Large Deviations 048944

Home assignment 2: 5 February 2008. Due in class of 19 February 2008

1. If $a > \mathbb{E}\, x \geq 0$ and $\mathbb{E}\, e^{\theta x} < \infty$ for all $|\theta|$ small enough then $e^{-\theta a}\, \mathbb{E}\, e^{\theta x} < 1$ for some θ. Hint: compute derivative w.r.t. θ at $\theta = 0$.

2. Define the functions $\pi_j^n(\cdot)$ on the integers $1, \dots, n$ as follows.

$$\pi_j^n(k) = \begin{cases} k + j & \text{if } k + j \leq n \\ k + j - n & \text{if } n < k + j \leq 2n\ . \end{cases}$$

   (a) Suppose the given real numbers $\{r_k\}_{k=1}^n$ satisfy $r_1 + \cdots + r_n \geq n\beta$ for some number β. Show that there is an integer j^* so that $r_{\pi_{j^*}^n(1)} + \cdots + r_{\pi_{j^*}^n(k)} \geq k\beta$ for all $k = 1, \dots, n$. Hint: choose j^* so as to minimize $r_1 + \cdots + r_k - k\beta$.

   (b) Now let $x_1, x_2, \dots$ be i.i.d. random variables with exponential moments of all orders and let $\beta > \mathbb{E}\, x_1$. Show that the limit

$$\frac{1}{n} \lim_{n\to\infty} \log \mathbb{P}\left\{x_1 + x_2 + \cdots + x_n \geq n\beta\right\}$$

   exist. What is this limit?

   (c) Now show that

$$\frac{1}{n} \lim_{n\to\infty} \log \mathbb{P}\left\{x_1 + x_2 + \cdots + x_n \geq n\beta\right\} = \frac{1}{n} \lim_{n\to\infty} \log \mathbb{P}\left\{x_1 + x_2 + \cdots + x_k \geq k\beta, \mathit{all}\ 1 \leq k \leq n\right\}. \tag{0.0.1}$$

   Hints: assume that $\mathbb{P}\{x_1 > \beta\} > 0$, for otherwise this is trivial. Now use the first part of the exercise: $x_1 + x_2 + \cdots + x_n \geq n\beta$ if and only if $x_{\pi_j^n(1)} + x_{\pi_j^n(2)} + \cdots + x_{\pi_j^n(k)} \geq k\beta$ for all $1 \leq k \leq n$ for some (random) j. Now use a union bound.