Geometric Sampling of Manifolds for Image Representation and Processing

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Abstract. It is often advantageous in image processing and computer vision to consider images as surfaces imbedded in higher dimensional manifolds. It is therefore important to consider the theoretical and applied aspects of proper sampling of manifolds. We present a new sampling theorem for surfaces and higher dimensional manifolds. The core of the proof resides in triangulation results for manifolds with or without boundary, not necessarily compact. The proposed method adopts a geometric approach that is considered in the context of 2-dimensional manifolds (i.e surfaces), with direct applications in image processing. Implementations of these methods and theorems are illustrated and tested both on synthetic images and on real medical data.

1 Introduction and Related Works

In recent years it became common amongst the signal processing community, to consider images as Riemannian manifolds embedded in higher dimensional spaces (see, e.g. [10], [12], [22]). Usually, the embedding manifold is taken to be $\mathbb{R}^n$ yet, other possibilities are also considered ([5]). For example, a gray scale image is a surface in $\mathbb{R}^3$, whereas a color image is a surface embedded in $\mathbb{R}^5$, each color channel representing a coordinate. In both cases the intensity, either gray scale or color, is considered as a function of the two spatial coordinates (x, y) and thus the surface may be equipped with a metric induced by this function. The question of smoothness of the function is in general omitted, if numerical schemes are used for the approximations of derivatives, whenever this is necessary. A major advantage of such a viewpoint of signals is the ability to apply mathematical tools traditionally used in the study of Riemannian manifolds, for image/signal processing as well. For example, in medical imaging it is often convenient to treat CT/MRI scans, as Riemannian surfaces in $\mathbb{R}^3$. One can then borrow techniques from differential topology and geometry and geometric analysis in the representation and analysis of the considered images.

Sampling is an essential preliminary step in processing of any continuous signal by a digital computer. This step lies at the heart of any digital processing of any (presumably continuous) data/signal. Undersampling causes distortions due to the aliasing of the post processed sampled data. Oversampling, on the other hand, results in time and memory consuming computational processes which, at the very least, slows down the analysis process. It is therefore important to have a

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measure which is instrumental in determining what is the optimal sampling rate. For 1-dimensional signals such a measure exists, and, consequently, the optimal sampling rate is given by the fundamental sampling theorem of Shannon, that yielded the foundation of information theory and led technology into the digital era. Shannon's theorem indicates that a signal can be perfectly reconstructed from its samples, given that the signal is band limited within some bound on its highest frequency. Ever since the introduction of Shannon's theorem in the late 1940's, deducing a similar sampling theorem for higher dimensional signals has become a challenge and active area of research, especially recently, in view of methods based on representation of images as manifolds (mostly surfaces) embedded in higher dimensional manifolds. This is further emphasized by the broad interest in its applications in image processing, and by the growing need for fast yet accurate techniques for processing high dimensional data such as medical and satellite images.

Recently a surge in the study of *fat* triangulations (Section 2 below) and manifold sampling in computational geometry, computer graphics and their related fields has generated many publications (see [1], [4], [8], [9], [13], [14], [17], to name just a few). For example, in [1] Voronoi filtering is used for the construction of fat triangulations of compact, $\mathcal{C}^2$ surfaces embedded in $\mathbb{R}^3$. Note that Voronoi cell partitioning is also employed in "classical" sampling theory (see [23]). Cheng et al. [8] used these ideas for manifold reconstruction from point samples. In [14] an heuristic approach to the problem of the relation between curvature and sampling density is given. Again, in these studies the manifolds are assumed to be smooth, compact n-dimensional hyper-surfaces embedded in $\mathbb{R}^{n+1}$.

In this paper we present new sampling theorems for manifolds of dimension ≥ 2. These theorems are derived from fundamental studies in three areas of mathematics: differential topology, differential geometry and quasi-regular maps. Both classical and recent results in these areas are combined to yield a rigorous and comprehensive sampling theory for such manifolds. Our approach lends itself also to a new, geometrical interpretation of classical results regarding proper interpretation of images.

In Section 3 we present geometrical sampling theorems for images/signals given as Riemannian manifolds, for both smooth and non-smooth images/signals. In preparation for that we provide, in Section 2, the necessary background regarding the main results on the existence of fat triangulations of manifolds, and the relation to sampling and reproducing of Riemannian manifolds. We also review the problem of smoothing of manifolds. Finally, in Section 4, we examine some delicate aspects of our study, and discuss extensions of this work, relating both to geometric aspects of sampling, as well as to its relationship to classical sampling theory.

2 Notations, Preliminaries and Background

2.1 Triangulation and Sampling

While for basic definitions and notation regarding triangulations and Piecewise Linear (PL) Topology we refer the reader to [15], we begin this section by recalling a few classical definitions:

Definition 1. *Let $f : K \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, and let $\delta : K \to \mathbb{R}^*_+$ be a continuous function. Then $g : |K| \to \mathbb{R}^n$ is called a δ-*approximation *to f iff:*
(i) There exists a subdivision K' *of K such that $g \in \mathcal{C}^r(K', \mathbb{R}^n)$;*
(ii) $d_{eucl}\big(f(x), g(x)\big) < \delta(x)$, for any $x \in |K|$;
(iii) $d_{eucl}\big(df_a(x), dg_a(x)\big) \leq \delta(a) \cdot d_{eucl}(x, a)$, for any $a \in |K|$ and for all $x \in \overline{St}(a, K')$.
(Here and below $|K|$ denotes the underlying polyhedron *of K.)*

Definition 2. *Let K' be a subdivision of K, $U = \overset{\circ}{U} \subset |K|$, and let $f \in \mathcal{C}^r(K, \mathbb{R}^n)$, $g \in \mathcal{C}^r(K', \mathbb{R}^n)$. g is called a δ-*approximation *of f on U, iff conditions (ii) and (iii) of Definition 2.6. hold for any $a \in U$.*

The most natural and intuitive δ-approximation to a given mapping f is the *secant map induced by* f:

Definition 3. *Let $f \in \mathcal{C}^r(K)$ and let s be a simplex, $s < \sigma \in K$. Then the linear map: $L_s : s \to \mathbb{R}^n$, defined by $L_s(v) = f(v)$, where v is a vertex of s, is called the* secant map induced by f.

Fat Triangulation. We now proceed to show that the apparent "naive" secant approximation of surfaces (and higher dimensional manifolds) represents a good approximation, insofar as distances and angles are concerned, provided that the secant approximation induced by a triangulations that satisfies a certain non-degeneracy condition called "fatness" (or "thickness").

We first provide the following informal, intuitive definition: A triangle in $\mathbb{R}^2$ is called *fat* (or *φ-fat*, to be more precise) iff all its angles are larger than some φ. In other words, fat triangles are those that do not "deviate" to much from being equiangular (regular), hence fat triangles are not too "slim". One can defined fat triangles more formally by requiring that the ratio of the radii of the inscribed and circumscribed circles of the triangle is bounded from bellow by φ, i.e. $r/R \geq \varphi$, for some $\varphi > 0$, where r denotes the radius of the inscribed circle of τ (*inradius*) and R denotes the radius of the circumscribed circle of τ (*circumradius*). This definition easily generalizes to triangulations in any dimension:

Definition 4. *A k-simplex $\tau \subset \mathbb{R}^n$, $2 \leq k \leq n$, is φ-*fat *if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$; where r denotes the radius of the inscribed sphere of τ and R denotes the radius of the circumscribed sphere of τ. A triangulation of a submanifold of $\mathbb{R}^n$, $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is φ-fat if all its simplices are φ-*fat. *A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is* fat *if there exists $\varphi \geq 0$ such that all its simplices are φ-fat; for any $i \in \mathbf{I}$.*

One recuperates the "big" angle characterization of fatness through the following proposition:

Proposition 1 ([7]). *There exists a constant $c(k)$ that depends solely upon the dimension k of τ such that*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\sigma < \tau} \measuredangle(\tau, \sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{1}$$

and

$$\varphi(\tau) \leq \frac{Vol_j(\sigma)}{diam^j\,\sigma} \leq c(k) \cdot \varphi(\tau)\,, \tag{2}$$

where φ denotes the fatness of the simplex τ, $\measuredangle(\tau, \sigma)$ denotes the (internal) dihedral angle of the face $\sigma < \tau$ and $Vol_j(\sigma)$; and where $diam\,\sigma$ stand for the Euclidian j-volume and the diameter of σ respectively. (If $dim\,\sigma = 0$, then $Vol_j(\sigma) = 1$, by convention.)

Condition (1) is just the expression of fatness as a function of dihedral angles in all dimensions, while Condition (2) expresses fatness as given by "large area/diameter". Diameter is important since fatness is independent of scale.

The importance of fatness of triangulations, for attaining good approximations, is underlined in the following proposition that represents the desired approximation result:

Proposition 2 ([15], Lemma 9.3). *Let $f : \sigma \to \mathbb{R}^n$ be of class $\mathcal{C}^k$. Then, for $\delta, \varphi_0 > 0$, there exists $\varepsilon > 0$, such that, for any $\tau < \sigma$, such that $diam(\tau) < \varepsilon$ and such that $\varphi(\tau) > \varphi_0$, the secant map L_τ is a δ-approximation of $f|\tau$.*

2.2 Fat Triangulation Results

In this section we review, in chronological order, existence theorems dealing with fat triangulations on manifolds. For detailed proofs see the original papers.

Theorem 1 (Cairns, [6]). *Every compact $\mathcal{C}^2$ Riemannian manifold admits a fat triangulation.*

Theorem 2 (Peltonen, [18]). *Every open (unbounded) $\mathcal{C}^\infty$ Riemannian manifold admits a fat triangulation.*

Theorem 3 (Saucan, [19]). *Let M^n be an n-dimensional $\mathcal{C}^1$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then, any uniformly fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Remark 1. Theorem 3 above holds, in fact, even without the finiteness and compactness conditions imposed on the boundary components (see [20]).

Corollary 1. *Let M^n be a manifold as in Theorem 3 above. Then M^n admits a fat triangulation.*

In low dimensions one can also discard the smoothness condition:

Corollary 2. *Let M^n be an n-dimensional, $n \leq 4$ (resp. $n \leq 3$), PL (resp. topological) connected manifold with boundary, having a finite number of compact boundary components. Then, any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

2.3 Smoothing of Manifolds

In this section we focus our attention on the problem of smoothing of manifolds. That is, approximating a manifold of differentiability class $\mathcal{C}^r$, $r \geq 0$, by manifolds of class $\mathcal{C}^\infty$. Of special interest is the case where $r = 0$. Later, when stating our sampling theorem we will make a use of this in two respects. One of them will be as a post-processing step where, after reproducing a PL manifold out of the samples, we can smoothen it to get a smooth reproduced manifold. Another aspect in which smoothing is useful is as a pre-processing step, when we wish to extend the sampling theorem to manifolds which are not necessarily smooth. Smoothing will take place followed by sampling of the smoothed manifold, yielding a sampling for the non-smooth one as well. As a major reference to this we use [15], Chap 4. Similar results can also be found in [11] and others.

Lemma 1. *For every $0 < \epsilon < 1$ there exists a $\mathcal{C}^\infty$ function $\psi_1 : \mathbb{R} \to [0,\ 1]$ such that, $\psi_1 \equiv 0$ for $|x| \geq 1$ and $\psi_1 = 1$ for $|x| \leq (1 - \epsilon)$. Such a function is called* partition of unity.

Let $c^n(\epsilon)$ be the ϵ cube around the origin in $\mathbb{R}^n$ (i.e. $X \in \mathbb{R}^n$; $-\epsilon \leq x_i \leq \epsilon$, $i = 1, ..., n$). We can use the above partition of unity in order to obtain a non-negative $\mathcal{C}^\infty$ function, ψ, on $\mathbb{R}^n$, such that $\psi = 1$ on $c^n(\epsilon)$ and $\psi \equiv 0$ outside $c^n(1)$. Define $\psi(x_1, ..., x_n) = \psi_1(x_1) \cdot \psi_1(x_2) \cdots \psi_1(x_n)$.

We now introduce the main theorem regarding smoothing of PL-manifolds.

Theorem 4 ([15]). *Let M be a $\mathcal{C}^r$ manifold, $0 \leq r < \infty$, and $f_0 : M \to \mathbb{R}^k$ a $\mathcal{C}^r$ embedding. Then, there exists a $\mathcal{C}^\infty$ embedding $f_1 : M \to \mathbb{R}^k$ which is a δ-approximation of f_0.*

The above theorem is a consequence of the following classical lemma concerning smoothing of maps:

Lemma 2 ([15]). *Let U be an open subset of $\mathbb{R}^m$. Let A be a compact subset of an open set V such that $\overline{V} \subset U$, is compact. Let $f_0 : U \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, $0 \leq r$. Let δ be a positive number. Then there exists a map $f_1 : U \to \mathbb{R}^n$ such that*

1. *f_1 is $\mathcal{C}^\infty$ on A.*
2. *$f_1 = f_0$ outside V.*
3. *f_1 is a δ-approximation of f_0*
4. *f_1 is $\mathcal{C}^r$-homotopic to f_0 via a homotopy f_t satisfying* (2) *and* (3) *above. i.e. f_0 can be continuously deformed to f_1.*

Remark 2. A modified version of the smoothing process presented herein was developed by Nash [16]. His idea was to define a radially symmetric convolution kernel φ, by taking its Fourier transform, $\hat{\varphi}$, to be a radially symmetric partition of unity. Nash's method renders an approximation that is faithful not only to the signal and its first derivative, as in the classical approach, but also to higher order derivatives (if they exist).

3 Sampling Theorems

We employ results regarding the existence of fat triangulations, to prove sampling theorems for Riemannian manifolds embedded in some Euclidean space.

Theorem 5. *Let $\Sigma^n, n \geq 2$ be a connected, not necessarily compact, smooth manifold, with finitely many compact boundary components. Then there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D}$ with respect to the volume element on Σ^n, $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\left(\frac{1}{k(p)}\right)$, where $k(p) = \max\{|k_1|, ..., |k_{2n}|\} > 0$, and where $k_1, ..., k_{2n}$ are the principal (normal) curvatures of Σ^n, at the point $p \in \Sigma^n$.*

Proof. The existence of the sampling scheme follows immediately from Corollary 1, where the sampling points(points of the sampling) are the vertices of the triangulation. The fact that the density is a function solely of $k = max\{|k_1|, ..., |k_{2n}|\}$ follows from the proof of Theorem 2 (see [18], [19]) and from the fact that the osculatory radius $\omega_\gamma(p)$ at a point p of a curve γ equals $1/k_\gamma(p)$, where $k_\gamma(p)$ is the curvature of γ at p; hence the maximal osculatory radius (of Σ) at p is: $\omega(p) = \max\{|k_1|, ..., |k_{2n}|\} = \max\{\frac{1}{\omega_1}, ..., \frac{1}{\omega_{2n}}\}$. (Here ω_{2i}, ω_{2i+1} , $i = 1, ..., n-1$ denote the minimal, respective maximal sectional osculatory radii at p.)

Corollary 3. *Let $\Sigma^n, \mathcal{D}$ be as above. If there exists $k_0 > 0$, such that $k(p) \leq k_0$, for all $p \in \Sigma^n$, then there exists a sampling of Σ^n of finite density everywhere.*

Proof. Immediate from the theorem above.

In particular we have:

Corollary 4. *If Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density.*

Proof. It follows immediately from a compactness argument and from the continuity of the principal curvature functions.

The implementation of Theorem 5 is illustrated in Figure 3. Note the fat triangulation and the good reconstruction (see below) obtained from it. Compare with the "flat" triangles in Figure 1, obtained by a "naive" sampling method.

Remark 3. Obviously, Theorem 5 above is of little relevance for the *space forms* $(\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n)$. Indeed, as noted above, this method is relevant for manifolds considered (by the Nash embedding theorem [16]) as submanifolds of $\mathbb{R}^N$, for some N large enough.

We approach the problem of sampling for non-smooth manifolds; a case that is of interest and practical importance in the context of image processing and computer vision (see, e.g. Figure 1). We begin by proposing the following definition:

Definition 5. *Let $\Sigma^n, n \geq 2$ be a (connected) manifold of class $\mathcal{C}^0$, and let Σ^n_δ be a δ-approximation to Σ^n. A sampling of Σ^n_δ is called a δ-*sampling* of Σ^n.*

Theorem 6. *Let Σ^n be a connected, non-necessarily compact manifold of class $\mathcal{C}^0$. Then, for any $\delta > 0$, there exists a δ-sampling Σ^n_δ of Σ^n, such that if $\Sigma^n_\delta \to \Sigma^n$ uniformly, then $\mathcal{D}_\delta \to \mathcal{D}$ in the sense of measures, where $\mathcal{D}_\delta$ and $\mathcal{D}$ denote the densities of Σ^n_δ and Σ^n, respectively.*

Proof. The proof is an immediate consequence of Theorem 2 and its proof, and the methods exposed in Section 2.3. We take the sampling of some smooth δ-approximation of Σ^n.

Corollary 5. *Let Σ^n be a $\mathcal{C}^0$ manifold with finitely many points at which Σ^n fails to be smooth. Then every δ-sampling of a smooth δ-approximation of Σ^n is in fact, a sampling of Σ^n apart of finitely many small neighborhoods of the points where Σ^n is not smooth.*

Proof. From Lemma 2 and Theorem 4 it folows that any such δ-approximation, Σ^n_δ, coincides with Σ^n outside of finitely many such small neighborhoods.

Remark 4. In order to obtain a better approximation it is advantageous, in this case, to employ Nash's method for smoothing, cf. Remark 5 of Section 3 (see [16], [2] for details).

Reconstruction. We use the secant map as defined in Definition 3 in order to reproduce a PL-manifold as a δ-approximation for the sampled manifold. As stated in the beginning of Section 2.3, we may now use smoothing in order to obtain a $\mathcal{C}^\infty$ approximation. This approach is illustrated in Figure 2, for the case of an analytic surface.

In the special case of surfaces (i.e. $n = 2$), more specific, geometric conditions can be obtained:

Corollary 6. *Let Σ^2 be a smooth surface. In the following cases there exist k_0 as in Corollary 3 above:*

1. *There exist H_1, H_2, K_1, K_2, such that $H_1 \le H(p) \le H_2$ and $K_1 \le K(p) \le K_2$, for any $p \in \Sigma^2$, where H, K denote the mean, respective Gauss curvature. (That is both mean and Gauss curvatures are* pinched.*)*
2. *The Willmore integrand $W(p) = H^2(p) - K(p)$ and K (or H) are pinched.*

Proof. 1. Since $K = k_1k_2$, $H = \frac{1}{2}(k_1 + k_2)$, the bounds for K and H imply the desired one for k.
2. Analogue reasoning to that of (*2.*), since $W = \frac{1}{4}(k_1 - k_2)^2$.

Remark 5. Condition (*ii*) on W is not only compact, it has the additional advantage that the *Willmore energy* $\int_\Sigma W dA$ (where dA represents the area element of Σ) is a *conformal invariant* of Σ.

Note that such geometric conditions, are hard to impose in higher dimensions, and the precise geometric constraints remain to be further investigated.

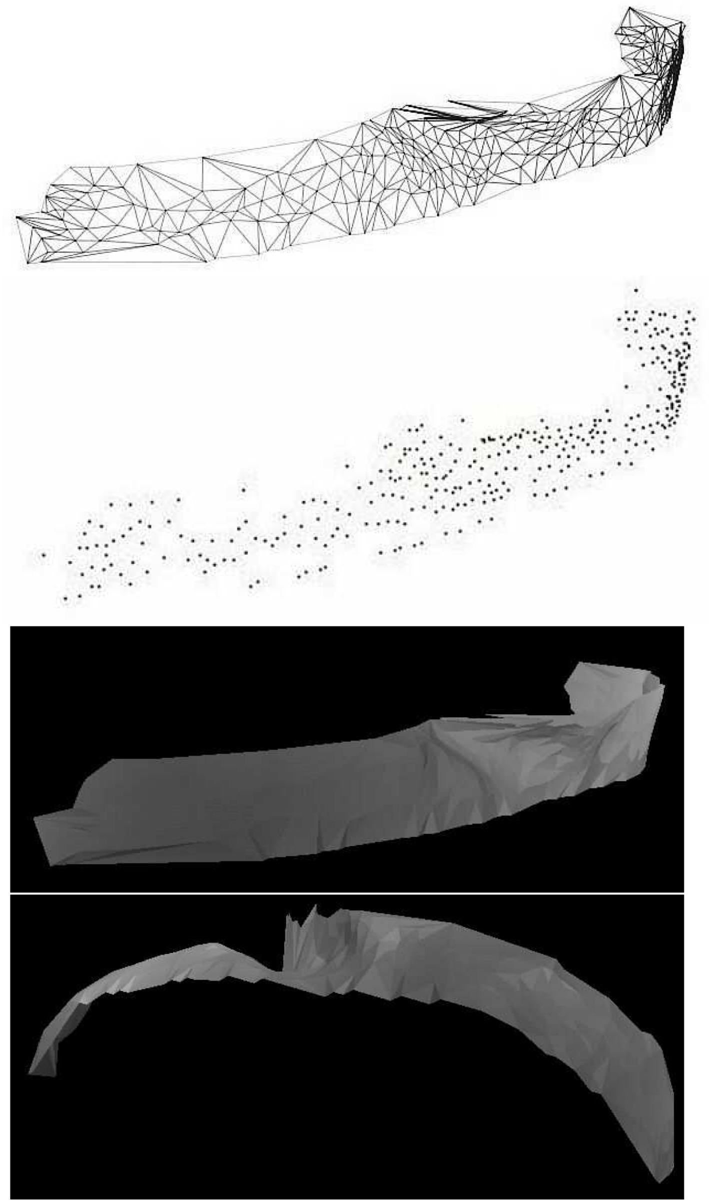

Fig. 1. The triangulation (upper image) obtained from a "naive" sampling (second image from above) resulting from a CT scan of the back-side of the human colon (second image from below). Note the "flat" triangles and the uneven mesh of the triangulation. This is a result of the high, concentrated curvature, as revealed in a view obtained after a rotation of the image (bottom). These and other images will be accessible through an interactive applet on the website [25].

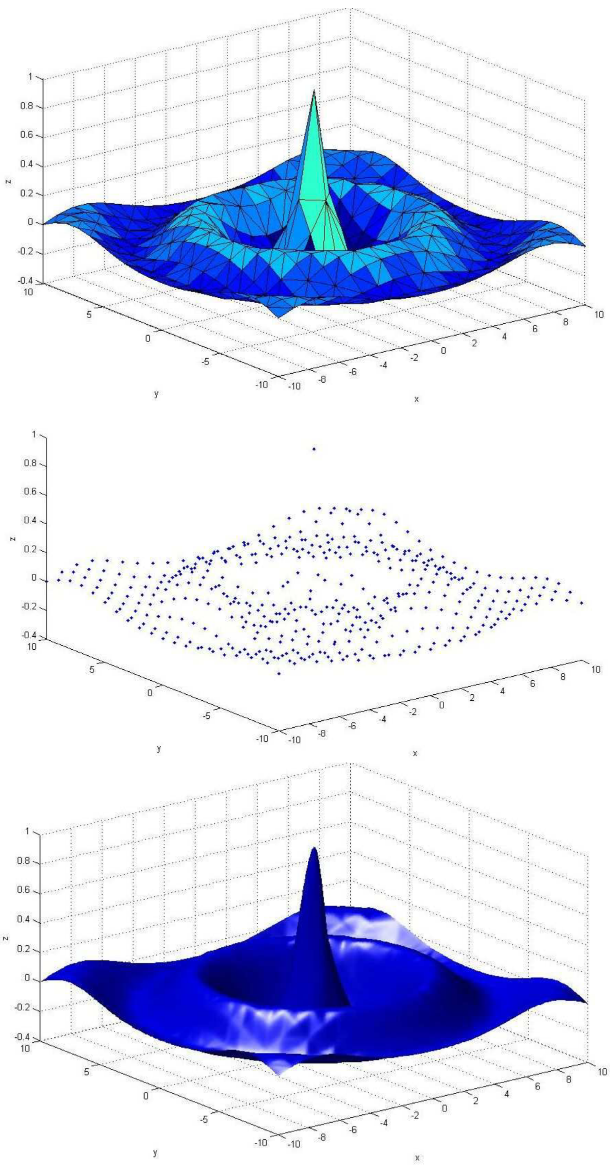

Fig. 2. The triangulation (upper image) obtained from the uniform sampling (second image from above) of the surface $S = \left(x, y, \cos\sqrt{x^2+y^2}/(1+\sqrt{x^2+y^2}\,)\right)$ (bottom image). Note the low density of sampling points in the region of high curvature.

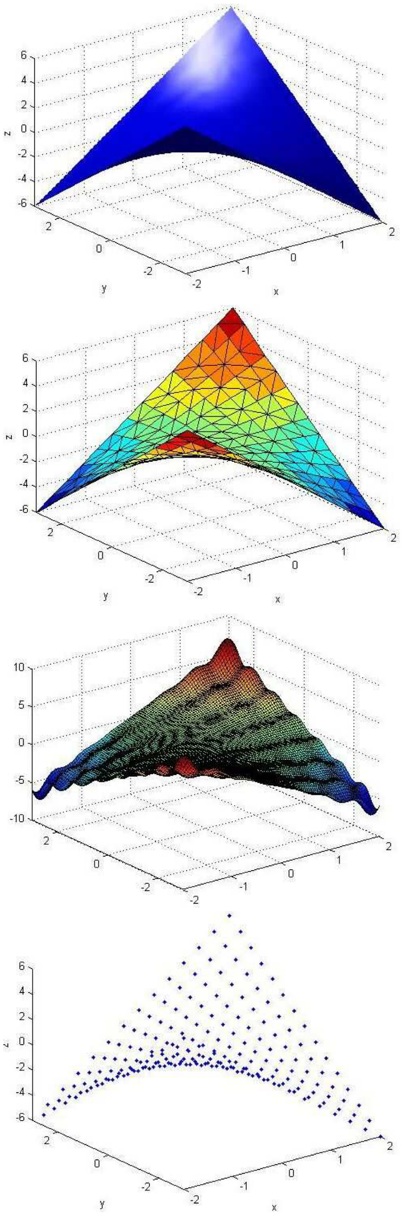

Fig. 3. Hyperbolic Paraboloid: Analytic representation, $z = xy$ – top image. Sampling according to curvature – second image from above. PL reconstruction – second image from bellow. Bottom – Nyquist reconstruction. To appreciate the triangulation results a full size display of color images [25] is required.

4 Conclusions and Discussion

The methods for sampling and reconstructing of images, introduced in this paper, extend previous studies based on the viewpoint that images and other types of data structures should be considered as surfaces and manifolds embedded in higher dimensional manifolds. In particular, the methods presented in this paper are based on the assertion that surfaces and manifolds should be properly sampled in Shannon's sense. This led to consideration of a sampling theorem for Riemannian manifolds. The sampling scheme presented in this paper, is based on the ability to triangulate such a manifold by a fat triangulation. This in turn, relies on geometric properties of the manifold and basically on its curvature. The sampling theorems are applicable to images/signals that can be represented as Riemannian manifolds, a well established viewpoint in image processing. Considering this viewpoint in rigorous manner still remains as a challenge for further study. It is common for instance, to consider a color image as a surface in $\mathbb{R}^5$ yet, it is more prone and probably more accurate to consider it as a three-dimensional manifold embedded in some higher dimensional Euclidian space. Another interesting issue currently under investigation, is whether the geometric framework for sampling of surfaces and manifolds present in this study can be degenerated to one-dimensional signals as an alternative to the classical sampling theorem of Shannon and how the two approaches are related. Some relevant results are already at hand [21]. Other theoretical and applied facets of this problem are currently under investigation.

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References

[1] Amenta, N. and Bern, M. *Surface reconstruction by Voronoi filtering*, Discrete and Computational Geometry, 22, pp. 481-504, 1999.

[2] Andrews, B. *Notes on the isometric embedding problem and the Nash-Moser implicit function theorem*, Proceedings of CMA, Vol. 40, 157-208, 2002.

[3] M. Berger, *A Panoramic View of Riemannian Geometry*, Springer-Verlag, Berlin, 2003.

[4] Bern, M., Chew, L.P., Eppstein, D. and Ruppert, J. : *Dihedral Bounds for Mesh Generation in High Dimensions*, 6th ACM-SIAM Symposium on Discrete Algorithms, 1995, 189-196.

[5] Bronstein, A. M., Bronstein, M. M. and Kimmel, R.: *On isometric embedding of facial surfaces into S3*, Proc. Intl. Conf. on Scale Space and PDE Methods in Computer Vision, pp. 622-631, 2005.

[6] Cairns, S.S.: *A simple triangulation method for smooth manifolds*, Bull. Amer. Math. Soc. **67**, 1961, 380-390.

[7] Cheeger, J., Müller, W., and Schrader, R.: *On the Curvature of Piecewise Flat Spaces*, Comm. Math. Phys., **92**, 1984, 405-454.

[8] Cheng, S. W., Dey, T. K. and Ramos, E. A.: *Manifold Reconstruction from Point Samples*, Proc. ACM-SIAM Sympos. Discrete Algorithms, 2005, 1018–1027.

[9] Edelsbrunner, H.: *Geometry and Topology for Mesh Generation*, Cambridge University Press, Cambridge, 2001.

[10] Hallinan, P. *A low-dimensional representation of human faces for arbitrary lighting conditions*, Proc. CVPR. pp. 995-999, 1994.

[11] Hirsh, M. and Masur, B.: *Smoothing of PL-Manifolds*, Ann. Math. Studies 80, Princeton University Press, Princeton, N.J., 1966.

[12] Kimmel, R. Malladi, R. and Sochen, N. *Images as Embedded Maps and Minimal Surfaces: Movies, Color, Texture, and Volumetric Medical Images*, International Journal of Computer Vision, **39**(2), pp. 111-129, 2000.

[13] Li, X.-Y. and Teng, S.-H.: *Generating Well-Shaped Delaunay Meshes in 3D*, SODA 2001, 28-37.

[14] Meenakshisundaram, G.: *Theory and Practice of Reconstruction of Manifolds With Bounderies*, PhD Thesis, Univ. North Carololina, 2001.

[15] Munkres, J. R.: *Elementary Differential Topology*, (rev. ed.) Princeton University Press, Princeton, N.J., 1966.

[16] Nash, J. *The embedding problem for Riemannian manifolds*, Ann. of Math. (2) **63**, 20-63, 1956.

[17] Pach, J. and Agarwal. P.K. *Combinatorial Geometry*, Wiley-Interscience, 1995.

[18] Peltonen, K. *On the existence of quasiregular mappings*, Ann. Acad. Sci. Fenn., Series I Math., Dissertationes, 1992.

[19] Saucan, E. *Note on a theorem of Munkres*, Mediterr. j. math., **2**(2), 2005, 215 - 229.

[20] Saucan, E. *Remarks on the The Existence of Quasimeromorphic Mappings*, to appear in Contemporary Mathematics.

[21] Saucan, E., Appleboim, E. and Zeevi, Y. Y. *Sampling and Reconstruction of Surfaces and Higher Dimensional Manifolds*, with Eli Appleboim and Yehoshua Y. Zeevi, Technion CCIT Report #591, June 2006 (EE PUB #1543 June 2006).

[22] Seung, H. S. and Lee, D. D. *The Manifold Ways of Perception*, Science 290, 2323-2326, 2000.

[23] Smale, S. and Zhou, D.X., *Shannon sampling and function reconstruction from point values*, Bull. Amer. Math. Soc., **41**(3), 2004, 279-305.

[24] Unser, M. and Zerubia, J.: *A Generalized Sampling Theory Without Band-Limiting Constrains*, IEEE Trans. Signal Processing, Vol. 45, No. 8, 1998, 956-969.

[25] http://visl.technion.ac.il/ImageSampling