Existence Conditions for Discrete Noncanonical Multiwindow Gabor Schemes

Nagesh K. Subbanna and Yehoshua Y. Zeevi

***Abstract*—A class of noncanonical duals for multiwindow Gabor (MWG) schemes, encompassing both rational and integer oversampling of the Gaborian combined time-frequency space, is considered. Using properties of Gabor frame matrices (GFM), block discrete Fourier transforms (BDFTs), and results from number theory, we use matrix factorization to establish existence conditions for noncanonical duals for both integer and rational oversampling rates, in the signal domain. For comparison and completeness of the results, we also obtain the equivalent results in the finite Zak transform (FZT) domain.**

***Index Terms*—Block circulant matrices, frames, multiwindow Gabor transforms, Zak transforms.**

I. Introduction

Multiwindow Gabor (MWG) expansion combines the advantages of localization in the combined time-frequency space, characteristic of the classical Gabor scheme, and scale–space properties inherent in wavelet representations. MWG expansions of signals (and images) find many applications in the fields of signal and image processing, computer vision, and recently in macromolecular sequence analysis.

Zibulski and Zeevi introduced MWG expansions in 1997 [1] and extended the concept to the finite, discrete-time case [2]. The coefficients of the MWG expansion are given by the projection of the finite signal $\mathbf{f} \in \mathbb{C}^L$ onto the combined space

$$c_{r,m,n} = \sum_{k=0}^{L-1} f[k] g_r[k-na] e^{-j2\pi mbk/L} \tag{1}$$

where $g_r[k]$, $r \in 0,\ldots,R-1$ are the window functions, and a and b are the combined space sampling intervals along the time and frequency axes, respectively, defining the discrete data lattice.

Given the coefficients $c_{r,m,n}$, the analysis windows $g_r[k]$, and the lattice constants a and b, one can reconstruct the signal $f[k]$ from the coefficients. We assume here, as in [3], that L is divisible by both a and b. The reconstruction of the signal $f[k]$ is given by [2]

$$f[k] = \sum_{r=0}^{R-1} \sum_{m=0}^{\overline{b}-1} \sum_{n=0}^{\overline{a}-1} c_{r,m,n} \gamma_r[k-na] e^{j2\pi mbk/L} \tag{2}$$

where $\{\gamma_r[k]\}$ are the dual windows, and $\overline{a} = L/a \in \mathbb{N}$ and $\overline{b} = L/b \in \mathbb{N}$ are the number of sampling intervals along the time and frequency axes, respectively.

In vector form, (1) can be written as

$$\mathbf{c} = \mathbf{G}^* \mathbf{f} \tag{3}$$

where $\mathbf{c}$ is the vector of coefficients, and $\mathbf{G}$ is the Gabor matrix

$$\mathbf{G} = \begin{bmatrix} g_{0,0,0}[0] & \cdots & g_{R-1,\overline{a}-1,\overline{b}-1}[0] \\ g_{0,0,0}[1] & \cdots & g_{R-1,\overline{a}-1,\overline{b}-1}[1] \\ \vdots & \ddots & \vdots \\ g_{0,0,0}[L-1] & \cdots & g_{R-1,\overline{a}-1,\overline{b}-1}[L-1] \end{bmatrix} \tag{4}$$

with $g_{r,m,n}[k] = g_r[k-na]e^{j2\pi mbk/L}$. The reconstruction, inverse of (1), has the following vector form version of (2):

$$\mathbf{f} = \mathbf{\Gamma} \mathbf{c} \tag{5}$$

where $\mathbf{\Gamma}$ is the dual of the Gabor matrix.

It was established in [2] that $R\overline{a}\overline{b} \geq L$ is a necessary condition for complete reconstruction in the case of MWG expansions. In the case of critical sampling and a single window, the reconstruction is unstable according to the Balian–Low theorem [4]. This theorem extends to well-behaved multiwindows [1]. We, therefore, consider only the oversampling case where $R\overline{a}\overline{b} > L$, which implies that the functions $g_{r,m,n}[k]$ are linearly dependent and the representation is overcomplete.

As the representation is overcomplete, there exists an infinite number of possible duals $\gamma_r[k]$. The canonical solution identifies the minimum norm dual of the set of generalized Gabor elementary functions $g_{r,m,n}[k]$ using [3]

$$\tilde{\gamma}_r[k] = (\mathbf{G}\mathbf{G}^*)^{-1} g_r[k]. \tag{6}$$

An attempt was made in [5] to find duals efficiently for various types of sampling, and yet retain the pseudoinverse. However, it is often advantageous to choose a different dual from a wider set of duals, as is shown in Section IV. Here, we extend noncanonical duals introduced in [6], and applied to single window Gabor expansions in [7], to MWG expansions.

Noncanonical duals [6] are given by

$$d_r[k] = d_r[k-na]e^{j2\pi mbk/L} = (\mathbf{H}\mathbf{G}^*)^{-1} h_r[k] \tag{7}$$

where $\mathbf{H}$ is another Gabor matrix of the same form as $\mathbf{G}$ such that $\mathbf{H}\mathbf{G}^*$ is invertible. We obtain existence conditions for noncanonical MWG frames, in the general context, and in integer and rational[1] oversampling cases. For comparison, we provide the equivalent results in the finite Zak transform domain and discuss some advantages of noncanonical duals.

We first discuss three properties of the discrete Gabor frame matrix (GFM) that are utilized in deriving existence conditions. Then, we consider the factorization of the frame-type matrix and generate an easy method to evaluate the existence conditions. These are discussed in the context of the finite Zak transform (FZT) domain. Finally, Examples of noncanonical duals are provided and their advantages are discussed.

II. Three Useful Properties of the Frame-Type Matrix

Condition for existence of the dual frame is equivalent to the invertibility of the GFM $\mathbf{P} = \mathbf{H}\mathbf{G}^*$ [7]. In [3], it was shown that the matrix $\mathbf{P} = \mathbf{H}\mathbf{G}^*$ is a block circulant matrix of the form depicted in Fig. 1.

1) **Block-Circulant Structure of Matrix $\mathbf{P}$:** $\mathbf{P}$ is a block circulant matrix having $\overline{a}-1$ blocks of size $a \times a$ [3].

[1]We refer to "rational oversampling" to distinguish the case where $\overline{b}/a$ is a rational number from the one where it is an integer. There is no irrational sampling in discrete case.

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The authors are with the Department of Biomedical Engineering, Technion—Israel Institiue of Technology, Haifa 32000, Israel (e-mail: nagesh@tx.technion.ac.il; zeevi@ee.technion.ac.il).

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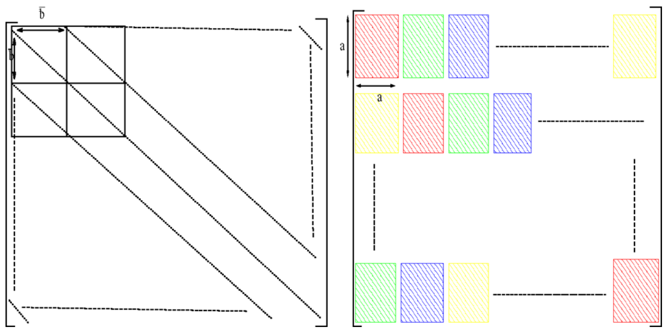

Fig. 1. Nonzero elements are $\overline{b}$ apart and a periodic, appearing along the diagonals, thereby indicating the (left) banded and (right) block circulant structures.

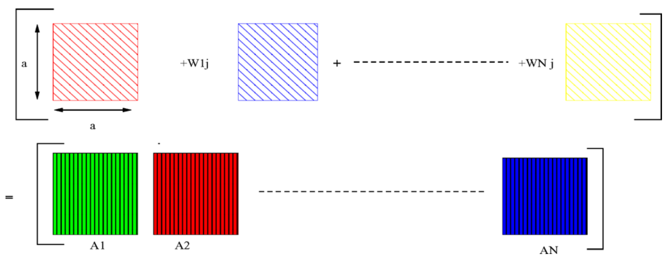

Fig. 2. Block discrete Fourier transform of a matrix having $a \times a$ blocks.

2) **Frame-Matrix Banded Structure**: The only nonzero entries in the frame matrix $\mathbf{P}$ are along the main diagonal and the $\overline{b}$th subdiagonals [3], as shown in Fig. 1. Each block $B_{k,l}$ is a diagonal matrix of size $\overline{b} \times \overline{b}$.
3) **Block Discrete Fourier Transform** (BDFT) (see Fig. 2): The BDFT of the block circulant matrix $\mathbf{P} = \mathcal{C}(A_0, A_1, \ldots, A_{\overline{a}-1})$ is given by $\mathcal{F}\mathbf{P} = \mathcal{C}(\hat{\mathbf{A}}_\mathbf{0}, \hat{\mathbf{A}}_\mathbf{1}, \ldots, \hat{\mathbf{A}}_{\overline{\mathbf{a}}-\mathbf{1}})$ [8], where $\mathcal{C}$ stands for circulation of the blocks in the block circulant matrix $\mathbf{P}$, and

$$\hat{\mathbf{A}}_q = \sum_{p=0}^{\overline{a}-1} \omega^{pq}\mathbf{A}_p, \quad 0 \leq q \leq \overline{a} - 1 \tag{8}$$

where $\omega = e^{-j2\pi/\overline{a}}$.

Using block-circulant properties of frame $\mathbf{P}$ leads to the following condition of invertibility.

Lemma 2.1 [8]: The matrix $\mathbf{P}$ is invertible if and only if all the $a \times a$ blocks of the BDFT of $\mathbf{P}$ are invertible individually.

III. Existence Conditions

Using the condition defined by Lemma 2.1, and the property of the banded structure of the GFM [3], we obtain a stronger (and more easily verifiable) condition on the invertibility of the matrix $\mathbf{P}$. The $a \times a$ BDFT submatrices have a very definite structure. The structure is given by the following theorem.

Theorem 3.1: Let the greatest common divisor (gcd) of $(a, \overline{b})$ be equal to α. Then, the nonzero elements in each row of the BDFT of the matrix $\mathbf{P}$ are α entries apart.

Proof: Consider the first block $a \times L$ of the matrix $\mathbf{P}$, which we use in the generation of the block-circulant Fourier matrix. It is apparent that structurally (i.e., the position of zeroes and nonzero elements), the subsequent rows are simply the first row shifted rightward by the appropriate distance from the first row. Thus, we find the block Fourier transform of this row considering only the first row in the block circulant matrix $\mathbf{P}$.

In the first row, we have the nonzero elements at positions $0, \overline{b}, \ldots, \overline{b}(b-1)$. We add the corresponding elements of the

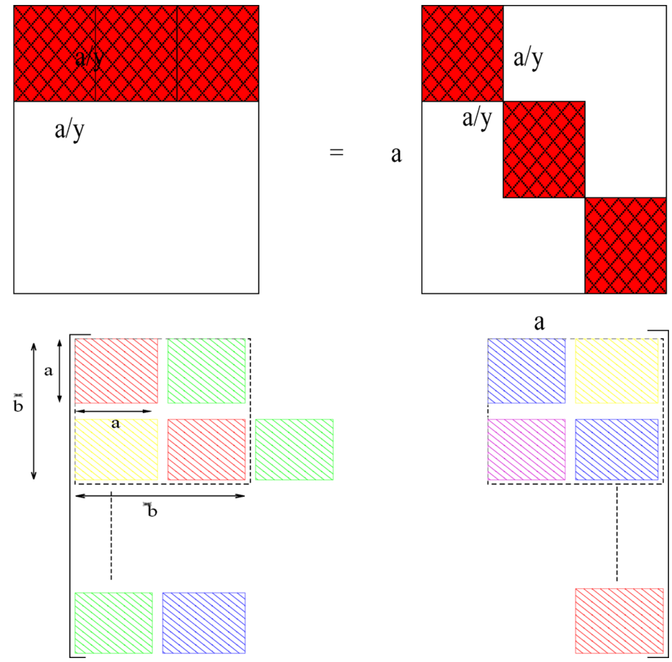

Fig. 3. (Top-Left) The $a \times a$ matrices have nonzero elements α apart in the rational case. (Top-Right) The matrices can be block-diagonalized as shown. (Bottom) In the integer case, the $a \times a$ blocks fit perfectly into the larger $\overline{b} \times \overline{b}$ blocks. Therefore, the $a \times a$ blocks become diagonal matrices.

$a \times a$ matrices after the multiplication with the appropriate phase $e^{-j2\pi mk/L}$, where $m, k \in 0, \ldots, \overline{a} - 1$.

The elements of the first row of the $a \times a$ matrix can be indexed by $\beta a + \nu$, where $\nu \in 0, \ldots, a-1$, $\beta \in 0, \ldots, \overline{a} - 1$. The nonzero elements are situated at $\mu\overline{b}$, where $\mu \in 0, \ldots, b-1$. Nonzero elements will occur when $\nu + \beta a = \mu\overline{b}$, for some $\beta \in 0, \ldots, \overline{a} - 1$ and $\mu \in 0, \ldots, b-1$, but from Euclid's algorithm, we know that the smallest positive number is $\alpha = gcd(a, \overline{b})$. Therefore, ν cannot be less than α. Thus, the nonzero elements of the block Fourier matrix of $\mathbf{P}$ have to be at least α apart. ■

It was shown in [9] and [10] that utilizing perfect shuffle matrices of the form

$$\mathbf{V}_{\alpha,a} = \mathcal{C}_\alpha \left[\mathbf{1}_0, \mathbf{1}_\alpha, \ldots, \mathbf{1}_{(q-1)\alpha}\right] \tag{9}$$

where $\alpha q = a$ and $\mathbf{1}_s$ is a column vector of length a having a 1 at row s and zeroes elsewhere, and $\mathcal{C}_\alpha$ denotes α times successive circular shifting of the entire block structured matrix, performed by down-shifting columnwise. The $a \times a$ blocks of matrix $\mathcal{F}(\mathbf{P})$ can be factorized to block diagonal matrices of size $a/\alpha \times a/\alpha$ (as shown in Fig. 3).

Theorem 3.2 [10]: An $a \times a$ matrix that has nonzero elements on the principal diagonal, and at a distance of $k\alpha, k \in 1, \ldots, (a/\alpha) - 1$ from the diagonal, can be factorized into α block-diagonal matrices of size $a/\alpha \times a/\alpha$ using a perfect shuffle matrix $\mathbf{V}_{\alpha,a}$. The block-diagonal matrix $\mathbf{W}$ is created using the formula

$$\mathbf{W} = \mathbf{V}^*_{\alpha,a}\tilde{\mathbf{A}}_s\mathbf{V}_{\alpha,a} \tag{10}$$

where $\tilde{\mathbf{A}}_s$ is the sth block of the BDFT of the matrix $\mathbf{P}$.

In the worst case of $\alpha = 1$, we have, of course, identical condition and matrix structure as in the previous problem, i.e., showing that an $a \times a$ matrix has to be invertible. We can show that the α matrices of size $a/\alpha \times a/\alpha$ are invertible in place of showing that an $a \times a$ matrix is invertible, lending itself to a simplification of the complexity of the problem.

The present technique presents also a more general framework and insight into the invertibility of GFMs in the case of integer oversampling [7]. In the latter case, we have $\bar{b}$ divisible by a, and thus the $gcd(\bar{b}, a)$ is a. In other words, the nonzero elements in the BDFT of $\mathbf{P}$ are a entries apart. This is tantamount to saying that the $a \times a$ block matrices of $\mathcal{F}(\mathbf{P})$ are diagonal. This leads to the condition of invertibility of $\mathbf{P}$ of having no zeroes on the ath subdiagonals of $\mathcal{F}(\mathbf{P})$. Setting $p = 1$ in Theorem 3.1 and expanding the matrix elements leads to the following formulation.

Corollary 3.3: For the matrix $\mathbf{P}$ to be invertible in the integer oversampling case, the following condition must be satisfied:

$$\sum_{q=0}^{b-1}\sum_{v=0}^{a-1}\sum_{r=0}^{R-1}\left\langle h_r[u], g_r[u+q\bar{b}]e^{j2\pi vu/a}\right\rangle e^{j2\pi sq/b}e^{j2\pi vk/a} \neq 0 \tag{11}$$

for any $k, s \in 0, \ldots, \bar{b}-1$.

Based on Corollary 3.3, and recognizing that $\bar{b}$ is divisible by a, we can develop an alternative condition that explicitly permits certain types of functions to generate frames for $\mathbb{C}^L$.

Corollary 3.4: A sufficient condition for the invertibility of $\mathbf{HG}^*$ is that the sequences $g_r[k - na - q\bar{b}]$, $q \in 0, \ldots, b-1$ be positive[2] (or negative) definite, when $h_r[k]$ are all of the same sign.

Proof: Writing the elements of the BDFT of the matrix $P = HG^*$, recognizing that $\bar{b}$ is divisible by a and rearranging the terms of the summation easily leads to the result. ■

Corollary 3.4 can be seen as the more general version of the result proved in [11].

A. Zak Transform Domain Results

The FZT of a function $\mathbf{f} \in \mathbb{C}^L$ denoted by $Z_{\bar{b}}$ is defined as the mapping $Z_{\bar{b}} : \mathbb{C}^L \to \mathbb{C}^{\bar{b}} \times \mathbb{C}^b$, given by the equation [12]

$$(Z_{\bar{b}})(r, v) = \sum_{k=0}^{b-1} f(r - \bar{b}k)e^{\frac{j2\pi\bar{b}kv}{L}} \tag{12}$$

where $\bar{b} \in \mathbb{N}$ is a fixed parameter, and $b\bar{b} = L$. Let $ab/L = p/q$, where p and q are mutually prime. Based on the definition of FZT, we adopt the approach presented in [13] and define a piecewise finite Zak transform (PFZT) as a vector-valued function of size p, as follows:

$$\mathbf{F}(r, v) = [F_0(r, v), \ldots, F_{p-1}(r, v)]^T \tag{13}$$

where

$$F_l(r, v) \triangleq (Zf)\left(r, v + l\frac{b}{p}\right), \quad 0, \ldots, p-1. \tag{14}$$

It is important to note that inner products are preserved in the PFZT domain.

Using these results, we can define the action of the frame operator in the Zak transform domain by [13]

$$(\mathcal{P}\mathbf{F})(r, v) = \mathcal{P}(r, v)\mathbf{F}(r, v) \tag{15}$$

where both $\mathcal{P}$ and $\mathbf{F}$ are the PFZTs of the frame operator $\mathbf{P}$ and of $\mathbf{f}$, respectively. The elements of the $p \times p$ matrix, constituting $\mathcal{P}_{k,l}(r, v)$ of the PFZT of $\mathcal{P}$, are given by

$$\frac{\bar{b}}{p}\sum_{r=0}^{R-1}\sum_{s=0}^{q-1}(Zh_r)(\tau - sa, v + kb/p)(Zg_r)^*(\tau - na, v + lb/p) \tag{16}$$

and $(Zg_r)^*(\tau, v)$ is given by (12).

Based on the results presented in [13], we now derive the condition for the existence of the Gabor frame in the general case.

[2]Positive-definite sequences are those sequences whose DFT is real and positive.

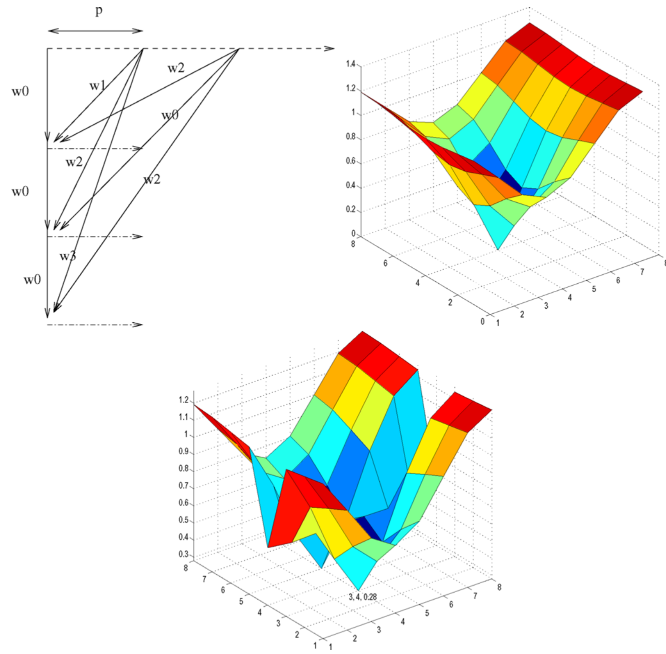

Fig. 4. (Top-Left) Structure of the Zak transform. Here, w stands for $e^{-j2\pi/b}$. (Top-Right) The zeroes of the Zak transform of a well-behaved window function, where there is exactly one zero, at position (4,4), and (bottom) the graph of the Zak transform of an ill-behaved window function, where there are no zeroes (the smallest value is 0.28, at (3,4)).

Theorem 3.5: Given that $g, h \in l^2(\mathbb{Z}_L)$, $\bar{a}\bar{b} \geq L$, and a matrix-valued function $P_{k,l}(\tau, v)$ given in (16), the matrix $\mathbf{HG}^*$ constitutes a frame operator if and only if $det(\mathcal{P})(\tau, v) \neq 0$ for all $\tau, v, \tau \in 0, \ldots, \bar{b}, v \in \lfloor b/p$.

It is instructive to examine the equivalence of results obtained in the Zak transform domain and by utilizing the block-circulant matrix methods. For the case of critical sampling, we have the interesting results that each of the $\mathbf{P}(s, v)$ turns into a scalar-valued function and that each of the $\mathbf{P}(s, v)$ should be nonzero for all values of s, v—paralleling the result in the canonical dual case. This yields the invertibility condition for the critical sampling case.

A similar result is obtained for integer oversampling, $p = 1$. Therefore, (16) becomes

$$\mathbf{P}(s, v) = \bar{b}\sum_{r=0}^{R-1}\sum_{w=0}^{q-1}(Zh_r)(s - wa, v)(Zg_r)^*(s - wa, v). \tag{17}$$

As long as (17) is not equal to zero, we have a simple way of determining the invertibility of the frame operators. In particular, (17) can be shown to be very similar to (11) [14].

There are some other interesting parallels between the computation of the dual frame using the Zak transform and the BDFT. In particular, both depend on the oversampling rate, $ab/L = p/q$ and require the $p \times p$ matrices to be inverted. The approach based on Zak transforms provides a more general framework with a rich theory (and it is applicable even to continuous domain results). Further, the Zak transform is very helpful in determining the suitability of window functions to constitute frames since the zeroes of the Zak transform provide insight into the properties of the window function (as illustrated in Fig. 4). This is very important in the case of noncanonical frames since two dissimilar windows are being used to generate the frame. The BDFT method is limited, however, to discrete signals.

IV. Results and Discussion

The technique used to invert the MWG frame operators is similar to the one used for the single window Gabor frame operators. In [7], the

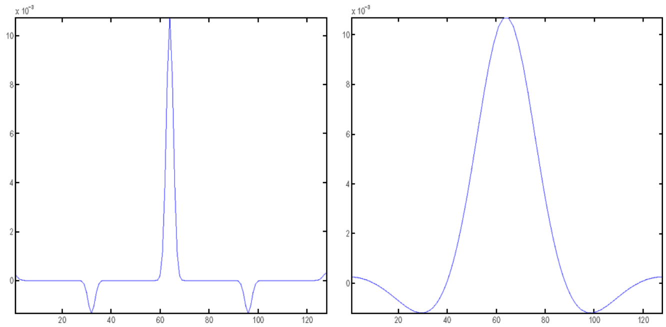

Fig. 5. Canonical dual for the multiple Gaussian windows with lattice constants ($a = 2, b = 4, \sigma_n g = \sigma_n h = 2, \sigma_w g = \sigma_w h = 16, L = 128$).

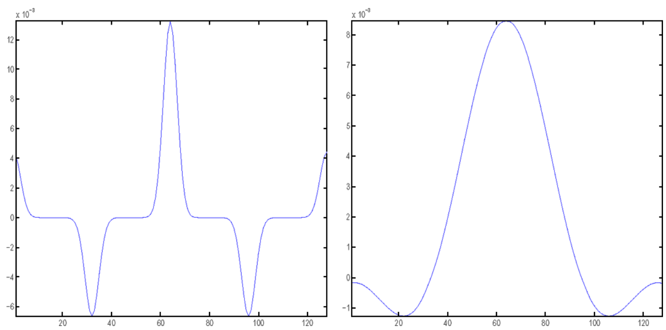

Fig. 6. Noncanonical dual for the multiple Gaussian windows with lattice constants ($a = 2, b = 4, \sigma_n g = 2, \sigma_n h = 4, \sigma_w g = 8, \sigma_w h = 16, L = 128$).

authors established an efficient technique to invert the Gabor frame operators in case of integer oversampling. In the present correspondence, we have a technique that allows us to simplify the problem of matrix inversion for rational oversampling. The computational complexity of finding the dual of the frame is $O(nlog(n))$, using the conjugate gradients techniques. The factorization technique we introduced for rational oversampling cases can be used to simplify the matrix inversion.

In the case of MWG expansions, the number of operations required to generate the matrix $\mathbf{P}$ is $2Rab$ multiplications and 0 additions. The complexity of computing the Fourier components and inverting $\mathbf{P}$ is given by $O(2Llog(b))$. The overall complexity of our algorithm is much better than that associated with the traditional methods, for both integer and rational oversampling.

As the simulations depicted in Figs. 5 and 6 illustrate, both the canonical and noncanonical duals retain the localization properties of multiple Gaussian windows. The noncanonical dual does not change much, especially in the wide window case. The parameters are considerably different, as shown in the figures. It is also worth mentioning that the ratio of the condition numbers of noncanonical duals to the condition number of the canonical dual is 0.23. This demonstrates that noncanonical duals may have greater stability than canonical duals, under certain conditions.

Other norms than the L_2 norm can be minimized. There is a particular case for minimizing the L_1 norm [15]. In this case, we assume that one set of windows is fixed (i.e., known) and it is the other set of windows that has to be found and optimized. Let us assume that the set of window functions that constitute the matrix $\mathbf{G}$ is known and that these functions constitute the expansion frame. We need to find the optimum $\mathbf{H}$. Mathematically, it can be written as

$$min \left\| \mathbf{H}^* (\mathbf{G}\mathbf{H}^*)^{-1} f \right\|_1 \tag{18}$$

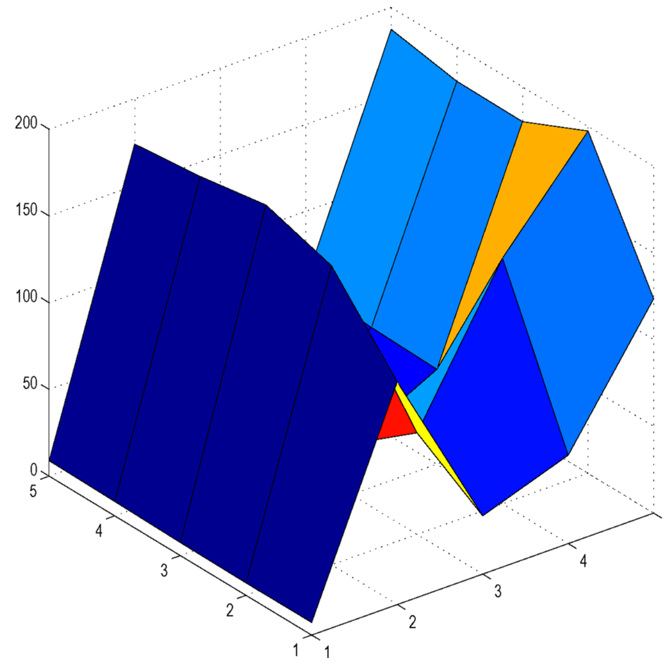

Fig. 7. L_1 norm for a signal, with $R = 2$, $L = 64$, $a = 4$, $b = 8$. The set of functions $h[\cdot]$ is varied, while $g[\cdot]$ is kept constant. In the example, both $g[\cdot]$ and $h[\cdot]$ are Gaussians. As can be seen, it does not follow any single, simple pattern.

subject to the condition that $\mathbf{G}\mathbf{H}^*$ is invertible. However, there are serious problems of obtaining an analytical solution. Firstly, computing the invertibility for all possible $\mathbf{H}$ is a very tall order. Secondly, the problem is not convex, so it is difficult to tell the local minima from the global minimum. Therefore, in order to reduce the complexity of the problem, we have assumed both $h[\cdot]$ and $g[\cdot]$ to be Gaussians, with effective spread[3] σ such that $0 < \sigma \leq L$. When the number of windows is small, the golden section technique [16] can be used with some success. Fig. 7 shows the local minima in the case of minimization of L_1 norm, when both $h[\cdot]$ and $g[\cdot]$ are Gaussian functions, with $g[\cdot]$ fixed, and $h[\cdot]$ having various effective spreads.

V. An Application

To complement our theoretical analysis, we illustrate a practical application where, with noncanonical duals, one can do better. Considering two images, we shall attempt to reconstruct them with 5% of the coefficients having the highest magnitude, obtained by means of the canonical and noncanonical frames.[4] We have incorporated four windows into our reconstruction frame (window functions are Gaussians with different spreads, whose spreads increase in arithmetic progression). The results of the reconstruction with the canonical and noncanonical duals are depicted in Fig. 8. We have three areas of interest. In the canonical case, the high frequency components have not been localized accurately and edges are blurred. Feathers decorating the hat cannot be identified clearly (segment B). The reflection in the mirror is blurred (segment C) and line features of the hat (segment A) are lost. In contrast, in the noncanonical case, they are clearly visible. Similarly, in the case of the nails (Fig. 9), we can see that noncanonical reconstruction is much clearer than in the canonical reconstruction. In both segments A and B, the delineation between the nails is clearer and sharper in the noncanonical case. We have clearly achieved better localization

[3]An alternative term for spread used often, is "effective support."

[4]Even though we have chosen only 5% of the coefficients, we have more than 90% of the total energy of the set of coefficients in the chosen coefficients. Further, all coefficients and windows were normalized to be able to choose the coefficients.

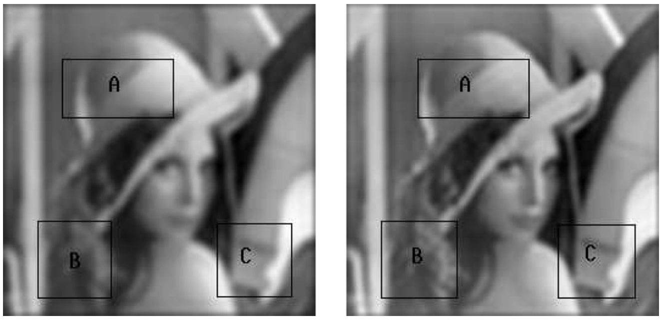

Fig. 8. Reconstruction with the canonical dual (left) and noncanonical dual (right). The marked areas A and C highlight the sharper contours of smooth segments, while B shows better preservation of texture in noncanonical reconstruction.

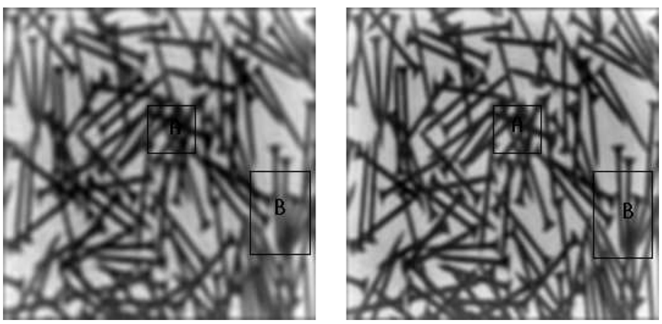

Fig. 9. Reconstruction of image of nails with canonical (left) and noncanonical duals (right). Here again, the marked areas (segments A and B) show significantly less blurring and the separation between the nails is visible. Even elsewhere (especially the bottom right of the image) the difference is noticeable

of frequencies in the case of noncanonical than with canonical duals. It is important to stress that noncanonical duals achieve better localization than canonical ones, regardless of the expansion frame used. One can achieve better reconstruction than we have achieved in this case using canonical duals, by choosing windows based on geometric progression. However, the better localization would again be limited to natural images. In cases where the expansion frame is fixed, it is better to choose noncanonical duals, and benefit from their inherent flexibility in localization of frequencies. Also, with noncanonical duals, it is possible to achieve better localization with noncanonical duals in cases where nothing is known about the set of signals/images *a priori*.

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Comments on "The Inverse S-Transform in Filters With Time-Frequency Localization"

C. R. Pinnegar

***Abstract*—The correspondence "The Inverse S-Transform in Filters With Time-Frequency Localization," by Schimmel and Gallart, presented a new method of time-frequency filtering with the S-transform. Their technique contains an error, which fortunately can be corrected. This correspondence describes the correction.**

***Index Terms*—Data-adaptive filter, noise attenuation, polarization, seismic signal processing, signal detection, time-frequency analysis, time-varying filters.**

I. Introduction

The authors of [1] presented a paper that introduced a new way of using the S-transform [2] to perform time-frequency filtering. The definition of the S-transform of a continuous function can be obtained by combining (1) and (2) of [1], to give

$$S(\tau, f) = \frac{|f|}{k\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) \exp\left(\frac{-f^2(\tau - t)^2}{2k^2}\right) e^{-i2\pi f t}\, dt. \quad (1)$$

Here, u is a continuous function of running time t, τ denotes the midpoint of the S-transform window (a Gaussian that gives the S-transform multiscale time resolution of the Fourier spectrum), f is frequency, and k is a user-defined constant (here, set to 1). From (1), it is not difficult to show that the S-transform is invertible via

$$u(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\tau, f) e^{i2\pi f t}\, d\tau\, df. \quad (2)$$

In analogy with Fourier-domain filtering, $S(\tau, f)$ can be multiplied by a filter, denoted $F(\tau, f)$, in (2) to give a filtered time series $u_{\mathrm{filt}_1}(t)$.

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The author is with Calgary Scientific, Inc., Calgary, AB T2G 1M8, Canada (e-mail: pinnegar@ucalgary.ca).

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