Risk-sensitive control for the parallel server model and an exponential version of the $c\mu$ rule*

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Abstract

A Markovian queueing model is considered in which servers of various types work in parallel to process jobs from a number of classes at rates μ_{ij} that depend on the class, i, and the type, j. The problem of dynamic resource allocation so as to minimize a risk-sensitive criterion is studied in a law-of-large-numbers scaling. Letting $X_i(t)$ denote the number of class-i jobs in the system at time t, the cost is given by

$$E \exp\Big\{ n\Big[\int_0^T h(\bar{X}(t))dt + g(\bar{X}(T))\Big]\Big\}$$

where $T > 0$, h and g are given functions satisfying regularity and growth conditions, and $\bar{X} = \bar{X}^n = n^{-1}X(n\cdot)$. It is well-known in an analogous context of controlled diffusion, and has been shown for some classes of stochastic networks, that the limit behavior, as $n \to \infty$, is governed by a differential game (DG) in which the state dynamics are given by a fluid equation for the formal limit φ of $\bar{X}$, while the cost consists of $\int_0^T h(\varphi(t))dt + g(\varphi(T))$ and an additional term that originates from the underlying large-deviation rate function. We prove that a DG of this type indeed governs the asymptotic behavior, that the game has value, and that the value can be characterized by the corresponding Hamilton-Jacobi-Isaacs equation. The framework allows for both fixed and growing number of servers $N \to \infty$, provided $N = o(n)$.

An additional contribution is the explicit solution of this DG in the case where the servers are homogenous ($\mu_{ij} = \mu_i$), $h = 0$ and g is linear, so that the cost takes the form $E \exp[\sum_i c_i X_i(nT)]$. An optimal strategy for the DG is identified, that assigns jobs following a fixed priority rule, specifically according to the index $(1 - e^{-c_i})\mu_i$. This is reminiscent of the $c\mu$ rule, that is known to be optimal under linear queue-length cost with weights c_i.

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1 Introduction

In the parallel server model (PSM), servers of various types process jobs from a number of classes, where each job requires service exactly once. Each class can be served at least by one of the types of servers, but not necessarily by all. A natural problem is to find a dynamic resource allocation policy to minimize a cost of interest. The model has been studied extensively in recent years due to its relevance in telephone call centers and in computer data systems. A sample of references treating this problem in fluid and diffusion regimes and via dynamic programming techniques is [2], [6], [7], [8], [10], [11], [14], [15], [22], [23], [24] and [28] (see [1] for a more comprehensive list).

The operation of queueing systems so as to avoid large exceedances of queue length and waiting time, such as for buffer overflow considerations or quality of service assurance, is of prime importance in practice. A natural way to address these considerations is to associate to the model a risk-sensitive cost criterion, that heavily penalizes such large exceedances. It is well-known for controlled diffusion models [21], [18] and has been shown for classes of stochastic networks [4], [16], [3] that considering a law-of-large-numbers scaling with this type of criterion brings into play large deviations phenomena, due to the fact that the most significant contribution to the cost originates from atypically large perturbations of the underlying state process. The dynamic control problem asymptotics can then be analyzed by a differential game (DG) associated with a perturbed fluid model. As a result, the asymptotic regime is different from the fluid or diffusion regimes. The goal of this paper is to study optimal dynamic resource allocation for the PSM under a risk-sensitive criterion. Our results show that a DG of the type alluded to above indeed governs the asymptotic behavior, that the game has value, and that the value can be characterized by the corresponding dynamic programming equation of Hamilton-Jacobi-Isaacs (HJI) type. Further, in a meaningful special case, we provide an explicit solution to the DG, that can be regarded as an exponential version of the classical $c\mu$ rule.

The model is treated in a Markovian setting, assuming that job arrival rates are proportional to a (large) parameter n, and that the total processing capacity for class-i jobs by type-j servers is given by $\mu_{ij}n$. Denoting by $X_i(t)$ the number of class-i jobs in the system at time t, the cost is given by

$$E\exp\Big\{n\Big[\int_0^T h(\bar X(t))dt+g(\bar X(T))\Big]\Big\} \tag{1}$$

where $T>0$ is fixed, h and g are given functions, and $\bar X=\bar X^n=n^{-1}X(n\cdot)$. Considering such a cost for large values of n puts emphasis on large values of $\int_0^T h(\bar X(t))dt+g(\bar X(T))$. Qualitatively it is obvious that the cost specified above is closely related to risk-sensitive costs for excessive waiting time, and, in a model with customer abandonment from the queue, for large abandonment count. This provided further motivation to study this problem. However, in this paper we do not make precise statements regarding these alternative measures of performance.

The paper [4] studies a class of controlled stochastic networks of re-entrant line structure, under a risk-sensitive cost associated with escape time (such as the time until the buffer limit is reached), establishing relations to the corresponding DG and HJI equation. A sequel [5] establishes explicit solutions in the case of a network of queues in tandem. Our techniques borrow much from [4]. However, there are several important aspects in which the treatments differ. First, the fixed time horizon form of (1) is different from one based on exit time. Second, the unboundedness of g makes the treatment of both the DG and the HJI equation more subtle. The most serious difference, however, is that the re-entrant line structure prohibits routing control, where a job could be handled by more than one server. In particular, a feature that makes the boundary analysis convenient in [4] is the spacial homogeneity of the controlled generator in the 'interior' of the domain (i.e., when $X_i > 0$ for all i), with boundary corrections given via a fixed, continuous Skorohod map. This is not valid for the PSM. Indeed, already in the case of a single class with two servers, there is difference between the set of possible jump intensities when there are two or more jobs in the system (the jump rate from x to $x-1$ could be as large as $\mu_1 + \mu_2$) and when there is only one (the jump rate to zero is at most $\mu_1 \vee \mu_2$). Although the Skorohod map plays an important role in the present treatment, its use is less straightforward. In our first main set of results we rigorously relate the control problem's asymptotics to the DG, prove that it has value, and characterize it in terms of a HJI equation.

For analogous treatments of other stochastic networks, explicit solutions were found by analysis of the value function [16] and by using the PDE [5]. In this paper we find explicit solutions by working directly with the DG. For that we focus on the case of homogenous servers (i.e., where there is only one type of servers, so $\mu_{ij} = \mu_i$), $h = 0$ and g linear. In this case the cost takes the form $E\exp[\sum_i c_i X_i(nT)]$. We show that the value of the game under the zero initial condition gives the leading term in the large-T behavior of the value under a general initial condition. Next, we provide exact analysis for the case of zero initial condition. We find an explicit expression for the value of the game and identify an optimal strategy for the minimizing player. This strategy selects jobs according to a fixed priority rule, in the order of the index $(1 - e^{-c_i})\mu_i$.

The strategy alluded to above is reminiscent of the classical $c\mu$ rule that is optimal under linear cost with weights c_i. See [13], [9] for exact optimality results of the $c\mu$ rule, and [28], [24] for asymptotic optimality in heavy traffic of a generalized version of this policy in the case of a nonlinear cost (we emphasize that, although this paper also studies a nonlinear cost, our results are different from those in the above references, and so is the structure of the policy). See also [6] for a variation of the $c\mu$ rule for queueing models with abandonment. Like the classical $c\mu$ rule, the strategy identified in this paper is simple and easy to implement. Another desired property is its independence of the λ_i parameters. It is reasonable to expect due to the game representation, and has been shown in [17] in a precise quantitative sense, that risk-sensitive control formulations give rise to robustness with respect to perturbations in the distribution of the underlying primitives. The fact that the strategy we have identified does not depend on the λ_i parameters is clearly a manifestation of a strong robustness property.

The organization of the paper is as follows. The next section introduces the model and states the first set of main results, that characterize the asymptotics in terms of the DG and HJI equation. Section 3 introduces tools required to prove these results. Section 4 addresses the stochastic control problem – DG relation, while Section 5 proves the DG – PDE relation. Finally, Section 6 analyzes the DG in the case where $h = 0$ and g is linear and identifies an optimal index strategy for the DG. The appendix contains an argument for showing that this strategy is well-defined.

2 Model and main results

The model is parameterized by $n \in \mathbb{N}$. It consists of I customer classes and J service stations, where station $j \in \mathcal{J} := \{1, 2, \dots, J\}$ contains $N_j(n) \geq 1$ identical servers. While I and J are fixed, $N_j = N_j(n)$ may vary with n. Denoting $N(n) = \sum_j N_j(n)$, it is assumed that

$$\lim_{n\to\infty} \frac{N(n)}{n} = 0. \tag{2}$$

Note that having $N_j(n) = N_j$ fixed (independent of n) is a legitimate special case.

Arrivals into the system occur according to independent Poisson processes, denoted by E_i, $i \in \mathcal{I} := \{1, 2, \dots, I\}$, with respective parameters $\lambda_i(n)$, where

$$\lambda_i(n) = \lambda_i n, \qquad i \in \mathcal{I}, \tag{3}$$

and $\lambda_i > 0$ are fixed. The servers are exponential, where a class-i customer can be served at rate $\mu_{ij}(n) \geq 0$ by a server from station j. Having $\mu_{ij}(n) = 0$ is possible, and means that a server from station j is unable to serve a class-i customer. It is assumed that the total processing capacity of class-i customers by station j, namely $N_j(n)\mu_{ij}(n)$ satisfies

$$N_j(n)\mu_{ij}(n) = \mu_{ij} n, \qquad i \in \mathcal{I},\ j \in \mathcal{J}, \tag{4}$$

where $\mu_{ij} \geq 0$ are fixed. Thus both the arrival rates and the per-station total service capacity scale like n.

Denote the number of class-i customers in the system at time t by $\Xi^{n,i}_t$ and write $\Xi^n = (\Xi^{n,i}_t)_{i\in\mathcal{I},t\geq 0}$ for the process taking values in $\mathbb{Z}^I_+$. A normalized version is

$$X^n_t = n^{-1}\Xi^n_t, \qquad t \geq 0, \tag{5}$$

which is a process taking values in $G^n := n^{-1}\mathbb{Z}^I_+$. Denote $G = \mathbb{R}^I_+$.

Control processes will be associated with service allocation. We first describe the action space. An *allocation matrix* is any member of

$$U := \Big\{ u \in \mathbb{R}^{I\times J}_+ : \sum_{i\in\mathcal{I}} u_{ij} \leq 1,\ j \in \mathcal{J} \Big\}. \tag{6}$$

If $u \in U$ and $N_j(n)u_{ij}$ is an integer for all i, j, this quantity represents the number of servers from station j allocated to serve class-i customers. For simplicity, the product $N_j(n)u_{ij}$ is not required to be integer, and thus a server may work on more than one job instantaneously. The precise formulation of control is based on the martingale approach. To describe it, introduce the *controlled generator* acting on the space of functions $G^n \to \mathbb{R}$. It is given, for each $n \in \mathbb{N}$ and $u \in U$, by

$$\mathcal{L}^{n,u} f(x) = \sum_{i\in\mathcal{I}} n\lambda_i\Big(f\big(x+\frac{1}{n}e_i\big)-f(x)\Big)+\sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} n\mu_{ij}u_{ij}\Big(f\big(x-\frac{1}{n}e_i\big)-f(x)\Big)\mathbf{1}_{\{x-\frac{1}{n}e_i\in\mathbb{R}^I_+\}}, \qquad x \in G^n, \tag{7}$$

where $\{e_i\}$ denote the standard basis of $\mathbb{R}^I$. Let a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given, supporting the processes defined below. Given n and an initial condition $(t, x) \in \mathbb{R}_+ \times G^n$, a *control system starting from* (t, x) is a triplet $S^n = (U^n, X^n, (\mathcal{F}_s)_{s\geq t})$, where U^n and X^n are processes defined on $[t, \infty)$, taking values in U and G^n, respectively, and having RCLL sample paths, $\mathcal{F}_s \subset \mathcal{F}$, $s \geq t$ forms a filtration to which these processes are adapted, and

- $\mathbb{P}(X_t^n = x) = 1$;
- One has
$$\sum_{j\in\mathcal{J}} N_j(n)U_{ij}^n \le \Xi^{n,i} \equiv nX^{n,i}, \qquad i \in \mathcal{I}; \tag{8}$$
- For each bounded $f : G^n \to \mathbb{R}$, the process
$$f(X_s^n) - \int_t^s \mathcal{L}^{n,U^n(r)} f(X_r^n)dr, \qquad s \ge t \tag{9}$$
is a martingale w.r.t. $(\mathcal{F}_s)_{s\ge t}$.

U^n is said to be a *control* and X^n the *associated controlled Markov process*. Given n and $(t,x) \in \mathbb{R}_+ \times G^n$, denote by $\mathcal{S}_{n,t,x}$ the corresponding class of control systems.

To present the risk sensitive control problem, let h and g be globally Lipschitz functions from $\mathbb{R}_+^I$ to $\mathbb{R}$, monotone nondecreasing with respect to the usual partial order on $\mathbb{R}_+^I$. Further, assume that the function h is bounded. Fix $T > 0$. The cost associated to a member $S = (U^n, X^n, (F_s^n))$ of $\mathcal{S}_{n,t,x}$ (where $t \in [0,T]$) is given by

$$C^n(t,x,S) = \frac{1}{n}\log \mathbb{E}[e^{n[\int_t^T h(X_s^n)ds + g(X_T^n)]}]. \tag{10}$$

The value function is given by

$$V^n(t,x) = \inf_{S\in\mathcal{S}_{n,t,x}} C^n(t,x,S), \qquad t \in [0,T),\, x \in G^n. \tag{11}$$

The first main result relates the limit of V^n, as $n \to \infty$, to a PDE of Hamilton-Jacobi-Isaacs type. To state it we need some notation. Set $m_0 = ((\lambda_i)_{i\in\mathcal{I}}, (\mu_{ij})_{i\in\mathcal{I},j\in\mathcal{J}})$. Then m_0 is a member of $M := \mathbb{R}_+^I \times \mathbb{R}_+^{I\times J}$. We write generic members of M as $m = ((\bar\lambda_i)_{i\in\mathcal{I}}, (\bar\mu_{ij})_{i\in\mathcal{I},j\in\mathcal{J}})$. While λ and μ denote the actual arrival and service parameters for the system, a possibly different member m of M will be interpreted as a perturbed set of parameters. For $u \in U$ and $m \in M$, let

$$v(u,m) = \sum_i \bar\lambda_i e_i - \sum_{ij} u_{ij}\bar\mu_{ij} e_i \tag{12}$$
$$\rho(u,m) = \sum_i \lambda_i\, l\Big(\frac{\bar\lambda_i}{\lambda_i}\Big) + \sum_{ij} u_{ij}\mu_{ij}\, l\Big(\frac{\bar\mu_{ij}}{\mu_{ij}}\Big), \tag{13}$$

where

$$l(x) = \begin{cases} x\log x - x + 1, & x \ge 0, \\ \\ +\infty, & x < 0, \end{cases}$$

with the convention $0 \log 0 = 0$ and $l(\varepsilon/0) = \infty$ for $\varepsilon > 0$. Let

$$H(p) = \inf_{u\in U}\sup_{m\in M}[\langle p, v(u,m)\rangle - \rho(u,m)], \qquad p \in \mathbb{R}^I. \tag{14}$$

Let $\mathbf{I} : \partial G \to 2^{\mathcal{I}}$ be defined by

$$\mathbf{I}(x) = \{i \in \mathcal{I} : x_i = 0\}.$$

The HJI equation, considered with boundary and terminal conditions, is as follows (denoting V_t as the derivative of V w.r.t. t, and DV the gradient of V w.r.t. x):

$$\begin{cases} V_t + H(DV) + h & = 0 \qquad \text{in } [0,T) \times G^o, \\ \langle DV(t,x), e_i \rangle & = 0 \qquad x \in \partial G, i \in \mathbf{I}(x), \\ V(T,x) & = g(x) \quad x \in G. \end{cases} \tag{15}$$

The precise definition of a solution to equation (15) is given in Section 3.

Theorem 2.1. *Given $t \in [0,T]$ and $G_n \ni x^n \to x \in G$,*

$$\lim_{n\to\infty} V^n(t,x^n) = V(t,x),$$

where V is the unique viscosity solution of (15)*.*

While the above result characterizes the limit behavior of V^n in terms of the HJI equation, we will have an additional characterization of it as the value of a DG (Theorem 3.1).

3 Preliminaries

We introduce the main tools on which the proof of Theorem 2.1 relies: (1) Two alternative queueing models, used to bound the performance of the original model. (2) Viscosity solutions of equation (15). (3) A differential game. At the end of this section we provide the proof of Theorem 2.1, that uses these tools, and present Theorem 3.1 regarding the relation to the DG.

3.1 Two alternative models

The constraint (8) is difficult to work with directly. We introduce two models that are more convenient, not having such a constraint. They will be used to treat the original model.

Model (a): Recall the definition of the class $\mathcal{S}_{n,t,x}$. We let $\mathcal{S}^{(a)}_{n,t,x}$ be defined the same way, except that the constraint (8) is removed, and call the resulting model Model (a). In the original model, the total processing rate for a given class i, namely $\sum_j n\mu_{ij}U^n_{ij}$, can get as large as $\sum_j n\mu_{ij}$, provided $\Xi^{n,i} \ge N(n)$ (see (7) and (8)). Indeed, this is achieved by selecting $U^n_{ij} = 1$ for all j, which corresponds to a situation where class i occupies all servers in every station of the system. When $\Xi^{n,i}$ is less than $N(n)$, the maximum possible total processing rate for class i decreases in the original model, while in Model (a) it remains at the same level. A physical interpretation of Model (a) could be that multiple servers can simultaneously work on a single job, having their processing rates sum up.

As is clear from the very definition of the two models, $\mathcal{S}_{n,t,x} \subset \mathcal{S}^{(a)}_{n,t,x}$.

Model (b): $\mathcal{S}^{(b)}_{n,t,x}$ is defined by replacing (8) by the requirement that, for each $i \in \mathcal{I}$,

$$\Xi^{n,i} \le N(n) \quad \text{implies} \quad \sum_{j\in\mathcal{J}} U^n_{ij} = 0. \tag{16}$$

The physical meaning of the resulting model, called Model (b), is that one simply ceases to serve class-i customers whenever they are too few. Based on (6), it is clear that (16) implies (8). As a result, $\mathcal{S}^{(b)}_{n,t,x} \subset \mathcal{S}_{n,t,x}$.

The two models automatically provide bounds on the original model. That is, for $n \in \mathbb{N}$ and $t \in [0,T), x \in G^n$,

$$Q^n(t,x) := \inf_{S\in\mathcal{S}^{(a)}_{n,t,x}} C^n(t,x,S) \le V^n(t,x) \le R^n(t,x) := \inf_{S\in\mathcal{S}^{(b)}_{n,t,x}} C^n(t,x,S). \tag{17}$$

Models (a) and (b) are quite similar: the controlled transition rates are of the same form $\sum_j n\mu_{ij}u_{ij}$ in direction $-e_i$, for arbitrary $u \in U$. Further, denote

$$\nu_n = \frac{N(n)}{n}.$$

Let $\bar{\nu}_n \in \mathbb{R}^I$ denote the vector $(\nu_n, \dots, \nu_n)$. Set

$$G^*_n = \{x + \bar{\nu}_n : x \in G_n\}, \qquad G^\#_n = G^n \setminus G^*_n. \tag{18}$$

Then by (16) and the form of the generator (7), under Model (b), if X^n starts in G^*_n it will never leave this set. This is analogous to the fact that under Model (a) each $\Xi^{n,i}$ satisfies a nonnegativity constraint.

The following useful estimates on these models are proved in Section 4. Throughout, $\|\cdot\|$ denotes the Euclidean norm.

Lemma 3.1. *There exists a constant c_0 such that, for all n, $t \in [0,T]$,*

$$|Q^n(t,x) - Q^n(t,x')| \le c_0\|x - x'\|, \qquad x, x' \in G_n, \tag{19}$$

$$|R^n(t,x) - R^n(t,x')| \le c_0\|x - x'\|, \qquad x, x' \in G^*_n, \tag{20}$$

$$R^n(t,x) \le R^n(t,x+\bar{\nu}_n) + c_0\nu_n, \qquad x \in G^\#_n, \tag{21}$$

$$R^n(t_1,x) \le R^n(t,x) + c_0(t_1 - t), \qquad x \in G^*_n,\ t_1 \in (t,T]. \tag{22}$$

3.2 Viscosity solutions

Solutions to equation (15) are defined in the viscosity sense.

Definition 3.1. *Let $V : [0,T]\times G \to \mathbb{R}$ be continuous in the first variable, uniformly over $[0,T]\times G$, and satisfy a global Lipschitz condition in the second, namely*

$$\sup\{|x-y|^{-1}|V(t,x) - V(t,y)| : t \in [0,T], x \ne y \in G\} < \infty.$$

Then V is said to be a sub (super) solution of (15) *if $V(T,\cdot)=g$, and, whenever $\theta\in C^\infty$ and $V-\theta$ has a local maximum (minimum) at $(t,x)\in[0,T)\times G$, the following holds*

$$\begin{aligned}[][\theta_t(t,x)+H(D\theta(t,x))+h(x)]\vee\max_{i\in\mathbf{I}(x)}\langle D\theta(t,x),e_i\rangle &\geq 0\\ \Big([\theta_t(t,x)+H(D\theta(t,x))+h(x)]\wedge\min_{i\in\mathbf{I}(x)}\langle D\theta(t,x),e_i\rangle &\leq 0\Big).\end{aligned}$$

A function is said to be a *solution* if it is both a sub- and a supersolution.

Proposition 3.1. *Let u be a subsolution and v be a supersolution. Then $u\leq v$.*

Note that this result, proved in Section 5, gives uniqueness of solutions.

3.3 A differential game

Fix $T>0$. Given $t\in[0,T)$, denote by $\mathbb{D}([t,T];\mathbb{R}^k)$ the space of RCLL functions mapping $[t,T]$ to $\mathbb{R}^k$. The *one-dimensional Skorohod map* $\Gamma_1=\Gamma_1^{t,T}$ from $\mathbb{D}([t,T]:\mathbb{R})$ to itself is given by

$$\Gamma_1[\psi](s)=\psi(s)-\inf_{r\in[t,s]}\psi(r)\wedge 0,\qquad s\in[t,T]. \tag{23}$$

Let $\Gamma=\Gamma^{t,T}$ mapping $\mathbb{D}([t,T]:\mathbb{R}^I)$ to itself be given by

$$\Gamma[\psi]_i=\Gamma_1[\psi_i],\qquad \text{for } i\leq I. \tag{24}$$

Γ is often called the *Skorohod map on G with normal constraint.* It is clear from the definition that, for $\psi,\phi\in\mathbb{D}([t,T];\mathbb{R}^I)$,

$$\sup_{[t,T]}\|\Gamma[\psi]-\Gamma[\phi]\|\leq 2\sup_{[t,T]}\|\psi-\phi\|. \tag{25}$$

Let

$$\bar{U}=\{u:[0,T]\to U; u \text{ is measurable}\},$$
$$\bar{M}=\{m:[0,T]\to M; m \text{ is measurable, } l\circ m \text{ is locally integrable}\}.$$

We describe a deterministic two-player zero-sum differential game where one player attempts to minimize a cost c (yet to be defined) by selecting a member of $\bar{U}$, corresponding to service allocation, and the other one chooses a member of $\bar{M}$, interpreted as perturbed arrival and service rates, to maximize c. To this end, consider the dynamics of the game,

$$\begin{cases}\psi(s)=x+\displaystyle\int_t^s v(u(r),m(r))dr, & s\in[t,T],\\[2ex] \varphi=\Gamma[\psi].\end{cases} \tag{26}$$

Let the cost be defined by

$$c(t,x,u,m)=\int_t^T[h(\varphi(s))-\rho(u(s),m(s))]ds+g(\varphi(T)), \tag{27}$$

where $\varphi = \varphi(\cdot; t, x, u, m)$ is given by (26). Note that neither the dynamics nor the cost are affected by the value of the controls u and m on $[0, t)$.

To define the game in the sense of Elliott and Kalton [19], we consider the notion of strategies. To this end, we endow $\bar{U}$ and $\bar{M}$ with the metric $d(v_1, v_2) = \int_0^T \|v_1(t) - v_2(t)\| dt$, and with the corresponding Borel σ-fields. A mapping $\alpha : \bar{M} \to \bar{U}$ is called a *strategy for the minimizing player* if it is measurable and if for every $m, \tilde{m} \in \bar{M}$ and $s \in [0, T]$,

$$m(r) = \tilde{m}(r) \qquad \text{for a.e. } r \in [0, s] \qquad \text{implies} \qquad \alpha[m](r) = \alpha[\tilde{m}](r) \qquad \text{for a.e. } r \in [0, s].$$

In a similar way a *strategy for the maximizing player* is defined by a mapping $\beta : \bar{U} \to \bar{M}$. The set of all strategies for the minimizing (respectively, maximizing) player will be denoted by A (respectively, B). The upper value for the game is defined as

$$V^+(t, x) = \sup_{\beta \in B} \inf_{u \in \bar{U}} c(t, x, u, \beta[u]),$$

and the lower value as

$$V^-(t, x) = \inf_{\alpha \in A} \sup_{m \in \bar{M}} c(t, x, \alpha[m], m).$$

The game is said to have value if the value functions V^+ and V^- coincide. The game is related to the stochastic control problem on the one hand, and to the PDE on the other hand, by the following two results.

Proposition 3.2. *Fix $x \in G$ and $t \in [0, T]$. Then*

$$\limsup_{n \to \infty} R^n(t, x^n) \le V^-(t, x) \qquad \textit{if} \qquad G_n^* \ni x^n \to x, \tag{28}$$

and

$$\liminf_{n \to \infty} Q^n(t, x^n) \ge V^+(t, x) \qquad \textit{if} \qquad G_n \ni x^n \to x. \tag{29}$$

Proposition 3.3. *Both V^+ and V^- are solutions of* (15).

Proposition 3.2 is proved in Section 4. Given the Lipschitz property of the value functions, that is proved in Lemma 5.1 at Section 5, the proof of Proposition 3.3 is analogous to that of Theorem 6 in [4], and therefore we omit it.

3.4 Proof of main results

Proof of Theorem 2.1. First, note that Propositions 3.1 and 3.3 imply that the game has value, and that the value function $V := V^+ = V^-$ uniquely solves the PDE (15). Next, fix $t \in [0, T]$ and $x \in G$. To prove the theorem, it suffices to consider only sequences of the form $G_n^* \ni x^n \to x$ and $G_n^\# \ni x^n \to x$. In the former case, the combination of (17) and Proposition 3.2 shows

$$\lim V^n(t, x^n) = V(t, x),$$

as required. Consider now the case $G_n^{\#} \ni x^n \to x$. By (17) and (29), we still have a lower bound of the form $V(t,x)$. While (28) does not directly apply as an upper bound, its combination with (17) and (21), noting that $y_n := x^n + \bar{\nu}_n \in G_n^*$ and $\nu_n \to 0$, gives

$$\limsup V^n(t,x^n) \le \limsup R^n(t,y_n) \le V(t,x).$$

This completes the proof of the theorem. $\square$

As a consequence, we obtain an alternative characterization of the asymptotic behavior of V^n.

Theorem 3.1. *For $G_n \ni x^n \to x \in G$ and $t \in [0,T]$, $\lim_{n\to\infty} V^n(t,x^n) = V(t,x)$, where V is the value of the DG.*

4 The stochastic control problem and the differential game

In this section we prove Lemma 3.1 and Proposition 3.2.

We begin with a result showing that the DG's value functions do not vary upon truncating the space M. For $b > 0$, denote

$$M_b = \{m = (\bar{\lambda}_i, \bar{\mu}_{ij}) \in M : \bar{\lambda}_i \le b, \bar{\mu}_{ij} \le b,\ i \in \mathcal{I},\ j \in \mathcal{J}\},$$

and let $\bar{M}_b$ be defined as $\bar{M}$, with M replaced by M_b. Let also B_b be defined similarly to B, with $\bar{M}$ replaced by $\bar{M}_b$. Thus a strategy $\beta \in B_b$ maps $\bar{U}$ into $\bar{M}_b$. Finally, analogously to V^+ and V^-, set

$$V_b^+(t,x) = \sup_{\beta \in B_b} \inf_{u \in \bar{U}} c(t,x,u,\beta[u]),$$

$$V_b^-(t,x) = \inf_{\alpha \in A} \sup_{m \in \bar{M}_b} c(t,x,\alpha[m],m).$$

The proof of the following lemma appears at the end of the section.

Lemma 4.1. *For sufficiently large b*

$$V^{b,\pm}(t,x) = V^{\pm}(t,x). \tag{30}$$

Next we prove Lemma 3.1. The proof uses a controlled generator similar to (7), for a process that need not be constrained to G_n but lives in $n^{-1}\mathbb{Z}^I$, namely

$$\mathcal{L}_0^{n,u} f(x) = \sum_{i\in\mathcal{I}} n\lambda_i (f(x + \frac{1}{n}e_i) - f(x)) + \sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} n\mu_{ij}u_{ij}(f(x - \frac{1}{n}e_i) - f(x)), \qquad x \in G^n, \tag{31}$$

for $f : n^{-1}\mathbb{Z}^I \to \mathbb{R}$.

Proof of Lemma 3.1. We first prove (19). Fix (t,x,x'). Fix also $S = (U^n, X^n, (F_s^n)) \in \mathcal{S}_{n,t,x}^{(a)}$. One can construct a process Y^n, a filtration $\bar{F}_s^n$ containing F_s^n, s.t. $\mathbb{P}(Y_t^n = x) = 1$, $X^n = \Gamma[Y^n]$, and for each bounded $f : n^{-1}\mathbb{Z}^I \to \mathbb{R}$,

$$f(Y_s^n) - \int_t^s \mathcal{L}_0^{n,U^n(r)} f(Y_r^n)dr, \qquad s \ge t,$$

is a martingale w.r.t. $(\bar{F}_s)$. Such a construction, that uses additional exponential clocks for jumps of Y^n that occur when X^n is on the boundary, is standard, and we omit the details.

Next let us construct on the new filtration a controlled process X'^n starting from x', simply by setting $X'^n = \Gamma[x' - x + Y^n]$. Then one directly verifies by the properties of Y^n and the Skorohod map, that $S' = (X'^n, U^n, (\bar{F}^n_s)) \in \mathcal{S}^{(a)}_{n,t,x'}$. Now, using (25), $\sup_{s\in[t,T]} \|X'^n(s) - X^n(s)\| \le 2\|x - x'\|$, and therefore by (10), $C_n(t, x, S) - C_n(t, x', S') \le c_0\|x - x'\|$, using the global Lipschitz property of h and g. Taking the infimum over $S \in \mathcal{S}^{(a)}_{n,t,x}$ shows $Q^n(t, x) \le Q^n(t, x') + c_0\|x - x'\|$, and the result (19) follows.

Toward proving (20), the following simple relation between Q^n and R^n will be useful. Write $Q_n(t, x, \tilde{g}, \tilde{h})$ for the value function Q^n of (17) where, in the cost function C^n (10), one replaces h and g by $\tilde{h}$ and $\tilde{g}$, respectively. Then

$$R^n(t, x + \bar{\nu}_n) = Q_n(t, x; g(\cdot + \bar{\nu}_n), h(\cdot + \bar{\nu}_n)), \qquad t \in [0, T],\, x \in G_n. \tag{32}$$

To obtain this identity, we will argue by correspondences between members of $\mathcal{S}^{(a)}_{n,t,x}$ and members of $\mathcal{S}^{(b)}_{n,t,x+\bar{\nu}_n}$. To this end, we first make the following observation. Recall that the way Model (a) is defined does not put any constraint on the process U^n (taking values in U), whereas under Model (b), (16) must be satisfied. Now, the form of the generator $\mathcal{L}^{n,u}$ (7) is such that whenever $x_i = 0$ for some i, the value of $u_{i,j}$, $j \in \mathcal{J}$ is immaterial. Hence given any control system $(U^n, X^n, (\mathcal{F}_s)) \in \mathcal{S}^{(a)}_{n,t,x}$, we may assume w.l.o.g., that, for each $i \in \mathcal{I}$, $\sum_j U^n_{ij}(s) = 0$ for a.e. s for which $X^n_i(s) = 0$. With this at hand, given a control system $S = (U^n, X^n, (\mathcal{F}_s)) \in \mathcal{S}^{(a)}_{n,t,x}$ starting from (t, x), $(U^n, X^n + \bar{\nu}_n, (\mathcal{F}_s))$ is clearly a member of $\mathcal{S}^{(b)}_{n,t,x+\bar{\nu}_n}$ (satisfying, in particular, (16)). On the other hand, given $S \in \mathcal{S}^{(b)}_{n,t,x+\bar{\nu}_n}$ one automatically has $(U^n, X^n, (\mathcal{F}_s)) \in \mathcal{S}^{(a)}_{n,t,x}$. These correspondences and the definition of R^n and Q_n yield (32).

Equipped with the above identity, the claim (20), regarding R^n, follows from the estimate just obtained on Q^n.

The proof of (21) is similar to that of (19), and thus omitted.

Finally, we prove (22). Fix $x \in G^*_n$ and $0 \le t < t_1 \le T$. By standard considerations, R^n, defined as the value function of a control problem (17), satisfies the dynamic programming principle, namely

$$R^n(t, x) = \inf \frac{1}{n} \log \mathbb{E}[e^{n[\int_t^{t_1} h(X^n_s)ds + R^n(t_1, X^n_{t_1})]}],$$

where, as in (17), the infimum is over $S \in \mathcal{S}^{(b)}_{n,t,x}$. Given $\varepsilon > 0$, let S and the corresponding controlled process X^n be such that

$$R^n(t, x) + \varepsilon \ge \frac{1}{n} \log \mathbb{E}[e^{n[\int_t^{t_1} h(X^n_s)ds + R^n(t_1, X^n_{t_1})]}].$$

Denote by $-c_1$ a lower bound on h. Using Jensen's inequality,

$$R^n(t, x) + \varepsilon \ge -c_1(t_1 - t) + \mathbb{E}[R^n(t_1, X^n_{t_1})].$$

Hence

$$R^n(t_1, x) - R^n(t, x) \le \varepsilon + c_1(t_1 - t) - \mathbb{E}[R^n(t_1, X^n_{t_1}) - R^n(t_1, x)].$$

The jump intensities of X^n are bounded, uniformly over all control systems S, by c_2n, where c_2 does not depend on S, n, t_1, ε. Hence the number of jumps that X^n performs over $[t, t_1]$ is dominated by a Poisson r.v. of mean $c_2n(t_1 - t)$, whereas the size of each jump is $1/n$. Along with the estimate (20) (and recalling that from $x \in G_n^*$ the process can only jump to sites in G_n^*), this shows that

$$|\mathbb{E}[R^n(t_1, X^n_{t_1}) - R^n(t_1, x)]| \le c_0\mathbb{E}[\|X^n_{t_1} - x\|] \le c_0\frac{1}{n}c_2n(t_1 - t) = c_0c_2(t_1 - t).$$

We obtain

$$R^n(t_1, x) - R^n(t, x) \le \varepsilon + (c_1 + c_0c_2)(t_1 - t).$$

Sending $\varepsilon \to 0$, the result follows. $\square$

The value function Q^n, defined in terms of the cost C^n (10), clearly satisfies a certain dynamic programming equation (DPE) of Bellman type on $[0, T] \times G_n$. It will be more convenient, however, to take advantage of the fact that, owing to the logarithmic transformation appearing in (10), Q^n also satisfies a DPE of Isaacs type, corresponding to the value of a game. In this game, an additional player is introduced, controlling the transition rates. By considering this equation we follow the approach of [21], see also [20], Chapter VI.

To this end, we introduce the following two controlled generators. They are similar to (7) and (31), but now $m = (\bar{\lambda}, \bar{\mu})$ are also controlled, namely

$$\mathcal{L}^{n,u,m}f(x) = \sum_{i\in\mathcal{I}} n\bar{\lambda}_i(f(x + \frac{1}{n}e_i) - f(x)) + \sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} n\bar{\mu}_{ij}u_{ij}(f(x - \frac{1}{n}e_i) - f(x))\mathbf{1}_{\{x-\frac{1}{n}e_i\in\mathbb{R}^I_+\}},$$

for $f : G_n \to \mathbb{R}$, and

$$\mathcal{L}_0^{n,u,m}f(x) = \sum_{i\in\mathcal{I}} n\bar{\lambda}_i(f(x + \frac{1}{n}e_i) - f(x)) + \sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} n\bar{\mu}_{ij}u_{ij}(f(x - \frac{1}{n}e_i) - f(x)),$$

for $f : n^{-1}\mathbb{Z}^I \to \mathbb{R}$. The function $Q^n : [0, T] \times G_n \to \mathbb{R}$ is continuously differentiable in t for every x, and satisfies the following Isaacs equation,

$$\begin{cases} \inf_{u\in U}\sup_{m\in M}(\mathcal{L}^{n,u,m}Q^n(t, x) + \frac{d}{dt}Q^n(t, x) + h(x) - \rho(u, m)) = 0, \\ \\ Q^n(T, x) = g(x). \end{cases} \tag{33}$$

The proof of these facts is very similar to that of Lemma 1 of [4], hence omitted.

Lemma 4.2 below regards the existence of processes governed by the generators just introduced. To state it, we need some additional notation.

Recall the definition of Γ based on the one-dimensional Skorohod map (23). We define a family of Skorokhod maps Γ^n, each mapping $\mathbb{D}([t, T] : \mathbb{R}^I)$ to itself, by

$$\Gamma^n[\psi]_i := \Gamma_1[\psi_i - \nu_n] + \nu_n, \qquad i \le I.$$

In fact, Γ^n is the Skorohod map on $G + \bar{\nu}_n \equiv \{x \in \mathbb{R}^I : x_i \ge \nu_n, i \le I\}$, with normal constraint. Note that

$$\|\Gamma^n[\psi] - \Gamma[\psi]\| \le I\nu_n, \qquad \psi \in \mathbb{D}([t, T] : \mathbb{R}^I). \tag{34}$$

For $f : G_n^* \to \mathbb{R}$, denote

$$\tilde{\mathcal{L}}^{n,u,m} f(x) = n \sum_i \bar{\lambda}_i (f(x + \frac{1}{n} e_i) - f(x)) + n \sum_{ij} \bar{\mu}_{ij} u_{ij} (f(x - \frac{1}{n} e_i) - f(x)) \mathbf{1}_{\{x - \frac{1}{n} e_i \in G_n^*\}}, \qquad x \in G_n^*,\ u \in U,\ m \in M. \quad (35)$$

The proof of the following result is similar to the proof of Lemmas 7 and 8 of [4], and thus omitted.

Lemma 4.2. *Fix n, $t \in [0, T)$, and $b > 0$.*
i. Fix $x \in G_n$. Let a measurable function $u : [t, T] \times G_n \to U$ and a strategy $\beta \in B_b$ be given. Then there exists a filtered probability space $(\bar{\Omega}, \bar{F}, \{\bar{F}_s\}_{[t,T]}, \bar{\mathbb{P}})$, and $(\bar{F}_s)$-adapted RCLL processes $\bar{X}$, $\bar{Y}$, $\bar{m}$ and $\bar{u}$, taking values in G_n, $n^{-1}\mathbb{Z}^I$, M and U, respectively, such that $\bar{\mathbb{P}}$-a.s., $\bar{m} = \beta[\bar{u}]$, $\bar{u}(s) = u(s, \bar{X}(s))$, $s \in [t, T]$, $\bar{X} = \Gamma(\bar{Y})$, $\bar{X}(t) = \bar{Y}(t) = x$ and

$$f(s, \bar{X}(s)) - \int_t^s \left(\mathcal{L}^{n, \bar{u}(r), \bar{m}(r)} f(r, \bar{X}(r)) + \frac{\partial}{\partial r} f(r, \bar{X}(r)) \right) dr \quad (36)$$

and

$$f(s, \bar{Y}(s)) - \int_t^s \left(\mathcal{L}_0^{n, \bar{u}(r), \bar{m}(r)} f(r, \bar{Y}(r)) + \frac{\partial}{\partial r} f(r, \bar{Y}(r)) \right) dr \quad (37)$$

are $(\bar{F}_s)$-martingales for all bounded f having continuous time derivative.

ii. Fix $x \in G_n^*$. Let a measurable function $m : [t, T] \times G_n^* \to M_b$ and a strategy $\alpha \in A$ be given. Then there exists a filtered probability space $(\bar{\Omega}, \bar{F}, \{\bar{F}_s\}_{[t,T]}, \bar{\mathbb{P}})$, and $(\bar{F}_s)$-adapted RCLL processes $\bar{X}$, $\bar{Y}$, $\bar{m}$ and $\bar{u}$, taking values in G_n^*, $n^{-1}\mathbb{Z}^I$, M and U, respectively, such that $\bar{\mathbb{P}}$-a.s., $\bar{u} = \alpha[\bar{m}]$, $\bar{m}(s) = m(s, \bar{X}(s)), s \in [t, T], \bar{X} = \Gamma_n[\bar{Y}], \bar{X}(t) = \bar{Y}(t) = x$, and the process defined as in (36)*, replacing $\mathcal{L}$ by $\tilde{\mathcal{L}}$, as well as the process defined as in* (37)*, are $(\bar{F}_s)$-martingales for all bounded f having continuous time derivative.*

Proof of Proposition 3.2. We first prove the second assertion, namely (29). Fix $t_0 \in [0, T)$, x_0 and $G_n \ni x^n \to x_0$. It follows from Lemma 4.1 that to show

$$\liminf_{n \to \infty} Q^n(t_0, x^n) \geq V^+(t_0, x_0), \quad (38)$$

it suffices that for each $\beta \in B_b$ (where b is sufficiently large),

$$\liminf_{n \to \infty} Q^n(t_0, x^n) \geq c(t_0, x_0, \beta) := \inf_{u \in \bar{U}} c(t_0, x_0, u, \beta[u]). \quad (39)$$

We fix such β and turn to prove (39).

Since U is compact and convex, and the objective function in (33) is affine in u and concave in m, the outer minimum in (33) is achieved. We denote by $u = u^n(t, x)$ a minimizer in (33). Furthermore, the minimizer $u^n(t, x)$ can be selected as a measurable function of t and x (see Theorem 2.2 in [25]). Then from (33) we have

$$\mathcal{L}^{n, u^n(t,x), m} Q^n(t, x) + \frac{d}{dt} Q^n(t, x) + h(x) - \rho(u^n(t, x), m) \leq 0, \qquad m \in M, t \in [t_0, T], x \in G_n. \quad (40)$$

We invoke Lemma 4.2(i) with $u = u^n$ and the given β. With replacing x by $\bar{X}^n_t$ in (40), $u^n(t,x)$ by $\bar{u}^n(t) = u^n(t, \bar{X}^n(t))$ and $m = \bar{m}^n(t) := \beta[\bar{u}^n](t)$, we have, $\bar{\mathbb{P}}$-a.s.,

$$\mathcal{L}^{n,\bar{u}^n(t),\bar{m}^n(t)} Q^n(t, \bar{X}^n(t)) + \frac{d}{dt} Q^n(t, \bar{X}^n(t)) + h(\bar{X}^n(t)) - \rho(\bar{u}^n(t), \bar{m}^n(t)) \le 0.$$

Consider the stopping times $\tau_L := \inf\{t \ge t_0 : \|\bar{X}^n_t - x_0\| \ge L\}$ for positive L. Set $Q^n_L := \max_{[t_0,T]\times B_L(x_0)} Q^n(t,x)$, where $B_L(x_0)$ is the intersection of G_n and the I-dimensional ball of radius L, centered at x_0. Substituting $s = T \wedge \tau_L$, $f = Q^n \wedge Q^n_L$ in (36), using the above inequality and taking expectation,

$$\bar{\mathbb{E}}^n \left[Q^n(T \wedge \tau_L, \bar{X}^n(T \wedge \tau_L)) - Q^n(t_0, x^n) + \int_{t_0}^{T\wedge\tau_L} (h(\bar{X}^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r)))dr \right] \le 0.$$

Denote by $\kappa_{n,L}$ the random variable inside the expectation. The assumed monotonicity of h and g implies that they are bounded below by $h(0)$ and $g(0)$, respectively. It follows by definition of Q^n that it is also bounded below. Since we also have $\bar{m}^n(r) \in M_b$, $\kappa_{n,L}$ is bounded below by a constant not depending on L. Hence, using Fatou's lemma, $\bar{\mathbb{E}}^n \liminf_L \kappa_{n,L} \le 0$. Since $\beta \in B_b$, the processes $\bar{X}^n$ are dominated in law by n^{-1} times a Poisson process with a given rate (that depends only on n). Hence $\lim_L \tau_L = \infty$, a.s., and

$$\bar{\mathbb{E}}^n \left[Q^n(T, \bar{X}^n(T)) - Q^n(t_0, x^n) + \int_{t_0}^{T} (h(\bar{X}^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r)))dr \right] \le 0.$$

Since $Q^n(T,x) = g(x)$ for all x, we obtain

$$Q^n(t_0, x^n) \ge \bar{\mathbb{E}}^n \left[\int_{t_0}^{T} (h(\bar{X}^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r)))dr + g(\bar{X}^n(T)) \right]. \tag{41}$$

From the definition of c we have $\bar{\mathbb{P}}$-a.s.,

$$\int_{t_0}^{T} \left[h(\varphi^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r)) \right] dr + g(\varphi^n(T)) \ge c(t_0, x_0, \beta), \tag{42}$$

where $\varphi^n = \Gamma(\psi^n)$ and $\psi^n(s) := x_0 + \int_{t_0}^s v(\bar{u}^n(r), \bar{m}^n(r))dr$, $s \in [t_0, T]$. Combining (41) and (42),

$$Q^n(t_0, x^n) \ge c(t_0, x_0, \beta) - \varepsilon_n,$$

where

$$\varepsilon_n = \bar{\mathbb{E}}^n \Big[\int_{t_0}^{T} |h(\bar{X}^n(r)) - h(\varphi^n(r))| dr + |g(\bar{X}^n(T)) - g(\varphi^n(T))| \Big]. \tag{43}$$

Using the Lipschitz continuity of h, g and the map Γ (in the sense of (25)), denoting $\|f\|^* := \sup_{[t_0,T]} \|f\|$, we have

$$\varepsilon_n \le c_1 \bar{\mathbb{E}}^n [\|\bar{Y}^n - \psi^n\|^*],$$

where c_1 is a constant not depending on n.

Toward proving that ε_n converges to zero, write $\bar{m}^n(s) = (\bar{\lambda}_i^n(s), \bar{\mu}_{ij}^n(s))$, and observe by (37) with $f(s,y) = y_i$, and (12), that

$$
\begin{aligned}
\bar{Y}^n(s) - x^n &= \int_{t_0}^{s} \left[\sum_i \bar{\lambda}_i^n(r) e_i - \sum_{ij} \bar{\mu}_{ij}^n(r) \bar{u}_{ij}^n(r) e_i \right] dr + \eta_1^n(s) \\
&= \int_{t_0}^{s} v(\bar{u}^n(r), \beta[\bar{u}^n](r)) dr + \eta_1^n(s) \\
&= \psi^n(s) - x_0 + \eta_1^n(s),
\end{aligned}
$$

where each of the components of η_1^n is a zero mean martingale. Given i, write M^n for the ith component $\langle e_i, \eta_1^n \rangle$. By the Burkholder-Davis-Gundy inequality,

$$
\bar{\mathbb{E}}^n\{(\|M\|^*)^2\} \le c_2 \bar{\mathbb{E}}^n\{[M^n, M^n]_T\},
$$

where c_2 is a universal constant, and $[M^n, M^n]$ is the quadratic variation process (see [26] p. 58, and p. 175). Note that M^n has sample paths that are piecewise absolutely continuous, null at zero. Hence $[M^n, M^n]_T$ is given by $\sum_{s \le T} \varDelta M^n(s)^2$ (see for example [26], Theorem 22(ii), p. 59). Each jump of M^n is of size n^{-1}. Hence $\bar{\mathbb{E}}^n\{(\|M^n\|^*)^2\} \le c_2 n^{-2} \bar{\mathbb{E}}^n[N^n]$, where N^n is the number of jumps of η_1^n (equivalently, $\bar{Y}^n$) in the interval. Since $\bar{m}^n$ is bounded, N^n is dominated by a Poisson r.v. of mean $O(n)$. This shows $\bar{\mathbb{E}}^n\{(\|M^n\|^*)^2\} \le O(n^{-1})$. As a consequence, $\bar{\mathbb{E}}^n \|\eta_1^n\|^* \to 0$, and since $x^n \to x_0$, $\varepsilon_n \to 0$. This shows (39) and completes the proof of the second assertion of the Proposition.

We next prove the first assertion of the proposition. To this end, we fix t_0 and a sequences $G_n^* \ni x^n \to x_0$. To prove

$$
\limsup_{n \to \infty} R^n(t_0, x^n) \le V^-(t_0, x_0), \tag{44}
$$

it suffices to show that, for any $\alpha \in A$,

$$
\limsup_{n \to \infty} R^n(t_0, x^n) \le \tilde{c}(t_0, x_0, \alpha) := \sup_{m \in \bar{M}} c(t_0, x_0, \alpha[m], m). \tag{45}
$$

Thus, fixing α, we will prove (45).

Using the relation between R^n and Q^n given in (32), it follows from (33) that the function $R^n : [0,T] \times G_n^* \to \mathbb{R}$ (that is, the restriction of R^n to $[0,T] \times G_n^*$, that we still denote by R^n) is continuously differentiable in t for every x, and satisfies

$$
\begin{cases}
\inf_{u \in U} \sup_{m \in M} (\tilde{\mathcal{L}}^{n,u,m} R^n(t,x) + \frac{d}{dt} R^n(t,x) + h(x) - \rho(u,m)) = 0, & t \in [0,T],\ x \in G_n^*, \\
R^n(T,x) = g(x), \qquad x \in G_n^*.
\end{cases} \tag{46}
$$

By [27], Corollary 37.3.2, we may interchange the order of infimum and supremum in (46). It is easy to see that the supremum over m is achieved, as that of a continuous function with compact super level sets. We denote by $m = m^n(t,x)$ a point where this maximum (with reversed order) is achieved. Thus

$$
\tilde{\mathcal{L}}^{n,u,m^n(t,x)} R^n(t,x) + \frac{d}{dt} R^n(t,x) + h(x) - \rho(u, m^n(t,x)) \ge 0 \qquad u \in U, t \in [t_0,T], x \in G_n^*. \tag{47}
$$

Toward using Lemma 4.2(ii), let us argue that m^n is bounded. Indeed, by the structure (35) of $\tilde{\mathcal{L}}$ and the estimate (20) on R^n, the first term on the r.h.s. of (47) is bounded by $C\|m^n\|$, where C is a constant and $\|m^n\| = \sup_{t,x} \|m^n(t,x)\|$. Since by (22) we also have that $\frac{d}{dt}R^n(t,x)$ is uniformly bounded, and h is bounded by assumption, this gives, for every u, t, x the inequality

$$\rho(u, m^n(t,x)) \le C(1 + \|m^n\|),$$

for some constant C independent of u, t, x, n. By the form (13) of ρ, noting that l is superlinear and selecting u bounded away from zero, it follows that $\gamma(\|m^n\|) \le C(1 + \|m^n\|)$ where γ is some function satisfying $\gamma(r)/r \to \infty$ as $r \to \infty$. This shows that m^n are bounded.

Consider Lemma 4.2(ii) with $m = m^n$ and the given α. Replace x by $\bar{X}^n_t$ in (47) (note that $\bar{X}^n$ takes values in G^*_n), $m^n(t,x)$ by $\bar{m}^n(t) = m^n(t, \bar{X}^n(t))$ and u by $\bar{u}^n := \alpha[\bar{m}^n](t)$, to obtain, $\bar{\mathbb{P}}$-a.s.,

$$\tilde{\mathcal{L}}^{n,\bar{u}^n(t),\bar{m}^n(t)} R^n(t, \bar{X}^n(t)) + \frac{d}{dt} R^n(t, \bar{X}^n(t)) + h(\bar{X}^n(t)) - \rho(\bar{u}^n(t), \bar{m}^n(t)) \ge 0.$$

Take expectation in (36), substitute $t = T$, $f = R^n$ and use $R^n(T, x) = g(x)$ for all x, to obtain

$$R^n(t, x^n) \le \bar{\mathbb{E}}^n \left[\int_t^T (h(\bar{X}^n(s)) - \rho(u^n(s), \bar{m}^n(s)))ds + g(\bar{X}^n(T)) \right]. \tag{48}$$

Here, we have omitted an argument to go from a truncated version of R^n to R^n, analogous to that used in the first part of the proof. By definition of $\tilde{c}$ we have $\bar{\mathbb{P}}$-a.s.,

$$\int_{t_0}^T [h(\varphi^n(s)) - \rho(\bar{u}^n(s), \bar{m}^n(s))] \, ds + g(\varphi^n_T) \le \tilde{c}(t_0, x_0, \alpha), \tag{49}$$

where $\varphi^n = \Gamma(\psi^n)$ and $\psi^n(s) := x_0 + \int_t^s v(\bar{u}^n(r), \bar{m}^n(r))dr$, $s \in [t_0, T]$. Thus

$$R^n(t_0, x^n) \le \tilde{c}(t_0, x_0, \alpha) + \tilde{\varepsilon}_n,$$

where $\tilde{\varepsilon}_n$ has the same form as ε_n of (43). The argument that $\tilde{\varepsilon}_n \to 0$ is similar to that for ε_n. This establishes (45) and completes the proof of assertion (44). $\square$

Proof of Lemma 4.1. Without loss of generality we set $t = 0$. We only prove $V^{b,+}(0,x) = V^+(0,x)$ (for b sufficiently large), because the proof regarding V^- is similar. Recall that B_b denotes the set of strategies of perturbed rates whose components are all bounded above by the constant b. Denote by $B_{b,\mu}, B_{b,\lambda} \subset B$ the sets of strategies of perturbed rates whose service and, respectively, arrival components are always bounded above by b. Thus $B_b = B_{b,\mu} \bigcap B_{b,\lambda}$.

Corresponding to a given $m \in \bar{M}$, we construct a specific truncation $\tilde{m}$, where only the service rates are truncated in the following way: $\tilde{\mu}_{ij} := \bar{\mu}_{ij} \wedge b$ for all i, j, whereas $\tilde{\lambda}_i = \bar{\lambda}_i$. Given $m \in \bar{M}$ and $u \in \bar{U}$, denote by φ and $\tilde{\varphi}$ the state dynamics for (m, u) and, respectively, $(\tilde{m}, u)$. Using vector relation $v(u(t), m(t)) \le v(u(t), \tilde{m}(t))$ for all $t \in [0, T]$, we have $\psi(t) \le \tilde{\psi}(t)$ (in the usual partial order on $\mathbb{R}^I$) and, using the monotonicity of Γ, $\varphi(t) \le \tilde{\varphi}(t)$. Note that for b sufficiently large,

$$\rho(u(t), m(t)) \ge \rho(u(t), \tilde{m}(t)) \qquad t \in [0, T].$$

Given $\beta \in B$, let $\tilde{\beta} \in B_{b,\mu}$ denote the corresponding modification of β. Since h and g are nondecreasing functions, by the above analysis we obtain $c(0,x,u,\beta[u]) \le c(0,x,u,\tilde{\beta}[u])$. Thus

$$\begin{aligned}\sup_{\beta\in B_{b,\lambda}} \inf_{\bar{U}} c(0,x,u,\beta[u]) &\le \sup_{\beta\in B_{b,\lambda}} \inf_{\bar{U}} c(0,x,u,\tilde{\beta}[u]) \\ &= \sup_{\beta\in B_{b,\lambda}\bigcap B_{b,\mu}} \inf_{\bar{U}} c(0,x,u,\beta[u]) = V^{b,+}(0,x).\end{aligned}$$

Hence, to show $V^+ = V^{b,+}$ it is sufficient to prove that

$$V^+(0,x) \le \sup_{\beta\in B_{b,\lambda}} \inf_{\bar{U}} c(0,x,u,\beta[u]). \tag{50}$$

Select b larger, if necessary, so that $b \ge b^* := \max_{i\le I} \lambda_i e^L > 0$ where $L = C_\Gamma(TC_h + C_g)$. This assures $\rho_i'(u,m)|_{\bar{\lambda}_i=b*} \ge L$ for all i, where we denote $\rho_i'(u,m) = \frac{\partial}{\partial\bar{\lambda}_i}\rho(u,m) = \log(\bar{\lambda}_i/\lambda_i)$.

We use the same notation, $\tilde{m}$, to specify a different modification of m, where now only the arrival rates are truncated as in $\tilde{\lambda}_i := \bar{\lambda}_i \wedge b$. We continue to use $\tilde{\psi}$ and $\tilde{\varphi}$ for the corresponding state dynamics. Given u and m,

$$\begin{aligned}&\int_0^T h(\varphi(t)) - h(\tilde{\varphi}(t))dt + g(\varphi(T)) - g(\tilde{\varphi}(T)) \\ &\le\ C_h \int_0^T \|\varphi(t) - \tilde{\varphi}(t)\|dt + C_g\|\varphi(T) - \tilde{\varphi}(T)\| \\ &\le\ L\sum_i \int_0^T (\bar{\lambda}_i(t) - \tilde{\lambda}_i(t))dt.\end{aligned} \tag{51}$$

By convexity of $m \mapsto \rho(u,m)$, we have for all t

$$\rho(u(t),m(t)) - \rho(u(t),\tilde{m}(t)) \ \ge\ \sum_i \rho_i'(u(t),\tilde{m}(t))(\bar{\lambda}_i(t) - \tilde{\lambda}_i(t)).$$

Note that $\bar{\lambda}_i(t) - \tilde{\lambda}_i(t)$ is nonnegative, and when it is positive one has $\tilde{\lambda}_i(t) = b \ge b^*$. Hence, due to the assigned value of b^*, the ith term in (52) is bounded below by $L(\bar{\lambda}_i(t) - \tilde{\lambda}_i(t))$. Integrating and using (51) gives $c(0,x,u,\beta[u]) \le c(0,x,u,\tilde{\beta}[u])$. Hence

$$\begin{aligned}V^+(0,x) = \sup_{\beta\in B} \inf_{\bar{U}} c(0,x,u,\beta[u]) &\le \sup_{\beta\in B} \inf_{\bar{U}} c(0,x,u,\tilde{\beta}[u]) \\ &= \sup_{\beta\in B_{b,\lambda}} \inf_{\bar{U}} c(0,x,u,\beta[u]).\end{aligned}$$

This gives (50) and completes the proof. □

5 The PDE and the differential game

In this section we establish uniqueness of solutions to the PDE (15) by proving Proposition 3.1, and state and prove Lemma 5.1 regarding regularity of V^+ and V^-.

Proof of Proposition 3.1. In this proof we write $H(p,x)$ for $H(p)+h(x)$. We will use the continuity of $p \mapsto H(p)$, that can be verified directly, using convexity of $H(p,u,m) = \langle p, v(u,m)\rangle - \rho(u,m)$ in u and concavity in m.

For $a > 0$ let

$$\begin{aligned} U(t,x) &:= u(t,x) - ae^{-\langle \mathbf{e},x\rangle} \\ V(t,x) &:= v(t,x) + ae^{-\langle \mathbf{e},x\rangle} \end{aligned}$$

where $\mathbf{e} = \sum_{i=1}^K e_i \in \mathbb{R}^K$. To prove that $u \le v$ we arguing by contradiction and assume that

$$\varrho := \sup_{[0,T]\times G} [u(t,x) - v(t,x)] > 0.$$

Hence there exists $(\tau, z) \in [0,T] \times G$ and $a_0 > 0$ such that for all $a \in (0, a_0)$,

$$U(\tau,z) - V(\tau,z) \ge \frac{2}{3}\varrho. \tag{52}$$

For $\varepsilon, \delta > 0$, introduce

$$\Phi(s,t,x,y) := U(s,x) - V(t,y) - \frac{1}{\varepsilon^2}\|x-y\|^2 - \frac{1}{\varepsilon^2}(t-s)^2 - \varepsilon(\|x\|^2 + \|y\|^2) - \delta(2T - s - t).$$

Note that $|U(s,x)| + |V(t,y)| \le c + c|x| + c|y|$, where $c = c(u,v,a)$. Thus $\Phi \downarrow -\infty$ as $(\|x\|^2 + \|y\|^2) \uparrow \infty$, and Φ admits a maximizer $(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon)$ over $[0,T]^2 \times G^2$. Therefore there exist positive numbers ε and δ such that

$$\Phi(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon) \ \ge \ \Phi(\tau,\tau,z,z) \ \ge \ \frac{2}{3}\varrho - 2\varepsilon\|z\|^2 - 2\delta(T-\tau) \ \ge \ \frac{\varrho}{2} \tag{53}$$

for all $a \in (0, a_0)$. In what follows, δ remains fixed while ε is made smaller (eventually, a will also be taken small). Since $\Phi(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon) > 0$, we have

$$U(s^\varepsilon, x^\varepsilon) - V(t^\varepsilon, y^\varepsilon) \ > \ \frac{1}{\varepsilon^2}\|x^\varepsilon - y^\varepsilon\|^2 + \frac{1}{\varepsilon^2}(t^\varepsilon - s^\varepsilon)^2 + \varepsilon(\|x^\varepsilon\|^2 + \|y^\varepsilon\|^2) + \delta(2T - s^\varepsilon - t^\varepsilon). \tag{54}$$

Since u and v both satisfy the terminal condition, namely, $u(T,\cdot) = v(T,\cdot) = g(\cdot)$ and U, V and g are Lipschitz, there are constants k_1 and k_2 such that

$$k_1\|x^\varepsilon - y^\varepsilon\| + k_2 \ge U(s^\varepsilon, x^\varepsilon) - V(t^\varepsilon, y^\varepsilon). \tag{55}$$

We argue that the left side is bounded for all positive ε. If not true, then there is a small enough ε such that $\frac{1}{\varepsilon^2}\|x^\varepsilon - y^\varepsilon\| \ge k_1\|x^\varepsilon - y^\varepsilon\| + k_2$. But this along with (55) and (54) lead to a contradiction. Thus the left side of (54) is bounded for all positive ε. Hence from (54) we conclude the following estimates:

$$\|x^\varepsilon - y^\varepsilon\| \le O(\varepsilon), \qquad |t^\varepsilon - s^\varepsilon| \le O(\varepsilon), \qquad \|x^\varepsilon\| \le O(\frac{1}{\sqrt{\varepsilon}}), \qquad \|y^\varepsilon\| \le O(\frac{1}{\sqrt{\varepsilon}}). \tag{56}$$

Next we show that

$$\|x^\varepsilon - y^\varepsilon\| + |t^\varepsilon - s^\varepsilon| = o(\varepsilon). \tag{57}$$

Using $\Phi(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon) \geq \Phi(s^\varepsilon, s^\varepsilon, x^\varepsilon, x^\varepsilon)$, we obtain

$$\begin{aligned}\frac{1}{\varepsilon^2}(\|x^\varepsilon - y^\varepsilon\|^2 + (t^\varepsilon - s^\varepsilon)^2) &\leq V(s^\varepsilon, x^\varepsilon) - V(t^\varepsilon, y^\varepsilon) + \varepsilon(\|x^\varepsilon\|^2 - \|y^\varepsilon\|^2) + \delta(t^\varepsilon - s^\varepsilon) \\ &\leq \omega_V(\|x^\varepsilon - y^\varepsilon\| + |t^\varepsilon - s^\varepsilon|) + \varepsilon\langle x^\varepsilon + y^\varepsilon, x^\varepsilon - y^\varepsilon\rangle + \delta(t^\varepsilon - s^\varepsilon)\end{aligned}$$

where ω_V is the modulus of continuity of V. Therefore (57) follows by using the estimates (56) in the above inequality. Next we show that

$$s^\varepsilon, t^\varepsilon < T \text{ for all sufficiently small } \varepsilon > 0. \tag{58}$$

To this end, note by (53) that

$$u(s^\varepsilon, x^\varepsilon) - v(t^\varepsilon, y^\varepsilon) \geq \frac{\varrho}{2}. \tag{59}$$

Now if any of s^ε and t^ε equals to T, then $|T - s^\varepsilon| = o(\varepsilon)$ and $|T - t^\varepsilon| = o(\varepsilon)$ hold. Thus using Lipschitz continuity of g and denoting by ω_u and ω_v the modulus of continuity of u and v, resp.,

$$\begin{aligned}|u(s^\varepsilon, x^\varepsilon) - v(t^\varepsilon, y^\varepsilon)| &\leq |u(s^\varepsilon, x^\varepsilon) - g(x^\varepsilon)| + |v(t^\varepsilon, y^\varepsilon) - g(y^\varepsilon)| + |g(x^\varepsilon) - g(y^\varepsilon)| \\ &\leq \omega_u(T - s^\varepsilon) + \omega_v(T - t^\varepsilon) + C_g\|x^\varepsilon - y^\varepsilon\| \to 0,\end{aligned}$$

by (57). This contradicts (59). Therefore (58) holds.

Let

$$\theta(s, x) := \frac{1}{\varepsilon^2}\|x - y^\varepsilon\|^2 + \frac{1}{\varepsilon^2}(t^\varepsilon - s)^2 + \varepsilon\|x\|^2 + \delta(T - s) + ae^{-\langle \mathbf{e}, x\rangle}.$$

By the definition of $(s^\varepsilon, x^\varepsilon)$, $(s, x) \mapsto u(s, x) - \theta(s, x)$ has local maximum at $(s^\varepsilon, x^\varepsilon)$. Since $D\theta(s^\varepsilon, x^\varepsilon) = \frac{2}{\varepsilon^2}(x^\varepsilon - y^\varepsilon) + 2\varepsilon x^\varepsilon - ae^{-\langle \mathbf{e}, x^\varepsilon\rangle}\mathbf{e}$ we have

$$\max_{i \in \mathbf{I}(x^\varepsilon)} \langle D\theta(s^\varepsilon, x^\varepsilon), e_i\rangle < 0. \tag{60}$$

Hence by definition of viscosity subsolution

$$\begin{aligned}0 &\leq \frac{\partial}{\partial s}\theta(s^\varepsilon, x^\varepsilon) + H(D\theta(s^\varepsilon, x^\varepsilon), x^\varepsilon) \\ &= -\frac{2}{\varepsilon^2}(t^\varepsilon - s^\varepsilon) - \delta + H(D\theta(s^\varepsilon, x^\varepsilon), x^\varepsilon).\end{aligned}$$

Similarly, for the following test function

$$\vartheta(t, y) := -\left(\frac{1}{\varepsilon^2}\|x^\varepsilon - y\|^2 + \frac{1}{\varepsilon^2}(t - s^\varepsilon)^2 + \varepsilon\|y\|^2 + \delta(T - t)\right) - ae^{-\langle \mathbf{e}, y\rangle}$$

the map $(t, y) \mapsto v(t, y) - \vartheta(t, y)$ has local minimum at $(t^\varepsilon, y^\varepsilon)$. Analogous to the prior, by the definition of viscosity supersolution we obtain

$$\begin{aligned}0 &\geq \frac{\partial}{\partial s}\vartheta(t^\varepsilon, y^\varepsilon) + H(D\vartheta(t^\varepsilon, y^\varepsilon), y^\varepsilon) \\ &= -\frac{2}{\varepsilon^2}(t^\varepsilon - s^\varepsilon) + \delta + H(D\vartheta(t^\varepsilon, y^\varepsilon), y^\varepsilon).\end{aligned}$$

From above two inequalities we obtain

$$2\delta + H\left(D\vartheta(t^\varepsilon, y^\varepsilon), y^\varepsilon\right) - H\left(D\theta(s^\varepsilon, x^\varepsilon), x^\varepsilon\right) \leq 0. \tag{61}$$

Again using (56) and (57),

$$\begin{aligned}
&\|D\vartheta(t^\varepsilon, y^\varepsilon) - D\theta(s^\varepsilon, x^\varepsilon)\| \\
= \;& \|\frac{2}{\varepsilon^2}(x^\varepsilon - y^\varepsilon) - 2\varepsilon y^\varepsilon + ae^{-\langle \mathbf{e}, y^\varepsilon\rangle}\mathbf{e} - \left(\frac{2}{\varepsilon^2}(x^\varepsilon - y^\varepsilon) + 2\varepsilon x^\varepsilon - ae^{-\langle \mathbf{e}, x^\varepsilon\rangle}\mathbf{e}\right)\| \\
= \;& \| - 2\varepsilon(x^\varepsilon + y^\varepsilon) + a\left(e^{-\langle \mathbf{e}, x^\varepsilon\rangle} + e^{-\langle \mathbf{e}, y^\varepsilon\rangle}\right)\mathbf{e}\| \\
\leq \;& O(\sqrt{\varepsilon}) + 2Ia.
\end{aligned}$$

Therefore

$$\limsup_{\varepsilon\to 0} \|D\vartheta(t^\varepsilon, y^\varepsilon) - D\theta(s^\varepsilon, x^\varepsilon)\| \leq 2Ia. \tag{62}$$

Recall that u and v are Lipschitz in x uniformly over $[0, T] \times G$, and denote the maximal of their Lipschitz constants by C. Let us argue that

$$\|D\theta(s^\varepsilon, x^\varepsilon)\| \leq C, \qquad \|D\vartheta(t^\varepsilon, y^\varepsilon)\| \leq C. \tag{63}$$

Because $(s, x) \mapsto u(s, x) - \theta(s, x)$ has a local maximum at $(s^\varepsilon, x^\varepsilon)$, we have for any $x \in G$,

$$\theta(s^\varepsilon, x) - \theta(s^\varepsilon, x^\varepsilon) \geq u(s^\varepsilon, x) - u(s^\varepsilon, x^\varepsilon) \geq -C\|x - x^\varepsilon\|,$$

and so $\|D\theta(s^\varepsilon, x^\varepsilon)\| \leq C$ provided x^ε is an interior point. If $x^\varepsilon \in \partial G$ then from the above display we can still deduce $|\langle e_i, D\theta(s^\varepsilon, x^\varepsilon)\rangle| \leq C$ for all $i \notin \mathbf{I}(x^\varepsilon)$. For $i \in \mathbf{I}(x^\varepsilon)$ we obtain $\langle D\theta(s^\varepsilon, x^\varepsilon), e_i\rangle \geq -C$ by the same inequality, which along with (60) again gives $|\langle D\theta(s^\varepsilon, x^\varepsilon), e_i\rangle| \leq C$. This shows (63) holds for θ, and for ϑ the argument is similar.

Thus using the uniform continuity of $(p, x) \mapsto H(p, x)$ on $B_C \times G$ and denoting the corresponding modulus of continuity by ω_C, we obtain from (57), (61) and (62)

$$2\delta \leq \omega_C(2Ia). \tag{64}$$

Note that the above holds for all $a \in (0, a_0)$ and choice of δ does not depend on a. This gives a contradiction for $\delta > 0$ fixed and small $a > 0$, and completes the proof of the result. $\square$

Lemma 5.1. *The functions $V^-, V^+ : [0, T] \times G \to \mathbb{R}$ are globally Lipschitz continuous.*

Proof. Fix $s, t \in [0, T]$ and $x, y \in G$. In view of (30), given a positive constant ε there exist $\beta^\varepsilon \in B_b$ such that

$$V^+(s, x) - \varepsilon \leq \inf_{u\in\bar{U}} c(s, x, u, \beta^\varepsilon[u]).$$

For this particular β^ε, there exists a $u^\varepsilon \in \bar{U}$ such that

$$\begin{aligned}
c(t, y, u^\varepsilon, \beta^\varepsilon[u^\varepsilon]) \;&\leq\; \inf_{\bar{U}} c(t, y, u, \beta^\varepsilon[u]) + \varepsilon \\
&\leq\; \sup_{\beta\in B_b} \inf_{\bar{U}} c(t, y, u, \beta[u]) + \varepsilon \\
&=\; V^+(t, y) + \varepsilon.
\end{aligned}$$

Therefore

$$V^+(s,x) \le c(s,x,u^\varepsilon,\beta^\varepsilon[u^\varepsilon]) + \varepsilon, \qquad V^+(t,y) \ge c(t,y,u^\varepsilon,\beta^\varepsilon[u^\varepsilon]) - \varepsilon.$$

Thus

$$V^+(s,x) - V^+(t,y) \le c(s,x,u^\varepsilon,m^\varepsilon) - c(t,y,u^\varepsilon,m^\varepsilon) + 2\varepsilon \tag{65}$$

where $m^\varepsilon := \beta^\varepsilon[u^\varepsilon]$. Note by the definition of $\bar{U}$ and $\bar{M}$ that u^ε and m^ε are defined over the interval $[0,T]$. Let $\varphi^i_\varepsilon := \Gamma(\psi^i_\varepsilon)$ for $i = 1,2$ where

$$\begin{aligned}
\psi^1_\varepsilon(\tau) &:= x + \int_s^\tau v(u^\varepsilon(z), m^\varepsilon(z))dz \qquad \tau \in [s,T] \\
\psi^2_\varepsilon(\tau) &:= y + \int_t^\tau v(u^\varepsilon(z), m^\varepsilon(z))dz \qquad \tau \in [t,T].
\end{aligned}$$

If $s < t$, set $\psi^2_\varepsilon = y$ on $[s,t]$. Otherwise, set $\psi^1_\varepsilon = x$ on $[t,s]$. Note that ψ^1_ε and ψ^2_ε are Lipschitz continuous due to the upper bound of each component of m^ε. Note also that $\|\psi^1_\varepsilon - \psi^2_\varepsilon\|^* := \max_{[s\wedge t,T]} \|\psi^1_\varepsilon - \psi^2_\varepsilon\| \le \|x-y\| + C|t-s|$, for some constant C. Thus by (27) we have

$$\begin{aligned}
&c(s,x,u^\varepsilon,m^\varepsilon) - c(t,y,u^\varepsilon,m^\varepsilon) \\
&= \int_t^T \big(h(\varphi^1_\varepsilon(z))dz - h(\varphi^2_\varepsilon(z))\big)\,dz \\
&\quad + \int_s^t \big(h(\varphi^1_\varepsilon(z)) - \rho(u^\varepsilon(z), m^\varepsilon(z))\big)\,dz + g(\varphi^1_\varepsilon(T)) - g(\varphi^2_\varepsilon(T)) \\
&\le \int_t^T \Big(h \circ \Gamma(\psi^1_\varepsilon)(z) - h \circ \Gamma(\psi^2_\varepsilon)(z)\Big)dz \\
&\quad + \int_{s\wedge t}^{s\vee t} [h \circ \Gamma(\psi^1_\varepsilon)(z) + \max_{U\times M_b} \rho(u,m)]dz + C_g\|\varphi^1_\varepsilon(T) - \varphi^2_\varepsilon(T)\| \\
&\le C_1\|\psi^1_\varepsilon - \psi^2_\varepsilon\|^* + C_2|t-s|
\end{aligned}$$

where C_1 and C_2 are constants (that may depend on T) and we used the Lipschitz continuity of g, h and Γ, and the boundedness of h. Hence the Lipschitz continuity of V^+ follows from (65) and the above inequality. Analogously, V^- can be shown to be Lipschitz continuous. □

6 An explicit solution

We find an explicit formula for V in the case where the terminal cost is of the form $g(x) = \sum_i c_i x_i$, and the running cost h vanishes. One could approach this by working with the PDE (as e.g. in [5]). We will find V by analyzing the DG.

We give an exact treatment for the case of a zero initial condition. We later consider the large time behavior of the game under an arbitrary initial condition, and show that the zero initial condition case plays a major role in this problem, in the sense that it provides the leading term, as $T \to \infty$.

Finally, we specialize to the case when the servers are homogenous (i.e., there is only one type of servers) and present the main result of this section, identifying an optimal strategy for the minimizing player for the zero initial condition case. This strategy is dictated by a fixed priority rule, that is reminiscent of the classical $c\mu$ rule.

Fix T and set $t = 0$. Then under the hypotheses of this section, the cost (27) is given by

$$c(u, m) = -\int_0^T \rho(u(s), m(s))ds + \sum_i c_i\varphi_i(T), \tag{66}$$

where

$$\begin{cases} \psi(s) = x + \displaystyle\int_0^s v(u(r), m(r))dr, \quad s \in [0, T], \\ \\ \varphi = \Gamma[\psi]. \end{cases} \tag{67}$$

Recall that $V = V^+ = V^-$. We will work with the lower value of the game, V^-, i.e.,

$$V = \inf_{\alpha\in A} \sup_{m\in\bar{M}} c(\alpha[m], m). \tag{68}$$

For $y \ge 0$, denote $l_i(y) = \lambda_i l(y/\lambda_i)$ and $l_{ij}(y) = \mu_{ij} l(y/\mu_{ij})$. By (68) and (66),

$$V = \inf_{\alpha\in A} \sup_{m\in\bar{M}} \Big[-\int_0^T \sum_i l_i(\bar\lambda_i(s))ds - \int_0^T \sum_{ij} u_{ij}(s) l_{ij}(\bar\mu_{ij}(s))ds + \sum_i c_i\varphi_i(T)\Big], \tag{69}$$

where $m = (\bar\lambda_i(\cdot), \bar\mu_{ij}(\cdot))$, $u = \alpha[m]$ and $\varphi = \varphi(u, m)$ is given by (67). Denote

$$\lambda_i^* = \lambda_i e^{c_i}, \qquad \mu_{ij}^* = \mu_{ij} e^{-c_i}, \qquad i \in \mathcal{I},\, j \in \mathcal{J}. \tag{70}$$

$$\hat\lambda_i = \lambda_i(e^{c_i} - 1), \qquad \hat\mu_{ij} = \mu_{ij}(1 - e^{-c_i}), \qquad i \in \mathcal{I},\, j \in \mathcal{J}. \tag{71}$$

Set

$$W = \min_{u\in U} \sum_i \Big(\hat\lambda_i - \sum_j u_{ij}\hat\mu_{ij}\Big)^+. \tag{72}$$

Theorem 6.1. *For $x = 0$, V is given by WT.*

Proof. Using (23) and (24) and the fact $x = 0$ gives

$$\varphi_i(T) = \sup_{s\in[0,T]} [\psi_i(T) - \psi_i(s)] = \sup_{s\in[0,T]} \int_s^T v_i(u(r), m(r))dr.$$

Using this in (69) and interchanging the order of suprema gives

$$V = \inf_{\alpha\in A} \sup_{\{s_i\}} \sup_{m\in\bar{M}} \Big[-\sum_i \int_0^T l_i(\bar\lambda_i(s))ds - \sum_{ij} \int_0^T u_{ij}(s) l_{ij}(\bar\mu_{ij}(s))ds + \sum_i c_i \int_{s_i}^T v_i(u(s), m(s))ds\Big], \tag{73}$$

where the outer supremum ranges over $\{s_i\} \in [0, T]^I$.

We argue that $V \le WT$. Denote by A_0 the subset of A of 'open loop' strategies that are constant in time, namely the collection of strategies α for which $\alpha[m](s) = u$ for all $s \in [0, T]$ and all $m \in \bar{M}$, where $u \in U$. Consider the value, that we denote by V_0, obtained upon replacing A by A_0 in (73). Clearly, $V \leq V_0$. Given the $\{s_i\}$, it is easy to solve for the supremum over m. First, owing to the fact that $\int_0^{s_i} l_i(\bar{\lambda}_i(s))ds \geq 0 = \int_0^{s_i} l_i(\lambda_i)ds$, and a similar fact about l_{ij}, an optimal m necessarily sets $\bar{\lambda}_i = \lambda_i$ on $[0, s_i]$ and $\bar{\mu}_{ij} = \mu_{ij}$ on that interval. Thus we can write the expression in square brackets (recalling u is constant) as

$$\begin{aligned}
&\sum_i \int_{s_i}^T \Big(- l_i(\bar{\lambda}_i(s)) - \sum_j u_{ij} l_{ij}(\bar{\mu}_{ij}(s)) + c_i v_i(u, m(s))\Big) ds \\
&= \sum_i \int_{s_i}^T \Big(- l_i(\bar{\lambda}_i(s)) + c_i \bar{\lambda}_i(s) + \sum_j u_{ij}[-l_{ij}(\bar{\mu}_{ij}(s)) - c_i \bar{\mu}_{ij}(s)]\Big) ds.
\end{aligned}$$

For any $\alpha \in A_0$, the constant $u = \alpha[m]$, appearing in the above expression, is independent of m. Hence, given $\{s_i\}$, the supremum of the above concave function of m is easy to calculate. The maximizer is $\bar{\lambda}_i(s) = \lambda_i^*$ and $\bar{\mu}_{ij}(s) = \mu_{ij}^*$ for $s \in [s_i, T]$. Recalling the notation (71), substituting in the above display gives

$$\sum_i \int_{s_i}^T \Big(\hat{\lambda}_i - \sum_j u_{ij} \hat{\mu}_{ij}\Big) ds = \sum_i \Big(\hat{\lambda}_i - \sum_j u_{ij} \hat{\mu}_{ij}\Big)(T - s_i).$$

Thus

$$V \leq \inf_{\alpha \in A_0} \sup_{\{s_i\}} \sum_i \Big(\hat{\lambda}_i - \sum_j u_{ij} \hat{\mu}_{ij}\Big)(T - s_i) = WT. \tag{74}$$

Next we prove $V \geq WT$. Bound V below by replacing the supremum over $\{s_i\} \in [0, T]^I$ in (73) by that over $\{s_i\} \in \{0, T\}^I$. Further, replace the supremum over all $m \in \bar{M}$ by the following particular choice of m (which depends on $\{s_i\}$)

$$(\bar{\lambda}_i(s), \bar{\mu}_{ij}(s)) = \begin{cases} (\lambda_i, \mu_{ij}), & s \in [0, s_i), \\ \\ (\lambda_i^*, \mu_{ij}^*), & s \in [s_i, T], \end{cases} \qquad i \in \mathcal{I},\ j \in \mathcal{J}. \tag{75}$$

Then

$$\begin{aligned}
V &\geq \inf_{\alpha \in A} \max_{\{s_i\} \in \{0,T\}^I} \sum_i \int_{s_i}^T \Big(- l_i(\lambda_i^*) - \sum_j u_{ij}(s) l_{ij}(\mu_{ij}^*) + c_i v_i(u(s), m)\Big) ds \\
&= \inf_{\alpha \in A} \max_{\{s_i\} \in \{0,T\}^I} \sum_i \Big(\hat{\lambda}_i - \sum_j \bar{u}_{ij} \hat{\mu}_{ij}\Big)(T - s_i),
\end{aligned}$$

where $\bar{u}_{ij} = T^{-1} \int_0^T u_{ij}(s) ds$ and, as before, $u(\cdot) = \alpha[m]$. Since $\bar{u}$ is always a member of U, the above expression is equal to WT. This shows $V \geq WT$. Along with (74), we have proved $V = WT$. $\square$

Let us now fix an arbitrary initial condition x. Rather than provide an exact analysis of $V(x) := V(0, x)$, we show that it is governed, to a large extent, by $V(0, 0)$, when T is large.

Proposition 6.1. *One has*

$$WT \le V(x) \le WT + \gamma, \qquad T > 0,$$

where $\gamma = \gamma(x)$ is linear in x and does not depend on T.

Proof. Write V^0 for the value under zero initial conditions (proved to be equal to WT). Then both V^0 and $V(x)$ are given by the formula (68), $c(u,m)$ is as in (66)–(67) (with $x = 0$ for the case V^0). Denote by $\varphi^0 = \varphi^0(u,m)$ the trajectory from (67) with zero initial condition, and by $c^0(u,m)$ the corresponding cost. The one-dimensional Skorohod map is monotone in the initial condition. Thus, using (25), for any $u \in \bar{U}$ and $m \in \bar{M}$,

$$\varphi_i^0(T) \le \varphi_i(T) \le \varphi_i^0(T) + 2x_i.$$

As a result, for any $u \in \bar{U}$ and $m \in \bar{M}$,

$$c^0(u,m) \le c(u,m) \le c^0(u,m) + \gamma,$$

where γ is linear in x and does not depend on T. The result follows from this relation and (68). $\square$

In the rest of this section we set $x = 0$. Further, we specialize to homogenous servers, namely $J = 1$. Note that as a consequence of the proof of Theorem 6.1, there exists an 'open loop' strategy that is optimal, namely the strategy setting $u(s) = u^*$ for all s, where u^* is a minimizer of (72). A perhaps more useful observation, from a practical viewpoint, is that there exists a fixed priority policy that is also optimal.

Consider the strategy that prioritizes according to the index $\hat{\mu}_i := \mu_i(1 - e^{-c_i})$. That is, let the labels be ordered so that

$$\hat{\mu}_1 \ge \hat{\mu}_2 \ge \cdots \ge \hat{\mu}_I, \tag{76}$$

and let α^* be the strategy that sends $m = (\bar{\lambda}_i(s), \bar{\mu}_i(s))_{s\in[0,T]} \in \bar{M}$ to $u \in \bar{U}$, where, for $s \in [0,T]$, $\bar{\rho}_i(s) = \bar{\lambda}_i(s)/\bar{\mu}_i(s)$, and

$$u_1(s) = \begin{cases} 1 & \text{if} \quad \varphi_1(s) > 0, \\ 1 \wedge \bar{\rho}_1(s) & \text{if} \quad \varphi_1(s) = 0, \end{cases} \tag{77}$$

$$u_i(s) = \begin{cases} 1 - \sum_{m=1}^{i-1} u_m(s) & \text{if} \quad \varphi_i(s) > 0, \\ \left(1 - \sum_{m=1}^{i-1} u_m(s)\right) \wedge \bar{\rho}_i(s) & \text{if} \quad \varphi_i(s) = 0, \end{cases} \qquad i \ge 2. \tag{78}$$

The fact that these relations give rise to a well-defined strategy is proved in the appendix.

Theorem 6.2. *α^* is optimal for V of (68). That is,*

$$\sup_{m\in\bar{M}} c(\alpha^*[m], m) = \inf_{\alpha\in A} \sup_{m\in\bar{M}} c(\alpha[m], m).$$

Toward proving the result, let us introduce some notation. Since $J = 1$, the functions l_{ij} depend only on i, and we denote them by $\tilde{l}_i$. Namely, $\tilde{l}_i(y) = \mu_i l(y/\mu_i)$ (while as before $l_i(y) = \lambda_i l(y/\lambda_i)$). Let

$$C_i(u,m) = -l_i(\bar{\lambda}_i) - u_i\tilde{l}_i(\bar{\mu}_i) + c_i(\bar{\lambda}_i - u_i\bar{\mu}_i), \qquad u \in U,\ m \in M. \tag{79}$$

Given $r \geq 0$, let

$$W(r) = \min\Big\{\sum_{i=1}^{I}\Big(\hat{\lambda}_i - v_i\hat{\mu}_i\Big)^+ : v_i \geq 0, \sum_{i=1}^{I} v_i \leq r\Big\}. \tag{80}$$

Remark 6.1. *It is easy to see that the following v is a minimizer in* (80)

$$v_1^* = r \wedge \hat{\rho}_1, \qquad v_i^* = \Big(r - \sum_{m=1}^{i-1} v_m^*\Big) \wedge \hat{\rho}_i, \qquad i \geq 2,$$

where $\hat{\rho}_i = \hat{\lambda}_i/\hat{\mu}_i$.

Lemma 6.1. *Given $r \geq 0$ and $m = (\bar{\lambda}_i, \bar{\mu}_i) \in M$, one has*

$$\sum_{i=1}^{I} C_i(u,m) \leq W(r),$$

provided that

$$u_1 \in \{r,\ r \wedge \bar{\rho}_1\}, \tag{81}$$

$$u_i \in \{r - u_{1,i-1},\ (r - u_{1,i-1}) \wedge \bar{\rho}_i\}, \qquad i \geq 2, \tag{82}$$

where $\bar{\rho}_i = \bar{\lambda}_i/\bar{\mu}_i$ (here, $r \wedge (y/0)$ is interpreted as r) and $u_{1,k} = \sum_1^k u_i$.

Before presenting the proof of the lemma, we show that the theorem follows.

Proof of Theorem 6.2. The fact that a strategy α^* exists, as well as that under this strategy one has $\psi_i(s) \geq 0$ for all s, is proved in Proposition A.1 in the appendix. Fix an arbitrary $m \in \bar{M}$ and set $u = \alpha^*[m]$. To prove the theorem it suffices to show that $c(u,m) \leq WT$. Since $\psi_i(s) \geq 0$ for all s, we have $\varphi(T) = \psi(T)$. Thus $c(u,m)$ is given by

$$c(u,m) = \int_0^T \sum_i C_i(u(s), m(s))ds.$$

By (77) and (78), for each s, $u(s)$ satisfies the hypotheses of Lemma 6.1, with data $m(s)$ and $r = 1$. Hence $c(u,m) \leq WT$, which completes the proof. $\square$

Proof of Lemma 6.1. The claim is proved by induction on I. The precise statement proved by induction involves an *arbitrary* set of parameters λ_i, μ_i, c_i. Namely, given I and r, and *any* $3I$-tuple of positive numbers λ_i, μ_i, c_i, for which the parameters $\hat{\mu}_i = \mu_i(1 - e^{-c_i})$ are ordered as in (76), the statement of the lemma is valid.

Consider first $I = 1$. We will show

$$C_1(u,m) \le \begin{cases} \hat{\lambda}_1 - r\hat{\mu}_1 & \text{if} \quad u_1 = r, \\ \\ 0 & \text{if} \quad u_1 = \bar{\rho}_1. \end{cases} \tag{83}$$

First, the inequalities

$$-l_i(\bar{\lambda}_i) + c_i\bar{\lambda}_i \le \hat{\lambda}_i, \quad -\tilde{l}_i(\bar{\mu}_i) - c_i\bar{\mu}_i \le -\hat{\mu}_i \tag{84}$$

hold for every $\bar{\lambda}_i, \bar{\mu}_i$, as can be directly verified. By (79), this gives the first line in (83). If $u_1 = \bar{\rho}_1$ then the last term in (79) is zero, hence $C_1(u,m) \le 0$. This shows (83), from which it follows that $C_1(u,m) \le W(r)$ in case $I = 1$.

Next, assuming that the claim holds for a given I, we show that it holds for $I+1$. Let then r and m be given, and let u be as in (81)–(82). Denote $C_{a,b} = \sum_{i=a}^{b} C_i(u,m)$. Also, let $W_{a,b}(r)$ be defined as in (80), where the sums range from a to b. The induction assumption implies

$$C_{2,I+1} \le W_{2,I+1}(r - u_1). \tag{85}$$

Case 1: $u_1 < v_1^*$. Then by (81), $u_1 = \bar{\rho}_1$. As a result, arguing as in the induction base, $C_1(u,m) \le 0$. Thus $C_{1,I+1} \le C_{2,I+1}$. Hence by the induction assumption, $C_{1,I+1} \le W_{2,I+1}(r-u_1)$. Clearly $W(r)$ is decreasing with r. Hence

$$C_{1,I+1} \le W_{2,I+1}(r - v_1^*) \le (\hat{\lambda}_1 - v_1^*\hat{\mu}_1) + W_{2,I+1}(r - v_1^*) = W_{1,I+1}(r).$$

Case 2: $\delta := u_1 - v_1^* \ge 0$. Using again (84), $C_{1,1} \le \hat{\lambda}_1 - u_1\hat{\mu}_1$. Hence by (85),

$$C_{1,I+1} \le \hat{\lambda}_1 - u_1\hat{\mu}_1 + W_{2,I+1}(r - u_1).$$

By definition of W, it is not hard to see that $|W(r_1) - W(r_2)| \le |r_1 - r_2|\hat{\mu}_{\max}$, where $\hat{\mu}_{\max}$ is the largest parameter $\hat{\mu}_i$ involved. Thus, recalling $\hat{\mu}_2 \ge \cdots \ge \hat{\mu}_I$,

$$|W_{2,I+1}(r_1) - W_{2,I+1}(r_2)| \le |r_1 - r_2|\,\hat{\mu}_2, \qquad r_1, r_2 \ge 0.$$

As a result,

$$\begin{aligned} C_{1,I+1} &\le \hat{\lambda}_1 - u_1\hat{\mu}_1 + W_{2,I+1}(r - u_1) \\ &= \hat{\lambda}_1 - v_1^*\hat{\mu}_1 + W_{2,I+1}(r - v_1^*) - \delta\hat{\mu}_1 + W_{2,I+1}(r - u_1) - W_{2,I+1}(r - v_1^*) \\ &\le \hat{\lambda}_1 - v_1^*\hat{\mu}_1 + W_{2,I+1}(r - v_1^*) - \delta\hat{\mu}_1 + \delta\hat{\mu}_2 \\ &\le \hat{\lambda}_1 - v_1^*\hat{\mu}_1 + W_{2,I+1}(r - v_1^*) \\ &= W_{1,I+1}(r). \end{aligned}$$

We have thus shown that $C_{1,I+1} \le W_{1,I+1}(r)$ and completed the argument. □

A Appendix

We argue that the relations (77)–(78) give rise to a well-defined strategy.

Proposition A.1. *There exists a strategy $\alpha^* \in A$ with the following properties. Given $m \in \bar{M}$, let $u = \alpha^*[m]$ and let ψ and φ be given by* (67) *(with $x = 0$). Then (ψ, φ, u) satisfy the relations* (77)*–*(78)*. Moreover, $\psi_i(t) \geq 0$ for all t and i, hence $\varphi = \psi$.*

Proof. We will use the following fact regarding Γ_1 (defined in (23)). Recall that if $q : \mathbb{R}_+ \to \mathbb{R}$ and $p = \Gamma_1[q]$ then $p = q + r$ where $r(t) = -\inf_{s \leq t} q(s) \wedge 0$. In case that q is absolutely continuous and $q(0) \geq 0$, the term r is given by

$$r(t) = \int_0^t \Big(\frac{dq}{ds}\Big)^- 1_{\{p(s)=0\}} ds. \tag{86}$$

The above is an immediate consequence of a general fact that solutions p of the Skorohod problem with absolutely continuous data q solve ODE of the form

$$\dot{p} = \pi(p, \dot{q}),$$

where $\pi(x, v)$ is a certain projection map, which in the one-dimensional case is given by

$$\pi(x, v) = v 1_{\{x>0\}} + v^+ 1_{\{x=0\}}.$$

For this fact and further details see [12].

Let $m = (\bar{\lambda}_i, \bar{\mu}_i)$ be given. We will construct ψ, φ and u satisfying relations (77)–(78) and (67), and then argue that the map $m \mapsto u$ is a strategy.

For $i = 1, \ldots, I$, denote $\bar{\rho}_i(s) = \bar{\lambda}_i(s)/\bar{\mu}_i(s)$. Let $q_1 = \int_0^\cdot (\bar{\lambda}_1 - \bar{\mu}_1) ds$ and $p_1 = \Gamma_1[q_1]$. Then $p_1 = q_1 + r_1$, where, by (86),

$$r_1(t) = \int_0^t (\bar{\lambda}_1(s) - \bar{\mu}_1(s))^- 1_{\{p_1(s)=0\}} ds.$$

As a result, $p_1 \geq 0$ and can be written as

$$p_1(t) = \int_0^t (\bar{\lambda}_1(s) - u_1(s)\bar{\mu}_1(s)) ds,$$

where

$$u_1(s) = \begin{cases} 1 & \text{if} \quad p_1(s) > 0, \\ \\ 1 \wedge \bar{\rho}_1(s) & \text{if} \quad p_1(s) = 0. \end{cases}$$

Now set $\psi_1 = p_1$. Then $\psi_1 \geq 0$ hence $\varphi_1 := \Gamma_1[\psi_1] = \psi_1$, and relations (67) and (77) hold. This gives a construction of (ψ_1, φ_1, u_1).

To proceed to (ψ_i, φ_i, u_i) for $i \geq 2$, we argue recursively. Fix $i \geq 2$. Denote $u_{1,i-1} = \sum_{m=1}^{i-1} u_m$. Set $q_i = \int_0^\cdot (\bar{\lambda}_i - (1 - u_{1,i-1})\bar{\mu}_i) ds$ and $p_i = \Gamma_1[q_i]$. Arguing as before, $p_i = q_i + r_i$ where

$$r_i(t) = \int_0^t (\bar{\lambda}_i - (1 - u_{1,i-1})\bar{\mu}_i)^- 1_{\{p_i=0\}} ds,$$

hence $p_i \geq 0$ and

$$p_i(t) = \int_0^t (\bar{\lambda}_i - u_i \bar{\mu}_i) ds,$$

where

$$u_i(s) = \begin{cases} 1 - u_{1,i-1}(s) & \text{if} \quad p_i(s) > 0, \\ \\ (1 - u_{1,i-1}(s)) \wedge \bar{\rho}_i(s) & \text{if} \quad p_i(s) = 0. \end{cases}$$

Setting $\psi_i = p_i$ and $\varphi_i = \Gamma_1[\psi_i]$ gives $\varphi_i = \psi_i$ and agrees with (67) and (78). This completes the construction of (ψ, φ, u). The construction has the property that for every $t \geq 0$, $m|_{[0,t]}$ uniquely defines $(\psi, \varphi, u)|_{[0,t]}$, and moreover, the map $m \mapsto u$ is measurable. Thus the map is a strategy. $\square$

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