C_3\mathcal{I}_{u_i-\tau}^{u_i+\tau}(\vec{r}) \\ &= -\mathcal{I}_{u_{i-1}-\tau}^{u_i+\tau}(\vec{r}) - 4C_1\tau - 2C_2\delta \\ &\quad - (C_3-1)\big(\mathcal{I}_{s_i-\tau}^{s_i+\tau}(\vec{r}) + \mathcal{I}_{u_i-\tau}^{u_i+\tau}(\vec{r})\big).\end{aligned}$$

III. Let $(u_{i-1}, u_i) \notin \mathcal{B}_\delta$. In this case it follows from the definition (54) of $\mathcal{B}_\delta$, that

$$\forall t \in (u_{i-1}, u_i) \quad d(\vec{r}(t), (0,0)) < \delta.$$

As we know, once at the origin, the most probable behavior of $\vec{z}_n$ is to stay near the origin, and this can be achieved at a zero cost. Therefore, we seek to restrict $\vec{z}_n$ to some small neighborhood of $(0,0)$.

We shall consider two cases.

a. Assume, that $\vec{r}$ doesn't visit the origin during the time $(u_{i-1}+\tau, u_i-\tau)$, i.e.

$$\{t \in (u_{i-1}+\tau, u_i-\tau) \mid \vec{r}(t)=\vec{0}\} = \emptyset.$$

Note that the interval $(u_{i-1}+\tau, u_i-\tau)$ is entirely covered by intervals in $\mathcal{A}(\vec{r}) \setminus \mathcal{B}_\delta$.

We apply Lemma 19 to the entire interval $(u_{i-1}+\tau, u_i+\tau)$ and obtain

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_i^n \\ &= \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i+\tau)=\vec{q}_i\Big) \\ &\geq -C_1\cdot(u_i-u_{i-1}) - C_2\delta - C_3\mathcal{I}_{u_{i-1}+\tau}^{u_i+\tau}(\vec{r}) \\ &= -\mathcal{I}_{u_{i-1}+\tau}^{u_i+\tau}(\vec{r}) - C_1\cdot(u_i-u_{i-1}) - C_2\delta - (C_3-1)\mathcal{I}_{u_{i-1}+\tau}^{u_i+\tau}(\vec{r}). \end{aligned} \tag{100}$$

b. If $\vec{r}$ does visit the origin during the time $(u_{i-1}+\tau, u_i-\tau)$, then we can define

$$\begin{aligned} s_{i1} &= \inf\{t \in (u_{i-1}+\tau, u_i-\tau) \mid \vec{r}(t)=\vec{0}\}, \\ s_{i2} &= \sup\{t \in (u_{i-1}+\tau, u_i-\tau) \mid \vec{r}(t)=\vec{0}\}. \end{aligned}$$

Like we noted in the previous case, the intervals $(u_{i-1}+\tau, s_{i1})$ and $(s_{i2}, u_i-\tau)$ are entirely covered by $\mathcal{A}(\vec{r}) \setminus \mathcal{B}_\delta$.

By the continuity of $\vec{r}$, $\vec{r}(s_{i1}) = \vec{r}(s_{i2}) = \vec{0}$. This time we denote $(u_{i-1}+\tau, s_{i1})$ and $(s_{i2}, u_i+\tau)$ as transitional intervals for the purpose of application of Lemma 19, and write

$$\begin{aligned} \mathcal{P}_i^n &= \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i+\tau)=\vec{q}_i\Big) \\ &\geq \mathbb{P}_{\vec{q}_{i-1}}\begin{pmatrix} \vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{s_{i1}}) \text{ and } \vec{z}_n(s_{i1})=\vec{0}, \\ \vec{z}_n \in B_\tau(\vec{0}|_{s_{i1}}^{s_{i2}}), \\ \vec{z}_n \in B_\epsilon(\vec{r}|_{s_{i2}}^{u_i+\tau}) \text{ and } \vec{z}_n(u_i+\tau)=\vec{q}_i \end{pmatrix} \\ &= \mathbb{P}_{\vec{q}_{i-1}}\Big(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{s_{i1}}) \text{ and } \vec{z}_n(s_{i1})=\vec{0}\Big) \\ &\quad \times \mathbb{P}_{\vec{0}}\Big(\vec{z}_n \in B_\tau(\vec{0}|_{s_{i1}}^{s_{i2}})\Big) \\ &\quad \times \mathbb{P}\left(\begin{matrix} \vec{z}_n \in B_\epsilon(\vec{r}|_{s_{i2}}^{u_i+\tau}), \\ \vec{z}_n(u_i+\tau)=\vec{q}_i \end{matrix} \;\middle|\; d(\vec{z}_n(s_{i2}),\vec{0}) < \tau\right). \end{aligned} \tag{101}$$

It readily follows from (12b), that the initial condition $\vec{z}_\infty(s_{i1}) = \vec{0}$ implies

$$\forall t > s_{i1} \quad \vec{z}_\infty(t) = \vec{0}.$$

Moreover, by Theorem 3

$$\begin{aligned}1 &\geq \mathbb{P}_{\vec{0}}\left(\vec{z}_n \in B_\tau(\vec{0}|_{s_{i1}}^{s_{i2}})\right)\\ &\geq \mathbb{P}_{\vec{0}}\Big(\sup_{s_{i1}\leq t\leq s_{i2}} d(\vec{z}_n(t),\vec{0}) < \tau\Big)\\ &> 1 - C_1' e^{-nC_2'(\tau)},\end{aligned}$$

for some C_1' and $C_2'(\delta)$, as appropriate. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{0}}\left(\vec{z}_n \in B_\tau(\vec{0}|_{s_{i1}}^{s_{i2}})\right) = 0. \tag{102}$$

The first and the third term of the righthand side of (101) are bounded in the usual manner:

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{q}_{i-1}}&\left(\vec{z}_n \in B_\epsilon(\vec{r}|_{u_{i-1}+\tau}^{s_{i1}}) \text{ and } \vec{z}_n(s_{i1}) = \vec{0}\right)\\ &\geq -C_1\cdot(s_{i1} - (u_{i-1}+\tau)) - C_2\delta - C_3\mathcal{I}_{u_{i-1}+\tau}^{s_{i1}}(\vec{r})\end{aligned} \tag{103}$$

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}&\left(\begin{array}{c}\vec{z}_n \in B_\epsilon(\vec{r}|_{s_{i2}}^{u_i+\tau}),\\ \vec{z}_n(u_i+\tau) = \vec{q}_i\end{array}\middle|\; d(\vec{z}_n(s_{i2}),\vec{0}) < \tau\right)\\ &\geq -C_1\cdot(u_i+\tau - s_{i2}) - C_2\delta - C_3\mathcal{I}_{s_{i2}}^{u_i+\tau}(\vec{r}).\end{aligned}$$

Putting (101), (102) and (103) together, we obtain this time

(104)

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_i^n &\geq -C_1\cdot(s_{i1} - (u_{i-1}+\tau)) - C_2\delta - C_3\mathcal{I}_{u_{i-1}+\tau}^{s_{i1}}(\vec{r})\\ &\quad - C_1\cdot(u_i+\tau-s_{i2}) - C_2\delta - C_3\mathcal{I}_{s_{i2}}^{u_i+\tau}(\vec{r})\\ &\geq -C_1((u_i-u_{i-1}) - (s_{i2}-s_{i1})) - 2C_2\delta\\ &\quad - C_3(\mathcal{I}_{u_{i-1}+\tau}^{s_{i1}}(\vec{r}) + \mathcal{I}_{s_{i2}}^{u_i+\tau}(\vec{r})) - \mathcal{I}_{s_{i1}}^{s_{i2}}(\vec{r})\\ &= -\mathcal{I}_{u_{i-1}+\tau}^{u_i+\tau}(\vec{r}) - C_1((u_i-u_{i-1}) - (s_{i2}-s_{i1})) - 2C_2\delta\\ &\quad - (C_3-1)(\mathcal{I}_{u_{i-1}+\tau}^{s_{i1}}(\vec{r}) + \mathcal{I}_{s_{i2}}^{u_i+\tau}(\vec{r})).\end{aligned}$$

IV. We have yet to deal with the two extreme cases $\mathcal{P}_1^n$ and $\mathcal{P}_{m+1}^n$.

Regarding $\mathcal{P}_1^n$, we can note that its underlying interval is of length τ, and therefore it can be estimated directly using Proposition 19:

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{P}_1^n \geq -C_1\tau - C_2\delta - C_3\mathcal{I}_0^\tau(\vec{r}). \tag{105}$$

The case of $\mathcal{P}_{m+1}^n$ seems to be a bit more complicated, but in fact it can be related to one of the already treated cases I-III, depending on where the underlying interval (u_m, T) belongs to. Without going into details, we only notice that $\mathcal{P}_{m+1}^n$ is defined over the interval $(u_m+\tau, u_{m+1})$, and therefore its estimate would be similar to one like e.g. (95) or (99), with $\mathcal{I}_{(\cdot)}^{u_i+\tau}(\vec{r})$ replaced with $\mathcal{I}_{(\cdot)}^{u_m}(\vec{r})$, as appropriate, and 2τ perhaps replaced with τ.

It is time now to finalize the proof. As we stated in (91), the desired lower bound for the expression

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}\Big(\vec{z}_n \in B_\epsilon(\vec{r})\Big)$$

can be expressed as a sum of similar expressions involving $\mathcal{P}_1^n$, $\mathcal{P}_2^n$, $\dots$, $\mathcal{P}_{m+1}^n$. The latter have been discussed in various situations, as described in cases I-IV, and the appropriate bounds (95), (99), (100), (104) and (105) were obtained.

Note that the constants C_1, C_2 and C_3 involved in all of them are the same, since by Lemma 19 they are fixed for any fixed path $\vec{r}$.

It has only left to sum up the particular bounds for $\mathcal{P}_i^n$ to obtain the total lower bound. Let us do that carefully step by step:

1. Each bound contains a term of either the form $-C_2\delta$ or $-2C_2\delta$. Since C_2 is positive, we can safely bound the sum of $m+1$ such terms by $-2(m+1)C_2\delta$ from below.
2. Each of the bounds for $\mathcal{P}_2^n, \dots, \mathcal{P}_m^n$ contains the term $-\mathcal{I}_{u_{i-1}+\tau}^{u_i+\tau}(\vec{r})$, and the bounds for $\mathcal{P}_1^n$ and $\mathcal{P}_{m+1}^n$ contain the terms $-\mathcal{I}_0^{\tau}(\vec{r})$ and $-\mathcal{I}_{u_m+\tau}^{T}(\vec{r})$ respectively. As we noted in (62), $\mathcal{I}$ is an integral operator, additive with respect to the domain of integration. Therefore the sum of these terms constitutes $-\mathcal{I}_0^T(\vec{r})$.
3. In each of the cases I-III we defined the transitional intervals, as appropriate for each particular case. For convenience we shall define $\mathcal{V}$ to be the collection of all transitional intervals. Each transitional interval (u,v) is reflected in a bound of some $\mathcal{P}_i^n$ in the form $-(C_3-1)\mathcal{I}_u^v(\vec{r})$ and in the form $-C_1(v-u)$.

It now follows from (91), that

$$\begin{aligned}(106)\quad &\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}\Big(\vec{z}_n \in B_\epsilon(\vec{r})\Big)\\ &\geq -\mathcal{I}_0^T(\vec{r}) - 2(m+1)C_2\delta - (C_3-1)\sum_{(u,v)\in\mathcal{V}} \mathcal{I}_u^v(\vec{r}) - C_1 \sum_{(u,v)\in\mathcal{V}} (v-u).\end{aligned}$$

Let us examine the contents of $\mathcal{V}$ in detail:

1. The case I defines a transitional interval of the form $(u_i-\tau, u_i+\tau)$, with length 2τ.
2. The case II defines two intervals $(s_i-\tau, s_i+\tau)$ and $(u_i-\tau, u_i+\tau)$ with total length 4τ.
3. The case IIIa defines an interval $(u_{i-1}+\tau, u_i+\tau)$, of which $(u_{i-1}+\tau, u_i-\tau)$ is covered by $\mathcal{A}(\vec{r})\setminus\mathcal{B}_\delta$, and the remaining part $(u_i-\tau, u_i+\tau)$ has length 2τ.
4. The case IIIb defines two intervals $(u_{i-1}+\tau, s_{i1})$ and $(s_{i2}, u_i+\tau)$ of which $(u_{i-1}+\tau, s_{i1})$ and $(s_{i2}, u_i-\tau)$ are covered by $\mathcal{A}(\vec{r})\setminus\mathcal{B}_\delta$, and the remaining part $(u_i-\tau, u_i+\tau)$ has again length 2τ.

Recall that $\mathcal{B}_\delta$ was chosen in such a manner, that the total length of intervals in $\mathcal{A}(\vec{r})\setminus\mathcal{B}_\delta$ would not exceed α. Besides that, the contribution of each case to the total length of the collection $\mathcal{V}$ doesn't exceed 4τ. Thus the total length of $\mathcal{V}$ doesn't exceed $\alpha + 4\tau m$, and thus

$$-C_1 \sum_{(u,v)\in\mathcal{V}} (v-u) \geq -C_1(\alpha + 4\tau m).$$

Furthermore, consider the union of all transitional intervals

$$V = \bigcup_{\mathcal{V}} (u,v).$$

Obviously,

$$\sum_{(u,v)\in\mathcal{V}} \mathcal{I}_u^v(\vec{r}) = \int_0^T \mathbf{1}_V l_{\vec{r}}(t)dt,$$

where $l_{\vec{r}}$ is defined in (61). Note that the set V depends on α, δ and τ. Every function of a kind $\mathbf{1}_V l_{\vec{r}}(t)$ is dominated by $l_{\vec{r}}(t)$, and the Lebesgue integral of $l_{\vec{r}}(t)$ exists and is finite. We intend to show that, as α, δ and τ tend to zero,

$$\lim \mathbf{1}_V l_{\vec{r}}(t) = 0 \quad \text{a.s. in } [0,T].$$

Indeed, any $t \in \cup_{i=0}^3 A_i(\vec{r})$, except the countable number of points of crossing in $\mathcal{A}_2(\vec{r})$ or $\mathcal{A}_3(\vec{r})$, satisfies

$$\vec{r}(t) \neq \vec{0}.$$

Therefore for some δ and τ sufficiently small, $\vec{r}(t)$ would belong to some interval $(u,v) \in \mathcal{B}_\delta$, and the distance of t from either u or v will be larger than τ. When it happens, t would not belong to V, and it will satisfy

$$\mathbf{1}_V l_{\vec{r}}(t) = 0 \cdot l_{\vec{r}}(t) = 0.$$

On the contrary, for any $t \in A_5(\vec{r})$ trivially

$$\mathbf{1}_V l_{\vec{r}}(t) = \mathbf{1}_V \cdot 0 = 0.$$

Thus by the Lebesgue dominated convergence theorem,

$$\lim_{\alpha,\delta,\tau\to 0} \sum_{(u,v)\in\mathcal{V}} \mathcal{I}_u^v(\vec{r}) = \int_0^T 0dt = 0.$$

Now we shall see that except for $\mathcal{I}_0^T(\vec{r})$, the remaining terms of the righthand side of (106) converge to zero, as α, δ and τ vanish. Indeed, by Lemma 8,

$$\lim_{\delta\to 0} \delta m \leq \lim_{\delta\to 0} \delta(2m_\delta + 1) = 0.$$

Therefore,

$$0 \geq \lim_{\alpha,\delta,\tau\to 0} \Big(-C_1 \sum_{(u,v)\in\mathcal{V}} (v-u)\Big) \geq \lim_{\alpha,\delta,\tau\to 0} \big(-C_1(\alpha + m\delta)\big) = 0,$$

$$\lim_{\alpha,\delta,\tau\to 0} \Big(-2(m+1)C_2\delta\Big) = 0.$$

The objective (80) now follows from (106) by bringing α, δ and τ to zero. $\square$

Proof of Theorem 16. Once we have in possession the claim of Proposition 17, the theorem becomes its trivial consequence.

Indeed, let $\vec{r} \in G$ such that $\vec{r}(0) = \vec{x}$. Since G is open, there exists $\epsilon > 0$ such that $B_\epsilon(\vec{r}) \subseteq G$. Therefore by Proposition 17,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}(\vec{z}_n \in G) \geq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}(\vec{z}_n \in B_\epsilon(\vec{r})) \geq -\mathcal{I}_0^T(\vec{r}).$$

As the above relation holds for all $\vec{r}$ in G with $\vec{r}(0) = \vec{x}$, the statement (78) follows immediately. $\square$

5.3. **A note on the uniformity of lower bound.** In the context of the Large Deviations, the uniformity of (78) in $\vec{x}$ actually means that

$$\lim_{\vec{y}\to\vec{x}} \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{y}}(\vec{z}_n \in G) \geq -\inf \left\{ \mathcal{I}_0^T(\vec{r}) \mid \vec{r} \in G, \vec{r}(0) = \vec{x} \right\}.$$

Since the general theory addresses such uniformity, we can easily conclude that for any $\vec{x}$ laying in the interior of either pane, there is also some small open neighborhood of $\vec{x}$ laying in the same pane, and thus (79) follows from the general theory without complications.

For the case $\vec{x} \in \partial\mathbb{D}$ the uniformity may sometimes not hold. The appropriate counterexample may be derived from the discussion which followed Proposition 13. Recall that there was considered the setting of $\vec{x} \in \partial_x D$ and a path $\vec{r}$ which starts at $\vec{x}$ and runs along $\partial_x D$ for some time. Then for a small neighborhood G of $\vec{r}$ it held

$$\inf\left\{\mathcal{I}_0^T(\vec{q}) \mid \vec{q} \in G, \vec{q}(0) = \vec{x}\right\} \approx I_0^T(\vec{r}),$$

$$\lim_{\vec{y}\to\vec{x}} \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{y}}\big(\vec{z}_n \in G\big) \approx J_0^s(\vec{r}) + I_s^T(\vec{r}),$$

and the uniformity statement (79) fails.

6. The upper bound

This chapter concludes the proof of the Large Deviations Principle by establishing the upper bound.

6.1. **The local upper bound.**

Like we did in Chapter 5, we will first establish some sort of local upper bound. We seek to prove the following statement:

Proposition 21. *For any $\vec{r} \in \mathcal{D}^2[0,T]$,*

$$\limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r})\Big) \le -\mathcal{I}_0^T(\vec{r}), \tag{107}$$

where $\vec{a} = \vec{r}(0)$.

The proof of Proposition 21 will basically follow the same scheme that was applied in the proof of the local lower bound (Proposition 17). We consider an interval from, say, $\mathcal{A}_0(\vec{r})$. For small ϵ, any sample path $\vec{z}_n(t)$ which belongs to the ϵ-neighborhood of $\vec{r}$ on that interval, would be confined to the pane D_0, except, possibly, for some small time near the ends of the interval. Thus we would be able to couple $\vec{z}_n$ with ζ_n, and estimate its probability to stay near $\vec{r}$ in terms of the general theory.

In this manner we will show that for each interval in $\mathcal{A}(\vec{r})$, as ϵ tends to zero, the probability for $\vec{z}_n$ to stay in ϵ-neighborhood during that interval becomes closer related to the cost of the interval. Besides that, for the parts of $[0,T]$ which are not covered by $\mathcal{A}(\vec{r})$, the relevant probabilities are simply bounded by 1, which naturally corresponds to zero cost.

Proof. Let $\vec{r} \in \mathcal{D}^2[0,T]$. If $\vec{r}$ is not feasible, then there exists $\epsilon > 0$ such that

$$\mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in B_{2\epsilon}(\vec{r})\Big) = 0$$

for arbitrarily large n. In this case,

$$\limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r})\Big) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in B_{2\epsilon}(\vec{r})\Big) = -\infty,$$

and (107) trivially holds. Therefore it suffices to consider only the case of a feasible path $\vec{r}$.

Now let $\vec{r} \in \mathcal{D}^2[0,T]$ be a feasible path. Recall that $\mathcal{A}(\vec{r})$, as defined in the section 4.1, is the collection of all intervals inside $[0,T]$, during which $\vec{r}$ somehow leaves the origin.

Fix some $\alpha > 0$ and let $\mathcal{C}_\alpha$ be the set of intervals in $\mathcal{A}(\vec{r})$, defined as follows:

(108)
$$\mathcal{C}_\alpha = \Big\{(u,v) \in \mathcal{A}_0(\vec{r}) \cup \mathcal{A}_1(\vec{r}) \;\Big|\; \exists t \in (u,v) :\ r_x(t) > \alpha, r_y(t) > \alpha\Big\}$$
$$\cup \Big\{(u,v) \in \mathcal{A}_2(\vec{r}) \cup \mathcal{A}_3(\vec{r}) \;\Big|\; \exists t \in (u,v) :\ d(\vec{r}(t),(0,0)) > \alpha\Big\}.$$

In other words, we choose from $\mathcal{A}_0(\vec{r})$ and $\mathcal{A}_1(\vec{r})$ the intervals at which $\vec{r}$ travels at least α away from both boundaries; and from $\mathcal{A}_2(\vec{r})$ and $\mathcal{A}_3(\vec{r})$ we choose such intervals, that $\vec{r}$ travels on them α away from the origin.

Note once again, that we don't consider the case of $\vec{x} \in \partial\mathbb{D}$ mentioned in the definition (52b) of $A_0(\vec{r})$. Its treatment differs from the general case in a negligible way.

The definition of $\mathcal{C}_\alpha$ implies, that $\mathcal{C}_\alpha \subseteq \mathcal{B}_\alpha$, and therefore by Lemma 8, $\mathcal{C}_\alpha$ is a finite set. For convenience, we enumerate the members of $\mathcal{C}_\alpha$ in the increasing order:

$$\mathcal{C}_\alpha = \big\{(u_i, v_i)\big\}_{i=1}^{k},$$
$$0 \le u_1 < v_1 \le u_2 < v_2 \le \ldots \le u_k < v_k \le T.$$

Choose any positive $\epsilon < \alpha$, and consider the expression

$$\mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r})\Big).$$

By the Markov property, for any $\vec{x} \in \overline{B}_\epsilon(\vec{r}(v_k))$

$$\mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}) \text{ and } \vec{z}_n(v_k) = \vec{x}\Big)$$
$$= \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_0^{v_k}) \text{ and } \vec{z}_n(v_k) = \vec{x}\Big)$$
$$\times \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{v_k}^{T}) \;\Big|\; \vec{z}_n(v_k) = \vec{x}\Big).$$

Therefore, by taking supremum and summing over $\vec{x}$ we obtain

$$\mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r})\Big)$$
$$\le \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_0^{v_k})\Big) \cdot \sup_{\vec{x} \in \overline{B}_\epsilon(\vec{r}(v_k))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{v_k}^{T}) \;\Big|\; \vec{z}_n(v_k) = \vec{x}\Big).$$

Proceeding further, we obtain in a similar fashion

$$\begin{aligned}
\mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r})\Big) \le \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_0^{u_1})\Big) \\
&\times \prod_{i=1}^{k} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u_i))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{u_i}^{v_i}) \mid \vec{z}_n(u_i) = \vec{x}\Big) \\
&\times \prod_{i=1}^{k-1} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(v_i))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{v_i}^{u_{i+1}}) \mid \vec{z}_n(v_i) = \vec{x}\Big) \\
&\times \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(v_k))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{v_k}^{T}) \mid \vec{z}_n(v_k) = \vec{x}\Big) \\
&\le \prod_{i=1}^{k} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u_i))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{u_i}^{v_i}) \mid \vec{z}_n(u_i) = \vec{x}\Big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\frac{1}{n}\log \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r})\Big) \\
&\qquad \le \sum_{i=1}^{k} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u_i))} \frac{1}{n}\log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{u_i}^{v_i}) \mid \vec{z}_n(u_i) = \vec{x}\Big),
\end{aligned}$$

and thus

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}_{\vec{a}}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r})\Big) \\
&\quad \le \limsup_{n\to\infty} \sum_{i=1}^{k} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u_i))} \frac{1}{n}\log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{u_i}^{v_i}) \mid \vec{z}_n(u_i) = \vec{x}\Big) \\
&\quad \le \sum_{i=1}^{k} \limsup_{n\to\infty} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u_i))} \frac{1}{n}\log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{u_i}^{v_i}) \mid \vec{z}_n(u_i) = \vec{x}\Big).
\end{aligned} \tag{109}$$

Note that the lefthand side of (109) closely resembles the one of the objective (107). Thus in order to prove the objective it would be useful to obtain a separate upper bound for each term of the righthand side of (109). For this purpose we consider two principally different cases:

I. Let $(u,v) \in \mathcal{C}_\alpha \cap \mathcal{A}_1(\vec{r})$. The obvious approach in this case would be to show that $\vec{z}_n$ behaves almost like ξ_n during the interval (u,v). Since the points $\vec{r}(u)$ and $\vec{r}(v)$ normally lie on the boundary, we will have to take some subinterval $(\tilde{u},\tilde{v})$ of (u,v) in order to eliminate irregularities at the endpoints. This would allow us to claim that $\vec{z}_n$ stays near $\vec{r}$ during the interval $(\tilde{u},\tilde{v})$, and also remains on the pane D_1 throughout that time. Using the coupling, we would be able to restate the problem for the free motion ξ_n instead of $\vec{z}_n$, and then apply the tools provided by the general theory.

Let us now carry this out in detail. Define a subinterval $(\tilde{u},\tilde{v}) \subseteq (u,v)$ as follows:

$$\tilde{u} = \inf\bigl\{t \in (u,v): \quad r_x(t) \geq \alpha,\ r_y(t) \geq \alpha\bigr\},$$
$$\tilde{v} = \sup\bigl\{t \in (u,v): \quad r_x(t) \geq \alpha,\ r_y(t) \geq \alpha\bigr\}.$$

The definition (108) of $\mathcal{C}_\alpha$ ensures that $\tilde{u}$ and $\tilde{v}$ both exist and satisfy $\tilde{u} < \tilde{v}$. Note also that the case $(u,v) \in \mathcal{C}_\epsilon \cap \mathcal{A}_0(\vec{r})$ is similar to the one we consider, and thus it will not be considered separately.

We intend to establish an upper bound for the expression

$$\limsup_{n\to\infty} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))} \frac{1}{n} \log \mathbb{P}\Bigl(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \bigm| \vec{z}_n(u) = \vec{x}\Bigr).$$

Obviously, an application of the Markov property gives us once again

$$\begin{aligned}(110)\qquad \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))} & \mathbb{P}\Bigl(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \bigm| \vec{z}_n(u) = \vec{x}\Bigr) \\ & \leq \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(\tilde{u}))} \mathbb{P}\Bigl(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \bigm| \vec{z}_n(\tilde{u}) = \vec{x}\Bigr).\end{aligned}$$

Let $\vec{x} \in \overline{B}_\epsilon(\vec{r}(\tilde{u}))$, and let $\vec{z}_n$ be a sample path satisfying

$$\vec{z}_n(\tilde{u}) = \vec{x},$$
$$\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}).$$

Note that our choice of $\tilde{u}$ and $\tilde{v}$ implies that $\vec{z}_n(\tilde{u})$ and $\vec{z}_n(\tilde{v})$ both belong to D_0. Indeed, one can see that both $\vec{r}(\tilde{u})$ and $\vec{r}(\tilde{v})$ are at least α away from the boundary $\partial\mathbb{D}$, and $\vec{z}_n(\tilde{u})$ and $\vec{z}_n(\tilde{v})$ are at most ϵ away from $\vec{r}(\tilde{u})$ and $\vec{r}(\tilde{v})$ respectively.

Moreover, since $\vec{r}$ is feasible, its x-coordinate must not decrease during the time $[\tilde{u},\tilde{v}]$. Therefore, $\vec{z}_n$ may not touch the boundary $\partial_y D$ during that time, and thus it lies entirely on D_0 (see Figure 22). Now $\vec{z}_n$ can be coupled with the free motion ξ_n on $[\tilde{u},\tilde{v}]$, and the following equality holds:

$$\begin{aligned}(111)\quad & \mathbb{P}\Bigl(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \bigm| \vec{z}_n(\tilde{u}) = \vec{x}\Bigr) \\ & \qquad = \mathbb{P}\left(\begin{array}{c} \max_{\tilde{u}\leq t\leq\tilde{v}} \bigl|\xi_n(t) - \vec{r}(t)\bigr| \leq \epsilon, \\ \forall \tilde{u} \leq t \leq \tilde{v} \quad \xi_{n,y}(t) > 0 \end{array} \;\middle|\; \xi_n(\tilde{u}) = \vec{x}\right).\end{aligned}$$

The last expression contains the initial condition $\xi_n(\tilde{u}) = \vec{x}$. In order to bring it to the form suitable for application of the LDP for the free motion ξ, we need to devise some path which also starts at $\vec{x}$. We use for this purpose the original path $\vec{r}$, with appropriate translation on the plane (see Figure 23). Since we discuss the motion ξ, the translated path can be viewed in the realm of the Euclidean space $\mathbb{R}^2$. In particular, there will be nothing wrong if it deviates from the 1st quadrant. It is also worth noting that the translated path bears the same cost as

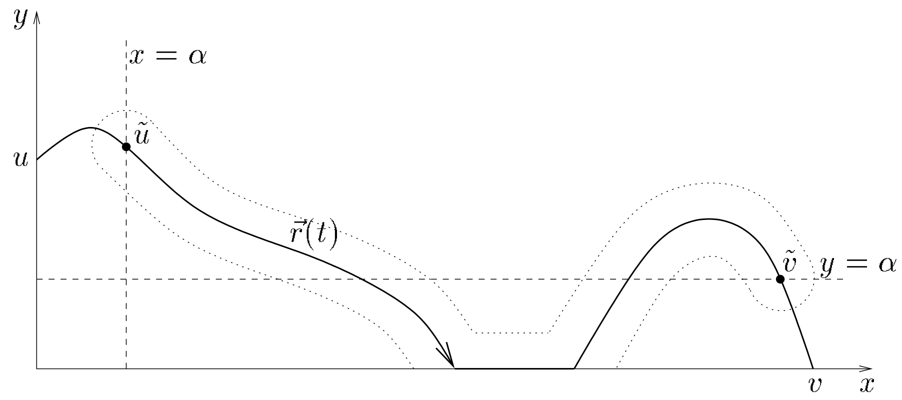


FIGURE 22. The ϵ-neighborhood of the path $\vec{r}(t)$.

the original, due to the fact that the generator of ξ is constant across the plane.

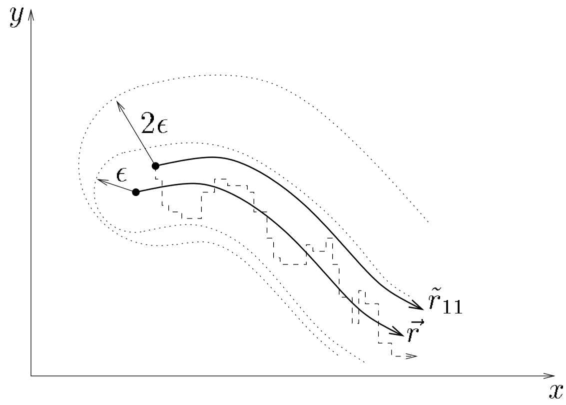


FIGURE 23. The translated path $\tilde{r}_{11}$ and its 2ϵ-neighborhood.

Define the translated path $\tilde{r}_{11}$ as follows:

$$\begin{aligned} \tilde{r}_1(t) &= \vec{r}(t+\tilde{u}), \\ \tilde{r}_{11}(t) &= \tilde{r}_1(t) + (\vec{x} - \tilde{r}_1(0)). \end{aligned}$$

Then

$$\begin{aligned} \mathbb{P}\left(\begin{matrix} \max_{\tilde{u}\le t\le \tilde{v}} \left|\xi_n(t) - \vec{r}(t)\right| \le \epsilon, \\ \forall \tilde{u} \le t \le \tilde{v} \quad \xi_{n,y}(t) > 0 \end{matrix} \;\middle|\; \xi_n(\tilde{u}) = \vec{x} \right) \tag{112} \\ \le \mathbb{P}\Big(\max_{\tilde{u}\le t\le \tilde{v}} \left|\xi_n(t) - \vec{r}(t)\right| \le \epsilon \;\Big|\; \xi_n(\tilde{u}) = \vec{x} \Big) \\ = \mathbb{P}_{\vec{x}}\Big(\max_{t\le \tilde{v}-\tilde{u}} \left|\xi_n(t) - \tilde{r}_1(t)\right| \le \epsilon \Big). \end{aligned}$$

Moreover, for any $t \le \tilde{v} - \tilde{u}$, if $\big|\xi_n(t) - \tilde{r}_1(t)\big| \le \epsilon$, then

$$\begin{aligned}\big|\xi_n(t) - \tilde{r}_{11}(t)\big| &= \big|\xi_n(t) - \tilde{r}_1(t) - (\vec{x} - \tilde{r}_1(0))\big| \\ &\le \big|\xi_n(t) - \tilde{r}_1(t)\big| + \big|\vec{r}(\tilde{u}) - \vec{x}\big| \\ &\le \epsilon + \epsilon = 2\epsilon,\end{aligned}$$

and thus

(113)

$$\begin{aligned}\mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \;\big|\; \vec{z}_n(\tilde{u}) = \vec{x}\Big) &\overset{(111),(112)}{\le} \mathbb{P}_{\vec{x}}\Big(\sup_{t \le \tilde{v} - \tilde{u}} \big|\xi_n(t) - \tilde{r}_1(t)\big| \le \epsilon\Big) \\ &\le \quad \mathbb{P}_{\vec{x}}\Big(\sup_{t \le \tilde{v} - \tilde{u}} \big|\xi_n(t) - \tilde{r}_{11}(t)\big| \le 2\epsilon\Big).\end{aligned}$$

But since the generator of ξ_n is constant across the entire plane, the last expression doesn't depend on $\vec{x}$, and thus one can take

$$\vec{x} = \vec{r}(\tilde{u}), \quad \tilde{r}_{11}(t) = \tilde{r}_1(t)$$

to get

$$\begin{aligned}&\mathbb{P}_{\vec{x}}\Bigg(\sup_{t \le \tilde{v} - \tilde{u}} \big|\xi_n(t) - \tilde{r}_{11}(t)\big| \le 2\epsilon\Bigg) \\ &\qquad = \mathbb{P}_{\tilde{r}_1(0)}\Bigg(\sup_{t \le \tilde{v} - \tilde{u}} \big|\xi_n(t) - \tilde{r}_1(t)\big| \le 2\epsilon\Bigg).\end{aligned}$$

Furthermore, using (113) we obtain

$$\begin{aligned}(114) \qquad &\sup_{\vec{x} \in \overline{B}_\epsilon(\vec{r}(\tilde{u}))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \;\big|\; \vec{z}_n(\tilde{u}) = \vec{x}\Big) \\ &\qquad \le \mathbb{P}_{\tilde{r}_1(0)}\Bigg(\sup_{t \le \tilde{v} - \tilde{u}} \big|\xi_n(t) - \tilde{r}_1(t)\big| \le 2\epsilon\Bigg),\end{aligned}$$

and

$$\begin{aligned}&\limsup_{n \to \infty} \sup_{\vec{x} \in \overline{B}_\epsilon(\vec{r}(u))} \frac{1}{n} \log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \;\big|\; \vec{z}_n(u) = \vec{x}\Big) \\ &\quad \overset{(110)}{\le} \limsup_{n \to \infty} \sup_{\vec{x} \in \overline{B}_\epsilon(\vec{r}(\tilde{u}))} \frac{1}{n} \log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \;\big|\; \vec{z}_n(\tilde{u}) = \vec{x}\Big) \\ &\quad \overset{(114)}{\le} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{\tilde{r}_1(0)}\Bigg(\sup_{t \le \tilde{v} - \tilde{u}} \big|\xi_n(t) - \tilde{r}_1(t)\big| \le 2\epsilon\Bigg).\end{aligned}$$

We can now apply the Large Deviation Principle [SW95, Thm 5.1(i)] to ξ_n and obtain

$$(115)\quad \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\tilde{r}_1(0)}\Big(\sup_{t\le \tilde{v}-\tilde{u}} \big|\xi_n(t) - \tilde{r}_1(t)\big| \le 2\epsilon \Big)$$
$$\le -\inf\Big\{ J_0^{\tilde{v}-\tilde{u}}(\vec{q}) : \sup_{t\le \tilde{v}-\tilde{u}} \big|\vec{q}(t) - \tilde{r}_1(t)\big| \le 2\epsilon,\ \vec{q}(0) = \tilde{r}_1(0) \Big\}$$
$$= -\inf\Big\{ J_{\tilde{u}}^{\tilde{v}}(\vec{q}) : \sup_{\tilde{u}\le t\le \tilde{v}} \big|\vec{q}(t) - \vec{r}(t)\big| \le 2\epsilon,\ \vec{q}(\tilde{u}) = \vec{r}(\tilde{u}) \Big\},$$

where J is the rate function associated with the free motion ξ (see section 2.3). The desired estimate thus takes the form

$$(116)\quad \limsup_{n\to\infty} \sup_{\vec{x}\in \overline{B}_\epsilon(\vec{r}(u))} \frac{1}{n} \log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \ \big|\ \vec{z}_n(u) = \vec{x} \Big)$$
$$\le -\inf\Big\{ J_{\tilde{u}}^{\tilde{v}}(\vec{q}) : \sup_{\tilde{u}\le t\le \tilde{v}} \big|\vec{q}(t) - \vec{r}(t)\big| \le 2\epsilon,\ \vec{q}(\tilde{u}) = \vec{r}(\tilde{u}) \Big\}.$$

II. Let $(u, v) \in \mathcal{C}_\alpha \cap \mathcal{A}_3(\vec{r})$. Once again, we consider some sample path $\vec{z}_n$. We wish to confine it to a single pane, in order to be able to perform estimation through coupling with a free motion. Unlike in the previous case, this time we face some uncertainity regarding the exact moment at which $\vec{z}_n$ touches the boundary $\partial_y D$ and passes from D_1 to D_0. We deal with this uncertainity by splitting (u, v) into small subintervals and considering separately each case where $\vec{z}_n$ touches the boundary $\partial_y D$ within some specific small interval. This approach allows us to confine $\vec{z}_n$ to some pane most of the time, with uncertainities only near the endpoints of (u, v) and within the specific subinterval. Besides that, the reasoning and the resulting estimate heavily resemble what we have seen in case I.

We choose $(\tilde{u}, \tilde{v}) \subseteq (u, v)$ to be the largest possible subinterval of (u, v) such that

$$\forall t \in (\tilde{u}, \tilde{v}) \quad r_x(t) > \alpha.$$

Again, the definition (108) of $\mathcal{C}_\alpha$ ensures that $(\tilde{u}, \tilde{v})$ exists and is non-empty. Note that in Figure 24 v and $\tilde{v}$ coincide, but this need not always be the case.

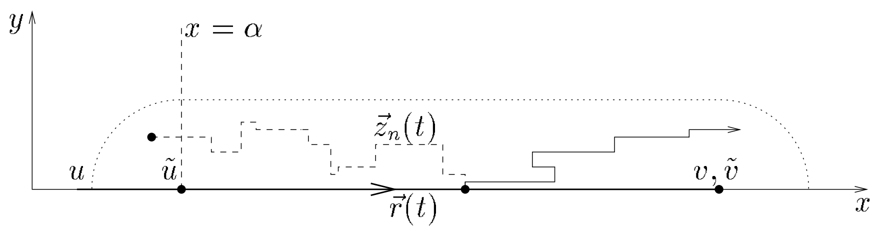


FIGURE 24. A sample path $\vec{z}_n(t)$ near the path $\vec{r}(t)$.

As in case I,

$$(117)\quad \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \mid \vec{z}_n(u) = \vec{x}\Big) \le \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(\tilde{u}))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \mid \vec{z}_n(\tilde{u}) = \vec{x}\Big).$$

Consider $\vec{x} \in \overline{B}_\epsilon(\vec{r}(\tilde{u}))$ and a sample path $\vec{z}_n$ such that

$$\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}),$$
$$\vec{z}_n(\tilde{u}) = x.$$

Since the distance between $\vec{r}(t)$ and the origin exceeds ϵ, it is clear that $\vec{z}_n$ can't arrive to the y-axis during the time $(\tilde{u}, \tilde{v})$. Therefore $\vec{z}_n$ may touch the x-axis at most once during that time, and then it will move from D_1 to D_0. For the sake of simplicity we shall say that $\vec{z}_n$ "touches" the x-axis at the moment $\tilde{u}$, when in fact it stays at D_0 all the time, and subsequently we shall also say that $\vec{z}_n$ touches the x-axis at the moment $\tilde{v}$, when in fact it stays at D_1. One can easily check that such "cheating" would not affect either the reasoning or the conclusions.

Let $\pi = \{\tilde{u} = w_0 < w_1 < w_2 < \ldots < w_{m-1} < w_m = \tilde{v}\}$ be a finite partition of $(\tilde{u}, \tilde{v})$ with some small diameter. The above discussion allows us to state, that

$$(118)\quad \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \mid \vec{z}_n(\tilde{u}) = \vec{x}\Big) \le \sum_{i=1}^{m} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}),\ \vec{z}_n \text{ crosses the } x\text{-axis} \\ \text{at a moment } s \in [w_{i-1}, w_i] \end{array} \,\middle|\, \vec{z}_n(\tilde{u}) = \vec{x}\right).$$

Consider now the expression

$$\mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}),\ \vec{z}_n \text{ crosses the } x\text{-axis} \\ \text{at a moment } s \in [w_{i-1}, w_i] \end{array} \,\middle|\, \vec{z}_n(\tilde{u}) = \vec{x}\right)$$

for some specific interval $[w_{i-1}, w_i]$. Each sample path $\vec{z}_n$ that satisfies such conditions, obviously stays on D_1 at the time $(\tilde{u}, w_{i-1})$, and stays on D_0 at the time $(w_i, \tilde{v})$. Therefore, we obtain using the Markov property:

$$
(119)\quad \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}),\ \vec{z}_n \text{ crosses the } x\text{-axis} \\ \text{at a moment } s \in [w_{i-1}, w_i] \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right)
$$

$$
= \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}), \\ \forall \tilde{u} < t < w_{i-1} \quad \vec{z}_n(t) \in D_1, \\ \forall w_i < t < \tilde{v} \quad \vec{z}_n(t) \in D_0 \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right)
$$

$$
\overset{MP}{\leq} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{w_{i-1}}), \\ \forall \tilde{u} < t < w_{i-1} \quad \vec{z}_n(t) \in D_1 \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right)
$$

$$
\times \max_{\vec{y} \in \overline{B}_\epsilon(\vec{r}(w_{i-1}))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{w_{i-1}}^{w_i}) \;\Big|\; \vec{z}_n(w_{i-1}) = \vec{y} \Big)
$$

$$
\times \max_{\vec{y} \in \overline{B}_\epsilon(\vec{r}(w_i))} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{w_i}^{\tilde{v}}), \\ \forall w_i < t < \tilde{v} \quad \vec{z}_n(t) \in D_0 \end{array} \;\middle|\; \vec{z}_n(w_i) = \vec{y} \right)
$$

$$
\leq \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{w_{i-1}}), \\ \forall \tilde{u} < t < w_{i-1} \quad \vec{z}_n(t) \in D_1 \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right)
$$

$$
\times \max_{\vec{y} \in \overline{B}_\epsilon(\vec{r}(w_i))} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{w_i}^{\tilde{v}}), \\ \forall w_i < t < \tilde{v} \quad \vec{z}_n(t) \in D_0 \end{array} \;\middle|\; \vec{z}_n(w_i) = \vec{y} \right).
$$

As long as $\vec{z}_n$ stays on a single pane, it can be coupled with an appropriate free motion. Thus the related probabilities can be computed in terms of that free motions. Specifically,

$$
\mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{w_{i-1}}), \\ \forall \tilde{u} < t < w_{i-1} \quad \vec{z}_n(t) \in D_1 \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right)
$$

$$
= \mathbb{P}\left(\begin{array}{c} \max_{\tilde{u} < t < w_{i-1}} \big|\xi_n(t) - \vec{r}(t)\big| \leq \epsilon, \\ \forall \tilde{u} < t < w_{i-1} \quad \xi_{n,y}(t) > 0 \end{array} \;\middle|\; \xi_n(\tilde{u}) = \vec{x} \right).
$$

As we did in case I, we can define

$$
\tilde{r}_1(t) = \vec{r}(t + \tilde{u}),
$$

and then

$$
(120)\quad \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{w_{i-1}}), \\ \forall \tilde{u} < t < w_{i-1} \quad \vec{z}_n(t) \in D_1 \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right)
$$

$$
\leq \mathbb{P}_{\tilde{r}_1(0)}\bigg(\sup_{t < w_{i-1} - \tilde{u}} \big|\xi_n(t) - \tilde{r}_1(t)\big| \leq 2\epsilon \bigg).
$$

The probability related to the second part of $\vec{z}_n$'s movement, namely that occurs during the time $(w_i, \tilde{v})$, can be estimated accordingly. Indeed, for any $\vec{y} \in \overline{B}_\epsilon(\vec{r}(w_i))$,

$$(121)\quad \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{w_i}^{\tilde{v}}), \\ \forall w_i < t < \tilde{v} \quad \vec{z}_n(t) \in D_0 \end{array} \;\middle|\; \vec{z}_n(w_i) = \vec{y} \right) \leq \mathbb{P}_{\tilde{r}_2(0)}\left(\sup_{t < \tilde{v} - w_i} \left|\zeta_n(t) - \tilde{r}_2(t)\right| \leq 2\epsilon \right),$$

where $\tilde{r}_2$ is defined as

$$\tilde{r}_1(t) = \vec{r}(t + w_i).$$

Since the last expression doesn't depend on $\vec{y}$, we can further state that

$$(122)\quad \max_{\vec{y} \in \overline{B}_\epsilon(\vec{r}(w_i))} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{w_i}^{\tilde{v}}), \\ \forall w_i < t < \tilde{v} \quad \vec{z}_n(t) \in D_0 \end{array} \;\middle|\; \vec{z}_n(w_i) = \vec{y} \right) \leq \mathbb{P}_{\tilde{r}_2(0)}\left(\sup_{t < \tilde{v} - w_i} \left|\zeta_n(t) - \tilde{r}_2(t)\right| \leq 2\epsilon \right).$$

We now conclude from the above discussion, that

$$\begin{aligned}(123)\quad & \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}),\ \vec{z}_n \text{ crosses the } x\text{-axis} \\ \text{at a moment } s \in [w_{i-1}, w_i] \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right) \\ & \overset{(119)}{\leq} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{w_{i-1}}), \\ \forall \tilde{u} < t < w_{i-1} \quad \vec{z}_n(t) \in D_1 \end{array} \;\middle|\; \vec{z}_n(\tilde{u}) = \vec{x} \right) \\ & \quad \times \max_{\vec{y} \in \overline{B}_\epsilon(\vec{r}(w_i))} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{w_i}^{\tilde{v}}), \\ \forall w_i < t < \tilde{v} \quad \vec{z}_n(t) \in D_0 \end{array} \;\middle|\; \vec{z}_n(w_i) = \vec{y} \right) \\ & \overset{(120),(122)}{\leq} \mathbb{P}_{\tilde{r}_1(0)}\left(\sup_{t < w_{i-1} - \tilde{u}} \left|\xi_n(t) - \tilde{r}_1(t)\right| \leq 2\epsilon \right) \\ & \quad \times \mathbb{P}_{\tilde{r}_2(0)}\left(\sup_{t < \tilde{v} - w_i} \left|\zeta_n(t) - \tilde{r}_2(t)\right| \leq 2\epsilon \right).\end{aligned}$$

As the last expression doesn't depend on $\vec{x}$,

$$
\begin{aligned}
(124)\quad & \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \;\Big|\; \vec{z}_n(u)=\vec{x}\Big) \\
& \overset{(117)}{\leq} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(\tilde{u}))} \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}) \;\Big|\; \vec{z}_n(\tilde{u})=\vec{x}\Big) \\
& \overset{(118)}{\leq} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(\tilde{u}))} \sum_{i=1}^{m} \mathbb{P}\left(\begin{array}{c} \vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_{\tilde{u}}^{\tilde{v}}), \\ \vec{z}_n \text{ crosses the } x\text{-axis at} \\ \text{a moment } s\in[w_{i-1},w_i] \end{array} \;\middle|\; \vec{z}_n(\tilde{u})=\vec{x}\right) \\
& \overset{(123)}{\leq} \sum_{i=1}^{m}\Bigg(\mathbb{P}_{\tilde{r}_1(0)}\Big(\sup_{t<w_{i-1}-\tilde{u}}\big|\xi_n(t)-\tilde{r}_1(t)\big|\leq 2\epsilon\Big) \\
& \qquad\qquad \times \mathbb{P}_{\tilde{r}_2(0)}\Big(\sup_{t<\tilde{v}-w_i}\big|\zeta_n(t)-\tilde{r}_2(t)\big|\leq 2\epsilon\Big)\Bigg) \\
& \leq m\cdot\max_{i=1,\dots,m}\Bigg(\mathbb{P}_{\tilde{r}_1(0)}\Big(\sup_{t<w_{i-1}-\tilde{u}}\big|\xi_n(t)-\tilde{r}_1(t)\big|\leq 2\epsilon\Big) \\
& \qquad\qquad \times \mathbb{P}_{\tilde{r}_2(0)}\Big(\sup_{t<\tilde{v}-w_i}\big|\zeta_n(t)-\tilde{r}_2(t)\big|\leq 2\epsilon\Big)\Bigg).
\end{aligned}
$$

Furthermore, we can obtain

$$
\begin{aligned}
& \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))} \frac{1}{n}\log\mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \;\Big|\; \vec{z}_n(u)=\vec{x}\Big) \\
& \leq \frac{1}{n}\log m + \max_{i=1,\dots,m}\Bigg[\frac{1}{n}\log\mathbb{P}_{\tilde{r}_1(0)}\Big(\sup_{t<w_{i-1}-\tilde{u}}\big|\xi_n(t)-\tilde{r}_1(t)\big|\leq 2\epsilon\Big) \\
& \qquad\qquad + \frac{1}{n}\log\mathbb{P}_{\tilde{r}_2(0)}\Big(\sup_{t<\tilde{v}-w_i}\big|\zeta_n(t)-\tilde{r}_2(t)\big|\leq 2\epsilon\Big)\Bigg].
\end{aligned}
$$

Since the free motions ζ and ξ satisfy the Large Deviations Principle [SW95, Thm 5.1(i)], it follows that

$$
\begin{aligned}
(125)\quad & \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{\tilde{r}_1(0)}\Big(\sup_{t<w_{i-1}-\tilde{u}}\big|\xi_n(t)-\tilde{r}_1(t)\big|\leq 2\epsilon\Big) \\
& \qquad \leq -\inf\Big\{J_0^{w_{i-1}-\tilde{u}}(\vec{q}):\ \big|\vec{q}(t)-\tilde{r}_1(t)\big|\leq 2\epsilon,\ \vec{q}(0)=\tilde{r}_1(0)\Big\}, \\
& \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{\tilde{r}_2(0)}\Big(\sup_{t<\tilde{v}-w_i}\big|\zeta_n(t)-\tilde{r}_2(t)\big|\leq 2\epsilon\Big) \\
& \qquad \leq -\inf\Big\{I_0^{\tilde{v}-w_i}(\vec{q}):\ \big|\vec{q}(t)-\tilde{r}_2(t)\big|\leq 2\epsilon,\ \vec{q}(0)=\tilde{r}_2(0)\Big\}.
\end{aligned}
$$

Note, that as we make n tend to infinity, $\frac{1}{n}\log m$ vanishes, so

$$(126)\quad \limsup_{n\to\infty} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))} \frac{1}{n}\log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \mid \vec{z}_n(u)=\vec{x}\Big)$$

$$\leq \limsup_{n\to\infty} \max_{i=1,\dots,m} \Bigg[\frac{1}{n}\log \mathbb{P}_{\tilde{r}_1(0)}\Big(\sup_{t<w_{i-1}-\tilde{u}} \big|\xi_n(t)-\tilde{r}_1(t)\big| \leq 2\epsilon\Big)$$

$$+\frac{1}{n}\log \mathbb{P}_{\tilde{r}_2(0)}\Big(\sup_{t<\tilde{v}-w_i} \big|\zeta_n(t)-\tilde{r}_2(t)\big| \leq 2\epsilon\Big)\Bigg]$$

$$\leq \max_{i=1,\dots,m} \Bigg[\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}_{\tilde{r}_1(0)}\Big(\sup_{t<w_{i-1}-\tilde{u}} \big|\xi_n(t)-\tilde{r}_1(t)\big| \leq 2\epsilon\Big)$$

$$+\limsup_{n\to\infty}\frac{1}{n}\log \mathbb{P}_{\tilde{r}_2(0)}\Big(\sup_{t<\tilde{v}-w_i} \big|\zeta_n(t)-\tilde{r}_2(t)\big| \leq 2\epsilon\Big)\Bigg]$$

$$\stackrel{(125)}{\leq} \max_{i=1,\dots,m} \Bigg[-\inf\Big\{J_0^{w_{i-1}-\tilde{u}}(\vec{q}):\ \big|\vec{q}(t)-\tilde{r}_1(t)\big| \leq 2\epsilon,\ \vec{q}(0)=\tilde{r}_1(0)\Big\}$$

$$-\inf\Big\{I_0^{\tilde{v}-w_i}(\vec{q}):\ \big|\vec{q}(t)-\tilde{r}_2(t)\big| \leq 2\epsilon,\ \vec{q}(0)=\tilde{r}_2(0)\Big\}\Bigg],$$

and this concludes the second case.

The important outcome of both cases I and II is that we successfully bounded each of the probabilities at the righthand side of (109) in the terms of the rate functions I and J of the appropriate free motions. Recall that the rate functions $I(\vec{q})$ and $J(\vec{q})$ are lower-semicontinuous with respect to $\vec{q}$ (see [SW95, Cor. 5.50]), and thus for any $a<b$

$$\limsup_{\epsilon\to 0}\Big(-\inf\Big\{I_a^b(\vec{q}):\ \sup_{a\leq t\leq b}\big|\vec{q}(t)-\vec{r}(t)\big| \leq 2\epsilon,\ \vec{q}(a)=\vec{r}(a)\Big\}\Big)$$

$$\leq -I_a^b(\vec{r}),$$

$$\limsup_{\epsilon\to 0}\Big(-\inf\Big\{J_a^b(\vec{q}):\ \sup_{a\leq t\leq b}\big|\vec{q}(t)-\vec{r}(t)\big| \leq 2\epsilon,\ \vec{q}(a)=\vec{r}(a)\Big\}\Big)$$

$$\leq -J_a^b(\vec{r}).$$

It now follows from (116) that for any $(u,v)\in\mathcal{C}_\alpha\cap\mathcal{A}_1(\vec{r})$ with appropriately defined $(\tilde{u},\tilde{v})$,

$$(127)\quad \limsup_{\epsilon\to 0}\limsup_{n\to\infty} \sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))} \frac{1}{n}\log \mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \mid \vec{z}_n(u)=\vec{x}\Big)$$

$$\leq -J_{\tilde{u}}^{\tilde{v}}(\vec{r}) = -\mathcal{I}_{\tilde{u}}^{\tilde{v}}(\vec{r}).$$

Accordingly, for any $(u,v) \in \mathcal{C}_\alpha \cap \mathcal{A}_3(\vec{r})$ with appropriate subinterval $(\tilde{u},\tilde{v})$ and its section $\pi = \{w_i\}_{i=0}^m$, it follows from (126), that

$$\text{(128)}\quad \limsup_{\epsilon\to 0}\limsup_{n\to\infty}\sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))}\frac{1}{n}\log\mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \mid \vec{z}_n(u)=\vec{x}\Big) \le \max_{i=1,\dots,m}\big\{-J_{\tilde{u}}^{w_{i-1}}(\vec{r}) - I_{w_i}^{\tilde{v}}(\vec{r})\big\}.$$

The results for the cases $(u,v) \in \mathcal{C}_\alpha \cap \mathcal{A}_0(\vec{r})$ and $(u,v) \in \mathcal{C}_\alpha \cap \mathcal{A}_2(\vec{r})$ are formulated in the same way, by switching I and J.

Now we shall perform the final step of the proof by taking α to zero.

Lemma. *In the above terms,*

$$\text{(129)}\quad \limsup_{\epsilon\to 0}\limsup_{n\to\infty}\sup_{\vec{x}\in\overline{B}_\epsilon(\vec{r}(u))}\frac{1}{n}\log\mathbb{P}\Big(\vec{z}_n \in \overline{B}_\epsilon(\vec{r}|_u^v) \mid \vec{z}_n(u)=\vec{x}\Big) \le -\mathcal{I}_u^v(\vec{r}).$$

for any $(u,v) \in \mathcal{A}(\vec{r})$.

Proof. As one can see from (108), for any member (u,v) of $\mathcal{A}(\vec{r})$ there exists some α small enougn such that $\mathcal{C}_\alpha$ contains (u,v). Moreover, for any such (u,v), the subinterval $(\tilde{u},\tilde{v})$ converges to (u,v) as α tends to zero. We shall prove (129) for all possible cases of (u,v).

Let $(u,v) \in \mathcal{A}_1(\vec{r})$. Consider a sequence of values of α which monotonically tends to zero (to avoid cumbersome notation, we don't employ indexes at the moment). By the definition of $\tilde{u}$ and $\tilde{v}$, they both converge monotonically to u and v respectively, and thus the interval $(\tilde{u},\tilde{v})$ grows to the outside towards (u,v).

Accordingly, the functions of form $l_{\vec{r}}(t)\cdot\mathbf{1}_{(\tilde{u},\tilde{v})}$ constitute a monotonically increasing sequence of non-negative measurable functions. Therefore we can apply to them the Lebesgue Monotone Convergence Theorem and obtain

$$\begin{aligned}
\lim_{\alpha\to 0} J_{\tilde{u}}^{\tilde{v}}(\vec{r}) &= \lim_{\alpha\to 0}\int_{\tilde{u}}^{\tilde{v}} l_{\vec{r}}(t)dt \\
&= \lim_{\alpha\to 0}\int_u^v l_{\vec{r}}(t)\cdot\mathbf{1}_{(\tilde{u},\tilde{v})}dt \\
&= \int_u^v \lim_{\alpha\to 0} l_{\vec{r}}(t)\cdot\mathbf{1}_{(\tilde{u},\tilde{v})}dt \\
&= \int_u^v l_{\vec{r}}(t)dt = \mathcal{I}_u^v(\vec{r}).
\end{aligned}$$

The objective (129) now follows immediately from (127). The treatment of the case $(u,v) \in \mathcal{A}_0(\vec{r})$ is, of course, similar.

For $(u,v) \in \mathcal{A}_2(\vec{r}) \cup \mathcal{A}_3(\vec{r})$ the situation is somewhat more complicated. Consider, for example, an interval $(u,v) \in \mathcal{A}_3(\vec{r})$. It is sufficient to prove that

$$\liminf_{\substack{\alpha\to 0\\ |\pi|\to 0}} \min_{i=1,\dots,m} \left\{ J_{\tilde{u}}^{w_{i-1}}(\vec{r}) + I_{w_i}^{\tilde{v}}(\vec{r}) \right\} \geq \mathcal{I}_u^v(\vec{r}), \tag{130}$$

and the combination of this result with (128) would immediately yield (129).

Fix some partition π. Just like we did in the previous case, we can show that for any specific interval $[w_{i-1}, w_i]$ in π,

$$\lim_{\alpha\to 0} J_{\tilde{u}}^{w_{i-1}}(\vec{r}) = J_u^{w_{i-1}}(\vec{r}),$$
$$\lim_{\alpha\to 0} I_{w_i}^{\tilde{v}}(\vec{r}) = I_{w_i}^{v}(\vec{r}).$$

Furthermore,

$$\lim_{\alpha\to 0}\left(J_{\tilde{u}}^{w_{i-1}}(\vec{r}) + I_{w_i}^{\tilde{v}}(\vec{r}) \right) = J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^{v}(\vec{r}),$$

(131)

$$\liminf_{\alpha\to 0} \min_{i=1,\dots,m} \left\{ J_{\tilde{u}}^{w_{i-1}}(\vec{r}) + I_{w_i}^{\tilde{v}}(\vec{r}) \right\} \geq \min_{i=1,\dots,m} \left\{ J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^{v}(\vec{r}) \right\}.$$

It has remained to show that

$$\liminf_{|\pi|\to 0} \min_{i=1,\dots,m} \left\{ J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^{v}(\vec{r}) \right\} \geq \mathcal{I}_u^v(\vec{r}).$$

Let us assume first, that $\mathcal{I}_u^v(\vec{r})$ is finite. Since $\mathcal{I}$ can be represented as an integral operator of a.s. finite measurable function $l_{\vec{r}}(t)$, the expression $\mathcal{I}_u^t(\vec{r})$ is increasing and continuous in t. Therefore for any $\delta > 0$ there exists a partition π such that for any interval $[w_{i-1}, w_i]$ in π, $\mathcal{I}_{w_{i-1}}^{w_i}(\vec{r}) < \delta$.

Recall the definition of $\mathcal{I}_u^v(\vec{r})$, and let s be a point at which the minimum of the expression (59d) is attained. Without loss of generality, we can assume that s is among the partition points of π. Assume also that $\min\{J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^{v}(\vec{r})\}$ is attained for some i_0. Now we shall consider separately the cases $w_{i_0} \leq s$ and $w_{i_0-1} \geq s$. Let us treat only the first case, and note that the second one is treated similarly.

For the case $w_{i_0} \le s$, we have that

$$
\begin{aligned}
J_{w_{i_0-1}}^{w_{i_0}}(\vec{r}) &= \int_{w_{i_0-1}}^{w_{i_0}} l_\zeta(t)dt \\
&= \int_{w_{i_0-1}}^{w_{i_0}} l_{\vec{r}}(t)dt \\
&= \mathcal{I}_{w_{i_0-1}}^{w_{i_0}}(\vec{r}) < \delta. \\
\min\Big\{ J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^v(\vec{r}) \Big\} &= J_u^{w_{i_0-1}}(\vec{r}) + I_{w_{i_0}}^v(\vec{r}) \\
&= J_u^{w_{i_0}}(\vec{r}) + I_{w_{i_0}}^v(\vec{r}) - J_{w_{i_0-1}}^{w_{i_0}}(\vec{r}) \\
&\ge J_u^s(\vec{r}) + I_s^v(\vec{r}) - J_{w_{i_0-1}}^{w_{i_0}}(\vec{r}) \\
&\ge \mathcal{I}_u^v(\vec{r}) - \delta.
\end{aligned}
$$

By choosing the appropriate π, the last inequality can be shown to hold for any $\delta > 0$. Therefore, the result (130) follows immediately.

Now consider the case $\mathcal{I}_u^v(\vec{r}) = \infty$. Then for any $s \in [u, v]$,

$$
J_u^s(\vec{r}) + I_s^v(\vec{r}) = \infty. \tag{132}
$$

Assume to the contrary, that

$$
\liminf_{|\pi| \to 0} \min_{i=1,\dots,m} \Big\{ J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^v(\vec{r}) \Big\} < \infty. \tag{133}
$$

Consider a sequence of partitions $\{\pi_j\}_{j=1}^\infty$ of (u, v) such that the diameter $|\pi_j|$ tends to zero and

$$
\lim_{j \to \infty} \min_{i=1,\dots,m_j} \Big\{ J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^v(\vec{r}) \Big\} = K < \infty.
$$

Without loss of generality we can assume that each π_j is a refinement of its predecessor, i.e. $\pi_{j-1} \subset \pi_j$.

Let

$$
\min_{i=1,\dots,m_j} \Big\{ J_u^{w_{i-1}}(\vec{r}) + I_{w_i}^v(\vec{r}) \Big\} = J_u^{w_{a_j-1}}(\vec{r}) + I_{w_{a_j}}^v(\vec{r}) < \infty.
$$

We assert that the sequence $\{[w_{a_j-1}, w_{a_j}]\}$ is monotone with respect to set inclusion. Indeed, if $w_{a_j} \le w_{a_{j-1}-1}$ for some j (see Figure 25), then

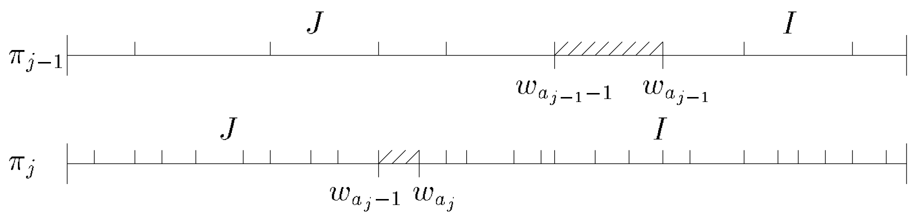


FIGURE 25. Monotone partitions of (u, v).

$$
J_u^{w_{a_{j-1}-1}}(\vec{r}) + I_{w_{a_{j-1}-1}}^v(\vec{r}) \le J_u^{w_{a_{j-1}-1}}(\vec{r}) + I_{w_{a_j}}^v(\vec{r}) < \infty,
$$

contrary to (132). Similarly, if $w_{a_{j-1}} \le w_{a_j - 1}$, then

$$J_u^{w_{a_j-1}}(\vec r) + I^v_{w_{a_{j-1}}}(\vec r) < \infty.$$

We have thus shown that the intervals $[w_{a_j-1}, w_{a_j}]$ indeed form a nested sequence, and thus

$$\lim_{j\to\infty} w_{a_j-1} = \lim_{j\to\infty} w_{a_j} = w$$

for some $w \in [u, v]$. Furthermore,

$$\begin{aligned} \lim_{j\to\infty} J_u^{w_{a_j-1}}(\vec r) &= J_u^w(\vec r) \\ \lim_{j\to\infty} I^v_{w_{a_j}}(\vec r) &= I^v_w(\vec r) \\ \lim_{j\to\infty}\Big(J_u^{w_{a_j-1}}(\vec r) + I^v_{w_{a_j}}(\vec r)\Big) &= J_u^w(\vec r) + I^v_w(\vec r) = K < \infty, \end{aligned}$$

and this contradicts (132). The contradiction shows that the assumption (133) is false, and this completes the proof of the lemma. □

By summing inequalities of the form (129) over all intervals $(u, v) \in \mathcal{A}(\vec r)$ we obtain that

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log \mathbb{P}_{\vec a}\Big(\vec z_n \in \overline{B}_\epsilon(\vec r)\Big) \le -\mathcal{I}_0^T(\vec r), \tag{134}$$

as required.

□

6.2. **Upper bound for general sets.**

Corollary 22. *Let $F \subseteq \mathcal{D}^2[0, T]$ be a compact set of paths, and let $\vec x \in \mathbb{D}$. Then*

$$\limsup_{n\to\infty}\frac{1}{n}\log \mathbb{P}_{\vec x}\big(\vec z_n \in F\big) \le -\inf\big\{\mathcal{I}_0^T(\vec r) : \ \vec r \in F, \ \vec r(0) = \vec x\big\}. \tag{135}$$

Proof. Note that we can safely assume that all paths $\vec r \in F$ originate in $\vec x$. Indeed, by neglecting paths that do not satisfy this condition, we stay with a compact subset of F, and neither side of (135) is affected.

Let $\delta > 0$. Consider some $\vec r \in F$. By Proposition 21, there exists some positive ϵ such that

$$\begin{aligned} \limsup_{n\to\infty}\frac{1}{n}\log \mathbb{P}_{\vec x}\big(\vec z_n \in B_\epsilon(\vec r)\big) &\le \limsup_{n\to\infty}\frac{1}{n}\log \mathbb{P}_{\vec x}\big(\vec z_n \in \overline{B}_\epsilon(\vec r)\big) \\ &\le -\mathcal{I}_0^T(\vec r) + \delta. \end{aligned}$$

Thus, for any $\vec r \in F$ there exists some small open neighborhood $B(\vec r)$ which satisfies

$$\limsup_{n\to\infty}\frac{1}{n}\log \mathbb{P}_{\vec x}\big(\vec z_n \in B(\vec r)\big) \le -\mathcal{I}_0^T(\vec r) + \delta. \tag{136}$$

The collection of such neighborhoods for all paths $\vec{r}$ is obviously an open cover of the compact set F, and therefore it has a finite subcover:

$$F \subseteq \bigcup_{i=1}^{k} B(\vec{r}_i).$$

Then

$$\begin{aligned}
\mathbb{P}_{\vec{x}}\big(\vec{z}_n \in F\big) &\le \sum_{i=1}^{k} \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in B(\vec{r}_i)\big) \\
&\le k \cdot \max_{i=1,2,\dots,k} \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in B(\vec{r}_i)\big), \\
\frac{1}{n}\log \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in F\big) &\le \frac{1}{n}\log k + \frac{1}{n}\log \max_{i=1,2,\dots,k} \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in B(\vec{r}_i)\big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(137) \quad \limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in F\big) & \\
&\le \limsup_{n\to\infty} \frac{1}{n}\log \max_{i=1,2,\dots,k} \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in B(\vec{r}_i)\big) \\
&= \max_{i=1,2,\dots,k} \limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in B(\vec{r}_i)\big) \\
&\overset{(136)}{\le} \max_{i=1,2,\dots,k} \big\{-\mathcal{I}_0^T(\vec{r}_i) + \delta\big\} \\
&= -\min_{i=1,2,\dots,k} \mathcal{I}_0^T(\vec{r}_i) + \delta \\
&\le -\inf\big\{\mathcal{I}_0^T(\vec{r}):\ \vec{r} \in F,\ \vec{r}(0) = \vec{x}\big\} + \delta.
\end{aligned}$$

Since the inequality (137) holds for any $\delta > 0$, it follows that

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in F\big) \le -\inf\big\{\mathcal{I}_0^T(\vec{r}):\ \vec{r} \in F,\ \vec{r}(0) = \vec{x}\big\}.$$

□

In order to extend the result of Corollary 22 to any closed set, we shall make use of the notion of exponential tightness (see definition in [DZ93, p. 8]). In the following lemma we establish the exponential tightness of $\{\vec{z}_n\}$, as it is formulated for families of random walks.

Lemma 23. *Let $C \in \mathbb{D}$ be a compact set. For each $\alpha < \infty$ there exists a compact set $\mathcal{K}_\alpha \in \mathcal{D}^2[0,T]$, such that for all $\vec{x} \in C$*

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}_{\vec{x}}\big(\vec{z}_n \notin \mathcal{K}_\alpha\big) \le -\alpha.$$

In this lemma we need to show that, roughly speaking, there are compact sets which cover a very large portion of $\mathcal{D}^2[0,T]$ on the exponential scale. For this purpose we convert our random motion to a free two-dimensional random motion by simply stripping the service events. Clearly, if a sample path of the free motion stays inside some large square next to the origin, then the appropriate sample path of the exhaustive polling will also stay inside that square (simply because it is being further slowed by service events).

We know that the family of scaled free motions is exponentially tight and thus satisfies the lemma in the first place. Given that, comparing the probabilities of the free motion and the exhaustive polling would give us the desired result.

Proof. Consider the two-dimensional free motion ψ with arrival rates λ_0 on the x-axis and λ_1 on the y-axis, i.e. the motion described by generator

$$L_\psi f(\vec{a}) = \lambda_0 f\big(\vec{a} + (1,0)\big) + \lambda_1 f\big(\vec{a} + (0,1)\big) - (\lambda_0 + \lambda_1) f(\vec{a}), \quad \vec{a} \in \mathbb{R}^2.$$

and the appropriate scaled motions ψ_n. By [SW95, Lemma 5.58], the family of random motions $\{\psi_n\}_{n=1}^{\infty}$ is exponentially tight.

Fix $\alpha \in \mathbb{R}$ and let $C_1 \in \mathbb{R}^2$ be the "flattening" of the set C:

$$C_1 = \big\{\vec{a} \in \mathbb{R}^2 \;\big|\; (a_x, a_y, s) \in C \text{ for some } s\big\}.$$

Due to the exponential tightness of $\{\psi_n\}$, there exists a compact set $\mathcal{K} \subset C^2[0,T]$ such that for all $\vec{x} \in C_1$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}\big(\psi_n \notin \mathcal{K}\big) \le -\alpha. \tag{138}$$

Let

$$x_{\max} = \sup\big\{x \;\big|\; x = q_x(t) \text{ for some } \vec{q} \in \mathcal{K}, t \in [0,T]\big\},$$
$$y_{\max} = \sup\big\{y \;\big|\; y = q_y(t) \text{ for some } \vec{q} \in \mathcal{K}, t \in [0,T]\big\},$$

and define

$$\mathcal{K}_1 = \left\{\vec{q} \in C^2[0,T] \;\middle|\; \forall t \in [0,T] \begin{matrix} 0 \le q_x(t) \le x_{\max} \\ 0 \le q_y(t) \le y_{\max} \end{matrix}\right\}.$$

Due to the compactness of $\mathcal{K}$, $x_{\max}$ and $y_{\max}$ are finite, and thus $\mathcal{K}_1$ is a compact set too. Note that $\mathcal{K}_1$ contains $\mathcal{K}$ as a subset.

Define $\mathcal{K}_\alpha \in \mathcal{D}^2[0,T]$ as the "two-fold instance" of the set $\mathcal{K}_1$:

$$\mathcal{K}_\alpha = \Big\{\vec{r} \in \mathcal{D}^2[0,T] \;\big|\; \big(r_x, r_y\big) = \vec{q} \text{ for some } \vec{q} \in \mathcal{K}_1\Big\}.$$

The set $\mathcal{K}_\alpha$ is obviously compact. Furthermore, we can couple $\vec{z}_n$ and ψ_n by considering the probability space of arrival/service waiting times and defining ψ_n as the motion which consists solely of arrivals. By the virtue of this coupling,

$$\mathbb{P}_{\vec{x}}\big(\vec{z}_n \in \mathcal{K}_\alpha\big) \ge \mathbb{P}_{\vec{x}}\big(\psi_n \in \mathcal{K}_1\big) \ge \mathbb{P}_{\vec{x}}\big(\psi_n \in \mathcal{K}\big),$$

and therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}\big(\vec{z}_n \notin \mathcal{K}_\alpha\big) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}\big(\psi_n \notin \mathcal{K}\big) \le -\alpha.$$

□

Theorem 24. *Let $F \subseteq \mathcal{D}^2[0,T]$ be a closed set of paths, and let $\vec{x} \in \mathbb{D}$. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{\vec{x}}\big(\vec{z}_n \in F\big) \leq -\inf\big\{\mathcal{I}_0^T(\vec{r}) : \ \vec{r} \in F, \ \vec{r}(0) = \vec{x}\big\}. \tag{139}$$

Proof. We have already shown that (139) is satisfied for any compact F (see Corollary 22), and that the family of scaled random walks $\{\vec{z}_n\}$ is exponentially tight (see Lemma 23). The objective of the theorem now immediately follows from [DZ93, Lemma 1.2.18(a)]. □

Appendix A. Properties of $\mathbb{D}$

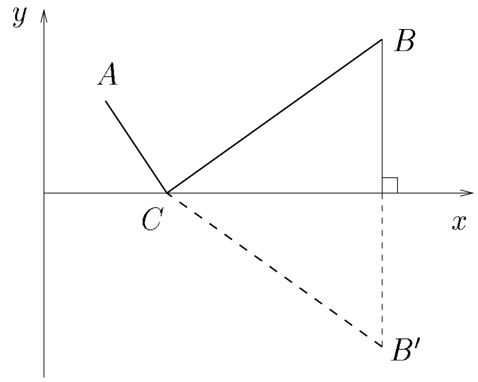
Proposition 25. *Let $x_0, y_0, x_1, y_1 \in \mathbb{R}^+$. Then the expression*

$$|(x_0, y_0) - (x, 0)| + |(x, 0) - (x_1, y_1)|$$

attains a minimum over $x \in \mathbb{R}^+$ at the point

$$(x, 0) = \left(x_0 \frac{y_1}{y_0 + y_1} + x_1 \frac{y_0}{y_0 + y_1}, 0 \right). \tag{140}$$

Proof. We shall present the reader with a simple geometric proof. Denote the points $A = (x_0, y_0)$, $B = (x_1, y_1)$ and let $C = (x, 0)$ for some $x \in \mathbb{R}^+$. Consider the point B' which is the reflection of B relatively



to the x-axis. By the triangle inequality,

$$\begin{aligned} &|(x_0, y_0) - (x, 0)| + |(x, 0) - (x_1, y_1)| \\ &\qquad = |AC| + |BC| = |AC| + |B'C| \geq |AB'| \end{aligned}$$

and the equivalence is attained only when C lies on AB'. The point of intersection of AB' with x-axis is precisely the one defined by (140). $\square$

Appendix B. Some elementary results

Lemma 26. *Let $\alpha \in [0, 1]$ and $k \in \mathbb{N}$, such that $1 - (1 - \alpha)^k \leq 1/2$. Then*

$$1 - (1 - \alpha)^k \geq \frac{k\alpha}{2}.$$

Proof. For the case $k = 1$ the statement is trivial. For any $k \geq 2$ let $f(x) = 1 - (1 - x)^k$ - strictly increasing and concave function on $[0, 1]$. Then

$$f'(x) = k(1 - x)^{k-1} = \frac{k}{1 - x} \cdot (1 - f(x)).$$

Let $\alpha_0 \in [0, 1]$ such that $1 - (1 - \alpha_0)^k = 1/2$. Clearly, $\alpha \leq \alpha_0$, since $f(x)$ increases on $[0, 1]$. Then

$$f'(\alpha_0) = \frac{k}{1 - \alpha_0}(1 - 1/2) = \frac{k}{2(1 - \alpha_0)} \geq \frac{k}{2}.$$

We further note that $f'(x)$ decreases on $[0,1]$, so for all $x \in [0, \alpha_0]$

$$f'(x) \geq \frac{k}{2}.$$

Now we can integrate both sides of this inequality over x and obtain

$$\int_0^\alpha f'(x)dx \geq \int_0^\alpha \frac{k}{2}dx$$
$$f(\alpha) \geq \frac{k\alpha}{2}.$$

□

Lemma 27. *Let $\{a_i\}_{i=1}^k$, $\{\alpha_i\}_{i=1}^k$ be positive constants and $\{A_i(\epsilon)\}_{i=1}^k$ be functions satisfying*

$$\forall 1 \leq i \leq k \quad \lim_{\epsilon\to 0} \frac{A_i(\epsilon)}{\epsilon^2} \in (0,\infty) \quad \textit{and} \quad \lim_{\epsilon\to\infty} \frac{A_i(\epsilon)}{\epsilon} = \infty$$

Then there exist a positive constant c and a function $C(\epsilon)$ satisfying the same conditions as $A_i(\epsilon)$, such that for all $n \geq 1$ and $\epsilon > 0$

$$\sum_{i=1}^k a_i e^{-nA_i(\alpha_i\epsilon)} \leq ce^{-nC(\epsilon)}$$

Proof. Define

$$c = k \cdot \max_{1\leq i\leq k} a_i$$
$$C(\epsilon) = \min_{1\leq i\leq k} A_i(\alpha_i\epsilon).$$

It can be easily shown that $C(\epsilon)$ satisfies both required conditions. Now, since e^{-x} is a decreasing positive function, we obtain

$$\begin{aligned}\sum_{i=1}^k a_i e^{-nA_i(\alpha_i\epsilon)} &\leq \max_{1\leq i\leq k} a_i \cdot \sum_{i=1}^k e^{-nA_i(\alpha_i\epsilon)} \\ &\leq \max_{1\leq i\leq k} a_i \cdot ke^{-n\min_{1\leq i\leq k} A_i(\alpha_i\epsilon)} \\ &= ce^{-nC(\epsilon)}.\end{aligned}$$

□

The next proposition demonstrates a simple exercise in analysis.

Proposition 28. *Let $\{f_m(x)\}$ be a uniformly absolutely continuous sequence of functions with values in some metric space $(\mathcal{X}, d)$, converging in* sup *to some function $f(x)$. Then $f(x)$ is absolutely continuous.*

Proof. Let $\epsilon > 0$. By the property of uniform absolute continuity, there exists $\delta > 0$ such that any collection of non-overlapping intervals

$$\{[s_n,t_n],\ n = 1,\ldots,N\} \text{ with } \sum_{n=1}^N (t_n - s_n) < \delta$$

satisfies

$$\forall m \in \mathbb{N} \quad \sum_{n=1}^{N} d\big(f_m(s_n), f_m(t_n)\big) < \frac{\epsilon}{2}.$$

We intend to show that

$$\sum_{n=1}^{N} d\big(f(s_n), f(t_n)\big) < \epsilon,$$

and the absolute continuity of f will follow immediately. Indeed, as $\{f_m\}$ converges to f, we can choose some $m \in \mathbb{N}$ such that

$$\sup_t d\big(f_m(t), f(t)\big) < \frac{\epsilon}{4N}.$$

Then for any $n = 1, 2, \ldots, N$

$$\begin{aligned} & d\big(f(s_n), f(t_n)\big) \\ & \qquad \leq d\big(f(s_n), f_m(s_n)\big) + d\big(f_m(s_n), f_m(t_n)\big) + d\big(f_m(t_n), f(t_n)\big) \\ & \qquad < \frac{\epsilon}{2N} + d\big(f_m(s_n), f_m(t_n)\big). \end{aligned}$$

By taking sum over n we obtain

$$\begin{aligned} \sum_{n=1}^{N} d\big(f(s_n), f(t_n)\big) &< N \cdot \frac{\epsilon}{2N} + \sum_{n=1}^{N} d\big(f_m(s_n), f_m(t_n)\big) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

as required. □

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