Orthogonalization of Circular Stationary Vector Sequences and Its Application to the Gabor Decomposition

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***Abstract*—Certain vector sequences in Hermitian or in Hilbert spaces, can be orthogonalized by a Fourier transform. In the finite-dimensional case, the discrete Fourier transform (DFT) accomplishes the orthogonalization. The property of a vector sequence which allows the orthogonalization of the sequence by the DFT, called circular stationarity (CS), is discussed in this paper. Applying the DFT to a given CS vector sequence results in an orthogonal vector sequence, which has the same span as the original one. In order to obtain coefficients of the decomposition of a vector upon a particular nonorthogonal CS vector sequence, the decomposition is first found upon the equivalent DFT-orthogonalized one and then the required coefficients are found through the DFT. It is shown that the sequence of discrete Gabor basis functions with periodic kernel and with a certain inner product on the space of N-periodic discrete functions, satisfies the CS condition. The theory of decomposition upon CS vector sequences is then applied to the Gabor basis functions to produce a fast algorithm for calculation of the Gabor coefficients.**

I. Introduction

In this paper, we present and utilize extensively properties of a class of vector sequences called circular stationary (CS). The definition and properties of CS vector sequences have correspondences in the theory of wide-sense stationary (WSS) random processes. Both the WSS process and the CS vector sequence are sequences in a Hilbert or Hermitian space and possess the characteristic of translation invariance. Therefore, many theorems from the theory of WSS processes can be directly applied to the CS vector sequences. In the finite-dimensional case, theorems pertaining to the CS sequences find equivalents in the theory of circulant matrices. The translation of relevant theorems into the language of vector Hermitian or Hilbert space allows the derivation of significant new properties. In the attempt to simplify certain proofs and ascertain whether or not a vector set is stationary, we present a group theoretical approach to stationarity. We then apply the approach to the Gabor decomposition and derive a new fast algorithm for its computation.

The paper is organized as follows: In Section II, the main theory of circular stationary sequences is expressed as vector language, orthogonalization and decomposition of these vector sequences are considered, and some examples are given. In Section III, some applications of the orthogonalization and decomposition theorems are shown. In Section IV, a group theoretical approach to the problem of circular stationarity is developed. In Section V, it is shown how to find a vector set that is biorthogonal to a given circular stationary vector set, even in the case of undersampling. The theory of orthogonalization and decomposition of stationary (not necessarily circular stationary) vector sequences is presented in Section VI. In Section VII, the decomposition theorems, the group theoretical approach, and biorthogonal basis determination are extended to multidimensional vector sequences and, in particular, to the discrete Gabor functions. In this section, a fast algorithm for decomposition on the Gabor basis is derived. The resulting formulas are expressed in Section VIII in terms of the Zak transform, and comparisons are made between our method and the ones developed in [10]–[12].

II. Theory and Definitions

Definition 1: Given a Hilbert or a Hermitian linear space $\mathcal{H}$, the continuous (discrete) 1-D vector sequence $\mathbf{x}(t)$ ($\mathbf{x}[n]$) is a mapping of continuous (discrete) variable t (n) into the space. The function that yields inner products between any two of such vectors $R_x(t_1,t_2) =< \mathbf{x}(t_1), \mathbf{x}(t_2) >$ ($R_x[n_1,n_2] =< \mathbf{x}[n_1], \mathbf{x}[n_2] >$) is called the autocorrelation function of the vector sequence.

For simplicity, we describe the discrete parameter case explicitly with analogous application to the continuous case understood.

Definition 2: The homogeneous (stationary) vector sequence is the vector sequence whose autocorrelation depends only on the difference between the second and the first arguments:

$$R[n_1, n_2] = R[n_1 - n_2, 0].$$

In this case, we call a single argument function $R[n] \triangleq R[n, 0]$ the autocorrelation function of the sequence.

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Examples of stationary vector sequences are described below.

Example 1—Wide Sense Stationary Stochastic Process

Any wide sense stationary stochastic process $x[n]$ with inner product given by $< x[n_1], x[n_2] >= E\{x[n_1]\bar{x}[n_2]\}$, where E denotes the expectation and $\bar{x}$ the conjugate of x, is a stationary vector sequence.

Example 2—Time Shifts

Consider a sequence $g_n(t) = g(t-n\Delta t)$ in $\mathcal{L}_2$ space, where $\mathcal{L}_2$ is a Hilbert space of all square integrable functions with inner product between any two functions $g(t)$ and $h(t)$ in the space given by the following formula:

$$< g, h >= \int_{-\infty}^{\infty} g(t)\bar{h}(t)dt$$

and $g(t)$ is a function in $\mathcal{L}_2$. From the definition and the substitution below, $g_n(t)$ is a stationary vector sequence. Indeed

$$\begin{aligned} R_g[n_1, n_2] &=< g_{n_1}, g_{n_2} > \\ &= \int_{-\infty}^{\infty} g(t-n_1\Delta t)\bar{g}(t-n_2\Delta t)dt \\ &= \int_{-\infty}^{\infty} g(t-(n_1-n_2))\Delta t)\bar{g}(t)dt \\ &= R_g[n_1-n_2, 0] \end{aligned}$$

Example 3—Frequency Shifts

Consider a sequence $g_n(t) = g(t)e^{jn\Omega t}$in $\mathcal{L}_2$ space. Again, the sequence is stationary, as follows:

$$\begin{aligned} R_g[n_1, n_2] &=< g_{n_1}, g_{n_2} > \\ &= \int_{-\infty}^{\infty} g(t)e^{jn_1\Omega t}\bar{g}(t)e^{-jn_2\Omega t}dt \\ &= \int_{-\infty}^{\infty} g(t)(t)e^{j(n_1-n_2)\Omega t}\bar{g}(t)dt \\ &= R_g[n_1-n_2, 0] \end{aligned}$$

Consider now a periodic vector sequence: $\ldots, \mathbf{x}[N-1], \mathbf{x}[0], \mathbf{x}[1], \cdots, \mathbf{x}[N-1], \ldots$ and the N unique vectors in this sequence $\mathbf{x}[0], \mathbf{x}[1], \cdots, \mathbf{x}[N-1]$.

Definition 3: The sequence of N vectors $\mathbf{x}[0], \mathbf{x}[1], \cdots, \mathbf{x}[N-1]$ is called a CS sequence with period N, if when continued periodically in both directions, it will produce a stationary sequence. Note that this condition can be reformulated in the following way: For n_1 and n_2, where n_1 and n_2 are any integers greater than or equal to zero and less than N, the following is true:

$$R_x[n_1, n_2] = R_x[(n_1-n_2) \bmod N, 0] \tag{1}$$

where $(n_1 - n_2) \bmod N$ is the remainder after division of $n_1 - n_2$ by N.

Note that the definition of CS sequences does not coincide with that of cyclostationarity. Actually, the CS sequences are precisely those that are periodic stationary and stationary at the same time.

Let us consider examples of CS vector sequences.

Example 4—Two Equal Norm Vectors with Real Inner Product. (Two Equal Norm Vectors on the Real Plane)

Let there be two vectors $\mathbf{x}[0]$ and $\mathbf{x}[1]$ in an Hermitian space $\mathcal{H}$ of the same norm ($||\mathbf{x}[0]||^2 = ||\mathbf{x}[1]||^2 = < \mathbf{x}[1], \mathbf{x}[1] >= a$), such that their inner product is real: $< \mathbf{x}[0], \mathbf{x}[1] >= b$. Extending periodically this vector sequence in both directions results in an infinite periodic vector sequence $\ldots, \mathbf{x}[0], \mathbf{x}[1], \mathbf{x}[0], \mathbf{x}[1], \ldots$, which is stationary. Indeed, since $< \mathbf{x}[0], \mathbf{x}[1] >$ is real and $< \mathbf{x}[0], \mathbf{x}[1] >=< \mathbf{x}[1], \mathbf{x}[0] >= b$, it is found that

$$R_x[n_1, n_2] = \begin{cases} a & \text{if } n_1 - n_2 \text{ is even} \\ b & \text{if } n_1 - n_2 \text{ is odd} \end{cases}.$$

Since the autocorrelation function depends only on the difference of its arguments, the extended sequence is stationary and the first sequence is circular stationary.

Example 5—Equally Spaced Three Vectors on a Real Plane

Consider a real plane with an Hermitian structure added by introducing multiplication by $j = \sqrt{-1}$. However, all of the inner products of the vectors of the plane will remain real. Take a vector sequence on that plane $\mathbf{x}[0], \mathbf{x}[1], \mathbf{x}[2]$ consisting of three vectors of the same norm, such that $angle(\mathbf{x}[0], \mathbf{x}[1]) = angle(\mathbf{x}[1], \mathbf{x}[2]) = angle(\mathbf{x}[2], \mathbf{x}[0]) = \frac{2\pi}{3}$, as shown in Fig. 2, where the function $angle$ is an ordinary angle taking values from 0 to 2π and is measured from the first argument vector to the second in the counterclockwise direction. The extended periodic sequence is stationary and the original sequence is CS.

Example 6—Time Shifts of a Periodic Function

Consider the same situation as in the Example 1, only assume that the function $g(t)$ is periodic with period $N\Delta t$. Now, there are only N unique functions in the sequence $g_n(t) = g(t - n\Delta t)$ for $n = 0, 1, \cdots, N-1$. Since $g(t)$ does not belong to $\mathcal{L}_2$, the space is taken now to be $P_2[0, N\Delta t]$ of $N\Delta t$ periodic functions and square integrable on the interval $[0, N\Delta t]$.[1] For any two functions $g(t)$ and $h(t)$ in $P_2[0, N\Delta t]$, $\int_0^{N\Delta t} g(t)\bar{h}(t)dt$ is defined to be their inner product. The periodically extended sequence is stationary (the proof of the fact is very similar to that of the Example 1) so that the functional sequence $g_n(t)$ is CS.

Example 7[2]—Frequency Shifts (Modulation) of a Discrete Function

Consider a space D_2 of discrete, square summable functions. For any two functions $g[k]$ and $h[k]$ in the space, define

[1] This actually means that the function is square integrable on any finite interval.

[2] Examples 6 and 7 are actually dual since the Fourier transform of a discrete function is periodic, and time shifts correspond to modulation in the frequency domain.

$\sum_{k=-\infty}^{\infty} g[k]\bar{h}[k]$ to be their inner product. The sequence $g_n[k] = g[k]e^{jn\frac{2\pi}{N}k}$ in the space is periodic since $g_{n+N}[k] = g[k]e^{j(n+N)\frac{2\pi}{N}k} = g[k]e^{jn\frac{2\pi}{N}k}e^{j2\pi k} = g_n[k]$, and the extended sequence is stationary:

$$\begin{aligned} R[n_1, n_2] &= \sum_{k=-\infty}^{\infty} g_{n_1}[k]\bar{g_{n_2}}[k] \\ &= \sum_{k=-\infty}^{\infty} g[k]e^{jn_1\frac{2\pi}{N}k}\bar{g}[k]e^{-jn_2\frac{2\pi}{N}k} \\ &= \sum_{k=-\infty}^{\infty} g[k]e^{j(n_1-n_2)\frac{2\pi}{N}k}\bar{g}[k] \\ &= R[n_1 - n_2, 0]. \end{aligned}$$

Therefore, the functional sequence $g_n[k]$, where $n = 0, 1, \cdots, N-1$, is CS.

The orthogonalization theorem for CS vector sequences can now be stated. This theorem has an analog not only in the theory of stochastic processes, but also in the theory of the circulant matrices, where it assumes the form of the diagonalization theorem for circulant matrices (see [5]).

Theorem 1 (Orthogonalization of CS Vector Sequences): Given a CS vector sequence $\mathbf{x}[0], \mathbf{x}[1], \cdots, \mathbf{x}[N-1]$ in Hermitian (Hilbert) space $\mathcal{H}$, we can obtain an orthogonal vector sequence $\mathbf{y}[0], \mathbf{y}[1], \cdots, \mathbf{y}[N-1]$ that spans the same linear space as the sequence $\mathbf{x}$, by applying to the original sequence the DFT of order N:

$$\mathbf{y}[k] = \sum_{i=0}^{N-1} \mathbf{x}[i]e^{-j\frac{2\pi}{N}ik}. \tag{2}$$

The number of nonzero vectors in the sequence $\mathbf{y}[0], \mathbf{y}[1], \cdots, \mathbf{y}[N-1]$ equals the maximal number of linearly independent vectors among the sequence $\mathbf{x}[0], \mathbf{x}[1], \cdots, \mathbf{x}[N-1]$. The sequence $\mathbf{x}[0], \mathbf{x}[1], \cdots, \mathbf{x}[N-1]$ can be recovered from $\mathbf{y}[0], \mathbf{y}[1], \cdots, \mathbf{y}[N-1]$ by applying to it the inverse discrete Fourier transform:

$$\mathbf{x}[i] = \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{y}[k]e^{j\frac{2\pi}{N}ki}. \tag{3}$$

First, we sketch the proof and then give the proof itself. To prove that the DFT (as in (2)) really orthogonalizes the basis, we can just take the vector product of two different vectors out of sequence $\mathbf{y}[0], \mathbf{y}[1], \cdots, \mathbf{y}[N-1]$ and use the CS condition to prove that it equals zero. The proofs of the two other statements of the theorem are also simple and straightforward. The inversion formula is proved by substituting into it the expression for $\mathbf{y}[i]$ given by (2). From here, we obtain the proof of the fact that the two sets of vectors are equivalent and therefore the number of nonzero vectors among the orthogonal sequence $\mathbf{y}[i]$ should equal the dimension of the span of the sequence $\mathbf{x}[i]$. It is important to notice that the vectors $\mathbf{y}[i]$ are not necessarily of the unit norm and not even necessarily of the same norm.

Proof: To prove that the vectors $\mathbf{y}[k]$ obtained according to (2) are mutually orthogonal, consider the inner product

$$\begin{aligned} <\mathbf{y}[k], \mathbf{y}[l]> &= <\sum_{i=0}^{N-1} \mathbf{x}[i]e^{-j\frac{2\pi}{N}ik}, \sum_{n=0}^{N-1} \mathbf{x}[n]e^{-j\frac{2\pi}{N}nl}> \\ &= \sum_{i=0}^{N-1}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}ik}e^{j\frac{2\pi}{N}nl} <\mathbf{x}[i], \mathbf{x}[n]> \\ &= \sum_{i=0}^{N-1}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(nl-ik)}R_x[i,n] \end{aligned} \tag{4}$$

and utilize the CS property. The expression above is the sum $e^{j\frac{2\pi}{N}(nl-ik)}R_x[i,n]$ over pairs (i,n); therefore, each of them is employed once and only once. The terms of the summation are rearranged in the following manner. Consider line segment $[0, N]$ reformed into a circle by putting together its ends: N and 0. Now, fix an integer p equal to the circular difference $(i-n) \bmod N$, and set up the requirement that when i cycles through values from 0 to $N-1$, n takes values lying p samples away from i, counterclockwise on the circle. For each $i = 1, 2, \cdots, N-1$, with p fixed in turn from 0 to $N-1$, each of the pairs (i,n) is traversed only once. Because of the CS property, $R_x[i,n]$ depends only on the circular difference between i and n.

If we fix p, we can rewrite the above expression as

$$\begin{aligned} &\sum_{p=0}^{N-1}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(nl-((n+p)\bmod N)k)}R_x[i,n] \\ &\quad \cdot \sum_{p=0}^{N-1}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(nl-((n+p)\bmod N)k)}R_x[p,0] \\ &\quad = \sum_{p=0}^{N-1} R_x[p,0]\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(nl-((n+p)\bmod N)k)}. \end{aligned} \tag{5}$$

Now, for a given p, consider $S_p = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(nl-((n+p)\bmod N)k)}$. We note that

$$\begin{aligned} e^{j\frac{2\pi}{N}(nl-((n+p)\bmod N)k)} &= e^{j\frac{2\pi}{N}(nl-((n+p)\bmod N)k)\bmod N} \\ &= e^{j\frac{2\pi}{N}(nl-(n+p)k)\bmod N} \\ &= e^{j\frac{2\pi}{N}(n(l-k)-pk)\bmod N} \\ &= e^{j\frac{2\pi}{N}(n(l-k))\bmod N-(pk)\bmod N} \\ &= e^{-j\frac{2\pi}{N}(pk)\bmod N}e^{j\frac{2\pi}{N}(n(l-k))\bmod N} \end{aligned} \tag{6}$$

and observe that

$$S_p = e^{-j\frac{2\pi}{N}(pk)\bmod N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(n(l-k))\bmod N}.$$

However, if $l-k \neq 0$, then $\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(n(l-k))\bmod N} = 0$, $S_p = 0$, and $<\mathbf{y}[k], \mathbf{y}[l]> = 0$. Therefore, the vectors $\mathbf{y}[k]$ are mutually orthogonal.

The straightforward calculation below proves that $\mathbf{x}[i]$ is the inverse DFT of the $\mathbf{y}[k]$ sequence:

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{y}[k]e^{j\frac{2\pi}{N}ki} &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1}\mathbf{x}[n]e^{-j\frac{2\pi}{N}nk}e^{j\frac{2\pi}{N}ki}\\ &= \frac{1}{N}\sum_{k,n=0}^{N-1}\mathbf{x}[n]e^{-j\frac{2\pi}{N}(n-i)k}\\ &= \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{x}[n]\sum_{k=0}^{N-1}e^{-j\frac{2\pi}{N}(n-i)k}.\end{aligned}$$

Because $\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(n-i)k}$ equals 0 for any $n-i\neq 0$ and equals N for $n=i$, the expression gives $\mathbf{x}[i]$.

Every vector of the sequence $\mathbf{y}[k]$ can be obtained as a linear combination of the vectors $\mathbf{x}[i]$. The opposite is also true: Every vector of the sequence $\mathbf{x}[i]$ can be obtained as a linear combination of the vectors $\mathbf{y}[k]$. Therefore, the two sequences must have the same span. Suppose that vectors $\mathbf{x}[i]$ span a space of dimension $D<N$. Therefore, there should be D linearly independent vectors among them, which constitute a basis of the span and $N-D$ vectors linearly dependent upon them. The same should be true about $\mathbf{y}[k]$ sequence, but since this sequence is orthogonal, the linearly dependent vectors will be zero, and D vectors will be nonzero. Therefore, the $\mathbf{y}[k]$ sequence has exactly $N-D$ zero vectors. □

Another way to prove this theorem is the following: One can notice that the autocorrelation of the circular stationary vector set is a circulant matrix and then use the circulant matrices diagonalization theorem (see [5]).

Suppose now that we have a vector $\mathbf{r}$ and want to find coefficients $c_0, c_1, \cdots, c_{N-1}$ such that $||\mathbf{r}-\sum_{k=0}^{N-1} c_k\mathbf{x}[k]||^2$ is minimal. The vector

$$\mathbf{r}_{pr}=\sum_{k=0}^{N-1}c_k\mathbf{x}[k] \tag{7}$$

is then the projection of $\mathbf{r}$ on the span of vectors $\mathbf{x}[0], \mathbf{x}[1], \cdots, \mathbf{x}[N-1]$. Since the orthogonal sequence $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ is equivalent to $\{\mathbf{x}[k]\}_{k=0}^{N-1}$ and the transformations to and from one to the other are known, we can find the decomposition of the vector $\mathbf{r}$ on the basis $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ and then obtain the coefficients $c_0, c_1, \cdots, c_{N-1}$ using the above transformations. More precisely, we have the following theorem.

Theorem 2 (Decomposition of CS Vector Sequences): With notations as above, the coefficients $c_0, c_1, \cdots, c_{N-1}$ are obtained according to the following formula:

$$c_k=\sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1}\frac{1}{||\mathbf{y}[i]||^2}<\mathbf{r},\mathbf{y}[i]>e^{-j\frac{2\pi}{N}ik}. \tag{8}$$

In other words, to obtain the coefficients of vector decomposition on the CS vector basis $\{\mathbf{x}[i]\}_{i=0}^{N-1}$, one can decompose the vector on the orthogonal basis $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ and take the DFT of the coefficients.

Proof: The projection of vector $\mathbf{r}$ on an orthogonal set $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ is found to be

$$\mathbf{r}_{pr}=\sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1}\frac{1}{||\mathbf{y}[i]||^2}<\mathbf{r},\mathbf{y}[i]>\mathbf{y}[i]. \tag{9}$$

Now, substituting for $\mathbf{y}[i]$, the expression given by (2) yields

$$\begin{aligned}\mathbf{r}_{pr}&=\sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1}\frac{1}{||\mathbf{y}[i]||^2}<\mathbf{r},\mathbf{y}[i]>\sum_{k=0}^{N-1}\mathbf{x}[k]e^{-j\frac{2\pi}{N}ki}\\ &=\sum_{k=0}^{N-1}\mathbf{x}[k]\sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1}\frac{1}{||\mathbf{y}[i]||^2}<\mathbf{r},\mathbf{y}[i]>e^{-j\frac{2\pi}{N}ki}.\end{aligned} \tag{10}$$

The inner summation in (10) is the coefficient c_k. □

Note that after substituting into (8) the expression for the $\mathbf{y}_i$ given by (2), the coefficients c_i of the decomposition can be expressed through inner products $<\mathbf{r},\mathbf{x}[i]>$ of the decomposed vector $\mathbf{r}$ and the vectors $\mathbf{x}_i$ of the original sequence:

$$\begin{aligned}c_k&=\Sigma_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1}\frac{1}{||\mathbf{y}[i]||^2}<\mathbf{r},\Sigma_{l=0}^{N-1}\mathbf{x}[l]e^{-j\frac{2\pi}{N}li}>e^{-j\frac{2\pi}{N}ik}\\ &=\Sigma_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1}\left(\frac{1}{||\mathbf{y}[i]||^2}\left(\Sigma_{l=0}^{N-1}<\mathbf{r},\mathbf{x}[l]>e^{j\frac{2\pi}{N}Nli}\right)\right)e^{-j\frac{2\pi}{N}ik}.\end{aligned} \tag{11}$$

The procedure for finding coefficients c_i based on (11) is, however, longer than that based on (8) because there are two FFT's involved instead of one. In the Gabor decomposition case treated later, the difference between the two procedures is even greater due to the structure of $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ and $\{\mathbf{y}[i]\}_{i=0}^{N-1}$.

Notice that the above theorem is applicable in the cases of undersampling as well, when the vector $\mathbf{r}$ does not belong to the span of $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ and oversampling when the vectors $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ are linearly dependent.

III. Examples of Application of Orthogonalization and Decomposition Theorems

Example 1—Two Equal Norm Vectors with Real Inner Product (Two Equal Norm Vectors on a Real Plane)

Consider the vector sequence given in Example 4 of the previous section. The vectors $\mathbf{x}[0]$ and $\mathbf{x}[1]$ are orthogonalized by DFT:

$$\mathbf{y}[0]=\mathbf{x}[0]e^{-j\frac{2\pi}{2}0\times 0}+\mathbf{x}[1]e^{-j\frac{2\pi}{2}1\times 0}=\mathbf{x}[0]+\mathbf{x}[1]$$

$$\mathbf{y}[1]=\mathbf{x}[0]e^{-j\frac{2\pi}{2}0\times 1}+\mathbf{x}[1]e^{-j\frac{2\pi}{2}1\times 1}=\mathbf{x}[0]-\mathbf{x}[1].$$

If $\mathbf{x}[0]$ and $\mathbf{x}[1]$ have the same norm and a real inner product, vectors $\mathbf{y}[0]$ and $\mathbf{y}[1]$ are orthogonal, as illustrated in Fig. 1. Given a vector $\mathbf{z}$, its projection on the $span(\mathbf{x}[0],\mathbf{x}[1])$ is

$$\mathbf{z}_{pr}=c_0\mathbf{x}[0]+c_1\mathbf{x}[1]$$

where

$$c_0=\frac{1}{||\mathbf{y}[0]||^2}<\mathbf{z},\mathbf{y}[0]>+\frac{1}{||\mathbf{y}[1]||^2}<\mathbf{z},\mathbf{y}[1]>$$

$$c_1=\frac{1}{||\mathbf{y}[0]||^2}<\mathbf{z},\mathbf{y}[0]>-\frac{1}{||\mathbf{y}[1]||^2}<\mathbf{z},\mathbf{y}[1]>.$$

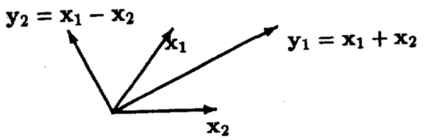

Fig. 1. Two vectors of equal length, $\mathbf{x}_1$ and $\mathbf{x}_2$ on a real plane and their orthogonalization. Note that $\mathbf{y}_1$ and $\mathbf{y}_2$ are orthogonal to one another.

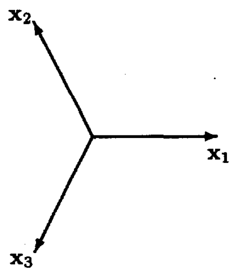

Fig. 2. Three equally spaced vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ on a real plane.

Example 2—Three Equally Spaced Vectors on a Real Plane

Consider the situation as in Example 5 of the previous section. This example is of interest because in this case the dimensionality of the space D is less than the number of vectors N in the sequence. $D = 2$, and $N = 3$, so that $N - D = 1$. We find that

$$\mathbf{y}[0] = \mathbf{x}[0] + \mathbf{x}[1] + \mathbf{x}[2] = 0$$

and the other two vectors $\mathbf{y}[1]$ and $\mathbf{y}[2]$ are nonzero. This example illustrates the fact that the number of nonzero vectors among the sequence $\mathbf{y}[i]$ equals the dimensionality of the space spanned by $(\{\mathbf{x}[i]\}_{i=0}^{i=N-1})$.

Example 3—Time Shifts of a Periodic Function

Consider the functional sequence g_i in Example 6 of the previous section. By applying the orthogonalization theorem, we obtain an orthogonal functional sequence h_k,

$$h_k(t) = \sum_{i=0}^{N-1} g_i(t) e^{-j\frac{2\pi}{N}ik} = \sum_{i=0}^{N-1} g(t - i\Delta t) e^{-j\frac{2\pi}{N}ik} \quad (12)$$

for $k = 0, 1 \cdots N-1$. Now, taking $||h_i||^2 = \int_0^{N\Delta t} |h_i(t)|^2 dt$, we can use the decomposition theorem and (8) to obtain the coefficients of decomposition of an arbitrary $N\Delta t$-periodic function $f(t) \in P_2[0; N\Delta t]$ (notation as in previous Example 6) on the sequence $g_n(t)$:

$$c_i = \sum_{k=0, ||h_k(t)|| \neq 0}^{N-1} \frac{1}{||h_k(t)||^2} \int_0^{N\Delta t} f(t) \bar{h}_k(t) dt e^{-j\frac{2\pi}{N}ik}. \quad (13)$$

Example 4—Frequency Shifts of a Discrete Function

Given the conditions of Example 7 of the previous section, by the use of the orthogonalization theorem, a set of orthogonal functions $h_n[k]$ can be found in D_2, which is the space of discrete, square summable functions, by taking the DFT of the original set

$$\begin{aligned} h_n[k] &= \sum_{i=0}^{N-1} g_i[k] e^{-j\frac{2\pi}{N}in} = \sum_{i=0}^{N-1} g[k] e^{j\frac{2\pi}{N}ik} e^{-j\frac{2\pi}{N}in} \\ &= g[k] \sum_{i=0}^{N-1} e^{j\frac{2\pi}{N}i(k-n)} \\ &= N g[k] \sum_{l=-\infty}^{\infty} \delta[k - n + lN] \end{aligned} \quad (14)$$

where $n = 0, 1 \cdots N-1$ and $\delta[k]$ is a discrete impulse function, which equals 1 at zero and 0 everywhere else. Analogous to (8), the coefficients of the decomposition of some function $f[k]$ on the sequence $g_i[k]$ are

$$\begin{aligned} c_i &= \sum_{n=0, ||h_n[k]|| \neq 0}^{N-1} \frac{1}{||h_n[n]||^2} \Big(\sum_{k=-\infty}^{\infty} f[k] \bar{h}_n[k] \Big) e^{-j\frac{2\pi}{N}in} \\ &= \sum_{n=0, ||h_n[k]|| \neq 0}^{N-1} \frac{1}{||h_n[n]||^2} N \\ &\quad \cdot \Big(\sum_{l=-\infty}^{\infty} f[n - Nl] \bar{g}[n - Nl] \Big) e^{-j\frac{2\pi}{N}in} \end{aligned} \quad (15)$$

and $||h_n[k]||^2 = N^2 \sum_{l=-\infty}^{\infty} |g[n - Nl]|^2)$.

IV. Group Theory Approach to CS

Proofs of the theorems regarding CS can be simplified by adopting a somewhat different viewpoint.

Definition 4: Given a Hermitian (Hilbert) space $\mathcal{H}$, a linear transformation P of the space onto itself is called a linear isometry, if it preserves the norm, i.e., for every $\mathbf{a} \in \mathcal{H}$, $||P\mathbf{a}|| = ||\mathbf{a}||$.

We shall need the following property of a linear isometry: With the conditions as above, for any two vectors $\mathbf{a}, \mathbf{b} \in \mathcal{H}$, we have

$$< P\mathbf{a}, P\mathbf{b} > = < \mathbf{a}, \mathbf{b} > . \quad (16)$$

To prove this, notice that $||P(\mathbf{a} + \mathbf{b})||^2 = ||\mathbf{a} + \mathbf{b}||^2$, i.e., $||P\mathbf{a}||^2 + 2Re < P\mathbf{a}, P\mathbf{b} > + ||P\mathbf{b}||^2 = ||\mathbf{a}||^2 + 2Re < \mathbf{a}, \mathbf{b} > + ||\mathbf{b}||^2$, or, since $||P\mathbf{a}||^2 = ||\mathbf{a}||^2$ and $||P\mathbf{b}||^2 = ||\mathbf{b}||^2$, we have $Re < P\mathbf{a}, P\mathbf{b} > = Re < \mathbf{a}, \mathbf{b} >$. By considering the difference $||\mathbf{a} - \mathbf{b}||^2$, we prove that $Im < P\mathbf{a}, P\mathbf{b} > = Im < \mathbf{a}, \mathbf{b} >$. Noticing that P^{-1} is also a linear isometry, by applying recursively (16), we obtain

$$< P^n\mathbf{a}, P^n\mathbf{b} > = < \mathbf{a}, \mathbf{b} > \quad (17)$$

for any integer n.

Definition 5: Given a linear isometry P, P^n is also a linear isometry for any integer n. Denoting the identity transformation (which is also a linear) isometry by I, one can see that the set $\ldots, P^{-1}, I, P, P^2, \cdots, P^n, \ldots$ constitutes an Abelian (commutative) group with respect to multiplication. We shall call P a basis of the group and say that the group is established by P since all of the nonzero elements of the group are orders of P. If $P^N = I$, and there is no positive $N_1 < N$ such that

$P^{N_1} = I$, the group established by P is called an N-cyclic group.

Theorem 3: A set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ is CS iff there is a linear isometry P, which is a basis of an N-cyclic group, such that $\mathbf{x}[i \bmod N] = P^i\mathbf{x}[0]$ for any integer i.

Proof: Suppose there is such an isometry P. Then, we have $R[n_1, n_2] =< \mathbf{x}[n_1], \mathbf{x}[n_2] >=< P^{n_1}\mathbf{x}[0], P^{n_2}\mathbf{x}[0] >$. Now, using (17), we obtain

$$\begin{aligned}< P^{n_1}\mathbf{x}[0], P^{n_2}\mathbf{x}[0] > &=< P^{-n_2}(P^{n_1}\mathbf{x}[0]), P^{-n_2}P^{n_2}\mathbf{x}[0] > \\ &=< P^{n_1-n_2}\mathbf{x}[0], \mathbf{x}[0] > \\ &= R[n_1 - n_2, 0]\end{aligned}$$

and the sequence is CS.

To prove the converse, assume that $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ is CS. Suppose it spans a subspace $S \in \mathcal{H}$. Applying DFT to the set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$,we obtain an orthogonal sequence of vectors $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ that may contain zero vectors. Consider now the linear transformation P_S of the subspace S onto itself, such that

$$P_S(\mathbf{y}[k]) = e^{j\frac{2\pi}{N}k}\mathbf{y}[k]. \tag{18}$$

The transformation is an isometry of S since for every vector $\mathbf{a} = \sum_{i=0}^{N-1} a_i\mathbf{y}[i]$, where a_i is assumed to be zero if $\mathbf{y}[i]$ is zero, we have

$$\begin{aligned}||P_S\mathbf{a}||^2 &=< P_S(\sum_{i=0}^{N-1} a_i\mathbf{y}[i]), P_S(\sum_{k=0}^{N-1} a_k\mathbf{y}[k]) > \\ &=< \sum_{i=0}^{N-1} a_iP_S\mathbf{y}[i]), \sum_{k=0}^{N-1} a_kP_S\mathbf{y}[k]) > \\ &=< \sum_{i=0}^{N-1} e^{j\frac{2\pi}{N}i}a_i\mathbf{y}[i]), \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k}a_k\mathbf{y}[k]) > \\ &= \sum_{i=0}^{N-1} a_i\bar{a}_i < \mathbf{y}[i], \mathbf{y}[i] >= ||\mathbf{a}||^2.\end{aligned} \tag{19}$$

P_S establishes a cyclic group since $P_S^N = I$. In addition, we have

$$\begin{aligned}P_S\mathbf{x}[i] &= \frac{1}{N}\sum_{k=0}^{N-1} P_S(\mathbf{y}[k])e^{j\frac{2\pi}{N}ki} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{y}[k]e^{j\frac{2\pi}{N}ki}e^{j\frac{2\pi}{N}k} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{y}[k]e^{j\frac{2\pi}{N}k(i+1)} \\ &= \mathbf{x}[(i+1) \bmod N].\end{aligned} \tag{20}$$

If $S^\perp$ is a complement of S in $\mathcal{H}$, then we can write $\mathcal{H} = S^\perp \bigoplus S$. If $I_{S^\perp}$ is the identity operator on $S^\perp$, then the operator $P = I_{S^\perp} \bigoplus P_S$ is a linear isometry of $\mathcal{H}$, and $P\mathbf{x}[i] = \mathbf{x}[(i+1) \bmod N]$, and therefore, $P^n\mathbf{x}[0] = \mathbf{x}[(n) \bmod N]$. □

The linear isometry P associated with a CS set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ is called the linear isometry of the set.

In addition, from the proof, one can see that the linear isometry P of a CS set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ will transform the elements of the set $\{\mathbf{y}[i]\}_{i=0}^{N-1}$, which is obtained by applying the DFT to $\{\mathbf{x}[i]\}_{i=0}^{i=N-1}$, according to the following formula:

$$P(\mathbf{y}[k]) = e^{j\frac{2\pi}{N}k}\mathbf{y}[k]. \tag{21}$$

V. Biorthogonal Basis and Proof of the Optimality of the Set of the Coefficients in Case of Oversampling

Suppose that there exists a finite set of vectors $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ that spans a subspace S of some Hilbert or Hermitian space $\mathcal{H}$. Suppose that there exists a set $\{\tilde{\mathbf{x}}[i]\}_{i=0}^{N-1}$, such that for every i $\tilde{\mathbf{x}}[i] \in S$, and for every $\mathbf{r} \in \mathcal{H}$, the following holds:

$$\mathbf{r}_{pr} = \sum_{i=0}^{N-1} < \mathbf{r}, \tilde{\mathbf{x}}[i] > \mathbf{x}[i] \tag{22}$$

where $\mathbf{r}_{pr}$ is the projection of $\mathbf{r}$ on S. Then, the set $\{\tilde{\mathbf{x}}[i]\}_{i=0}^{N-1}$ is called a biorthogonal vector set of the original vector set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$. Note that it need not be unique.

Rearranging (8)

$$\begin{aligned}c_k &= \sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1} \frac{1}{||\mathbf{y}[i]||^2} < \mathbf{r}, \mathbf{y}[i] > e^{-j\frac{2\pi}{N}ik} \\ &=< \mathbf{r}, \sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1} \frac{\mathbf{y}[i]}{||\mathbf{y}[i]||^2}e^{j\frac{2\pi}{N}ik} >\end{aligned} \tag{23}$$

and one can see that the set $\{\tilde{\mathbf{x}}[k]\}_{i=0}^{N-1}$, which is given by

$$\tilde{\mathbf{x}}[k] = \sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1} \frac{\mathbf{y}[i]}{||\mathbf{y}[i]||^2}e^{j\frac{2\pi}{N}ik} \tag{24}$$

is a biorthogonal set of the CS set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$. Indeed, every $\tilde{\mathbf{x}}[k]$ is in the span of the set $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ and, therefore, in the span of the original set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$. In addition, using (10), one can see that (22) also holds, and the coefficients of the decomposition of an arbitrary vector $\mathbf{r} \in \mathcal{H}$ on the set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ are given by

$$c_i =< \mathbf{r}, \tilde{\mathbf{x}}[i] > . \tag{25}$$

Theorem 4: With notations as above, given that P is the linear isometry of the CS vector set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$, the biorthogonal set $\{\tilde{\mathbf{x}}[k]\}_{i=0}^{N-1}$ is also CS with the same linear isometry P.

Proof: As was shown, P acts on the set $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ according to (21), and therefore

$$\begin{aligned}P(\tilde{\mathbf{x}}[k]) &= P(\sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1} \frac{\mathbf{y}[i]}{||\mathbf{y}[i]||^2}e^{j\frac{2\pi}{N}ik}) \\ &= \sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1} \frac{P(\mathbf{y}[i])}{||\mathbf{y}[i]||^2}e^{j\frac{2\pi}{N}ik} \\ &= \sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1} \frac{\mathbf{y}[i]}{||\mathbf{y}[i]||^2}e^{j\frac{2\pi}{N}ik}e^{j\frac{2\pi}{N}i} \\ &\cdot \sum_{i=0,||\mathbf{y}[i]||\neq 0}^{N-1} \frac{\mathbf{y}[i]}{||\mathbf{y}[i]||^2}e^{j\frac{2\pi}{N}i(k+1)} \\ &= \tilde{\mathbf{x}}[(k+1) \bmod N].\end{aligned} \tag{26}$$

□

Note that finding the coefficients of the decomposition either through the biorthogonal set or through the Fourier transform is equivalent and also works in the cases of undersampling and oversampling.

Suppose that there is a Hilbert or Hermitian space $\mathcal{H}$ and a CS set of vectors $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ in it. Suppose that this set spans a subspace S of $\mathcal{H}$ and that the $dim(S) < N$, i.e., vectors $\{\mathbf{x}[i]\}_{i=0}^{N-1}$ are linearly dependent. Then, given a vector $\mathbf{r}$, one can see that the set of the coefficients $\{c_i\}_{i=0}^{N-1}$ of the decomposition of the vector on the set $\{\mathbf{x}[i]\}_{i=0}^{N-1}$, such that

$$\mathbf{r}_{pr} = \sum_{i=0}^{N-1} c_i \mathbf{x}[i] \tag{27}$$

will not be unique. However, we can prove the following theorem concerning the optimality of the representation obtained through (8) or (25).

Theorem 5: Under the above conditions, (8) and (25) provide the optimal set of coefficients, i.e., the set with minimal sum of the squares of their norms:

$$\sum_{i=0}^{N-1} ||c_i||^2 = \min. \tag{28}$$

Proof: Since (8) adduces (25), it is sufficient to prove the fact only for coefficients obtained by means of (8).

Consider

$$d_i = \sum_{i=0, ||\mathbf{y}[i]|| \neq 0}^{N-1} \frac{1}{||\mathbf{y}[i]||^2} < \mathbf{r}, \mathbf{y}[i] >.$$

Then, using (9), one can write

$$\mathbf{r}_{pr} = \sum_{i=0, ||\mathbf{y}[i]|| \neq 0}^{N-1} d_i \mathbf{y}[i] \tag{29}$$

or taking $d_i = 0$ for such i that $||\mathbf{y}[i]|| = 0$, one can rewrite the above equation as

$$\mathbf{r}_{pr} = \sum_{i=0}^{N-1} d_i \mathbf{y}[i]. \tag{30}$$

Since $\mathbf{y}[i]$ is an orthogonal basis and d_i's are zero for $||\mathbf{y}[i]|| = 0$, these d_i coefficients are optimal in the mean square sense for the basis $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ (the sum of their squares is minimal). Indeed, the only other possibility for choosing the d_i coefficients, so that (30) would still yield a projection, would be to choose nonzero coefficients for such i's that $||\mathbf{y}[i]|| = 0$, which could only increase the norm. Because of (8), one can see that the c_k coefficients are the DFT of the d_i coefficients. Suppose that another set $\tilde{c}_k$ has a smaller sum of the squares than the c_k's, and the formula analogous to (7) still holds with $\tilde{c}_k$ instead of c_k. Then, the corresponding coefficients $\tilde{d}_i$ of the decomposition on the $\{\mathbf{y}[i]\}_{i=0}^{N-1}$ basis would be obtained through the inverse DFT of the $\tilde{c}_k$. Since the inverse DFT preserves the norms up to a scalar, the sum of the square norms of the $\tilde{d}_i$ would be less then the sum of d_i's, which is a contradiction. □

VI. Comparison with Stationary Sequences

Consideration of the orthogonalization and decomposition technique for the CS vector sequences was actually prompted by the existence of a theorem stating that any stationary vector sequence can be orthogonalized by the Fourier transform, which in turn is related to the fact that the Fourier transform of a wide sense stationary (WSS) random process is a white (uncorrelated) process.

Theorem 6—Orthogonalization of Stationary Stochastic Sequences: The Fourier transform of a continuous (discrete) WSS random process $x(t)$ ($x[n]$) is a white noise process $y(\omega)$, where ω assumes values from $-\infty$ to ∞ (from $-\pi$ to π in the discrete case), with power equal to the power spectral density (psd) of the process $x(t)$ ($x[n]$). The psd is the Fourier transform of the autocorrelation function of the process. (For proof see [1].)

From here, we obtain the following theorem regarding any stationary vector sequence:

Theorem 7—Stationary Vector Sequences Orthogonalization Theorem: The Fourier transform of a continuous (discrete) stationary vector sequence $\mathbf{x}(t)$ ($\mathbf{x}[n]$) is an orthogonal vector sequence

$$\mathbf{y}(\omega) = \int_{-\infty}^{\infty} \mathbf{x}(t) e^{-j\omega t} dt \tag{31}$$

or (for the discrete case)

$$\mathbf{y}(\omega) = \sum_{n=-\infty}^{\infty} \mathbf{x}[n] e^{-j\omega n} \tag{32}$$

where ω assumes values from $-\infty$ to ∞ (from $-\pi$ to π in the discrete case), with the square of its norm equal to the psd of the vector sequence $\mathbf{x}(t)$ ($\mathbf{x}[n]$):

$$||\mathbf{y}(\omega)||^2 = S_x(\omega). \tag{33}$$

The psd is the Fourier transform of the autocorrelation function of the vector sequence

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(t) e^{-j\omega t} dt \tag{34}$$

or in the discrete case

$$S_x(\omega) = \sum_{n=-\infty}^{\infty} R_x[n] e^{-j\omega n}. \tag{35}$$

We can also derive a decomposition theorem for stationary vector sequences, which we state here only for the discrete case.

Theorem 8—Decomposition of Stationary Vector Sequences: Given a stationary vector sequence $\mathbf{x}[n]$ in a Hilbert space $\mathcal{H}$ and a vector $\mathbf{r}$ whose projection $\mathbf{r}_{pr}$ on $span\{\mathbf{x}[n]\}_{n=-\infty}^{\infty}$ is to be decomposed on the sequence $\mathbf{x}[n]$, the coefficients c_n of the decomposition are found by projecting $\mathbf{r}$ first on the orthogonal sequence $\mathbf{y}(\omega)$ and then taking the Fourier transform, the result of which is

$$c_n = \int_{-\pi}^{\pi} \frac{1}{||\mathbf{y}(\omega)||^2} < \mathbf{r}, \mathbf{y}(\omega) > e^{-j\omega n} d\omega. \tag{36}$$

The integration in (36) is performed only over regions of the interval $[-\pi,\pi]$, where $||\mathbf{y}(\omega)|| \neq 0$. It is important to notice that in this case, vectors $\mathbf{y}$ may not be in the space $\mathcal{H}$, but they must be linear functions on $\mathcal{H}$. The proper definition of the norm, which is utilized in our examples, is a subject by itself and will not be treated here.

Example 1—Time Shift Sequence

Consider a functional sequence $g_n(t) = g(t-n\Delta t)$ as in Example 1 of Section II. The Fourier transform of the sequence is

$$h_\omega(t) = \sum_{n=-\infty}^{\infty} g(t-n\Delta t)e^{-j\omega n}. \quad (37)$$

When attempting to decompose a function $f(t)$ on the basis $g_n(t)$, it is desirable to decompose it first upon the sequence $h_\omega(t)$ and then take the Fourier transform.

Example 2—Frequency Shift Sequence

Consider the sequence of frequency shifts of a function as in Example 3 of Section II. Orthogonalization of the sequence $g_n(t) = g(t)e^{jn\Omega t}$ is accomplished by application of the Fourier transform to yield

$$\begin{aligned} h_\omega(t) &= \sum_{n=-\infty}^{\infty} g(t)e^{jn\Omega t}e^{-j\omega n} = g(t)\textstyle\sum_{n=-\infty}^{\infty} e^{jn(\Omega t-\omega)} \\ &= 2\pi g(t)\sum_{n=-\infty}^{\infty}\delta(\Omega t-\omega-2\pi n) \\ &= \frac{2\pi}{\Omega}g(t)\sum_{n=-\infty}^{\infty}\delta(t-\frac{\omega}{\Omega}-\frac{2\pi}{\Omega}n) \\ &= \frac{2\pi}{\Omega}\sum_{n=-\infty}^{\infty} g(\frac{\omega}{\Omega}+\frac{2\pi}{\Omega}n)\delta(t-\frac{\omega}{\Omega}-\frac{2\pi}{\Omega}n) \quad (38) \end{aligned}$$

where $\delta(t)$ denotes the Dirac function. One may verify that

$$||h_\omega(t)||^2 = \frac{4\pi^2\sum_{n=-\infty}^{\infty}|g(\frac{\omega}{\Omega}+\frac{2\pi}{\Omega}n)|^2}{\Omega^2}. \quad (39)$$

The coefficients of the decomposition of any function $f(t)$ on the basis $g_n(t)$ may therefore be obtained through the following formula:

$$c_n = \int_{-\pi}^{\pi}\frac{\Omega^2\sum_{l=-\infty}^{\infty} f(\frac{\omega}{\Omega}+\frac{2\pi}{\Omega}l)\bar{g}(\frac{\omega}{\Omega}+\frac{2\pi}{\Omega}l)}{4\pi^2\sum_{i=-\infty}^{\infty}|g(\frac{\omega}{\Omega}+\frac{2\pi}{\Omega}i)|^2}e^{-j\omega n}d\omega. \quad (40)$$

VII. Multidimensional Vector Sequences

A. Definitions

The theory presented above can easily be generalized to consideration of multi-dimensional vector sequences.

Definition 6: The continuous (discrete) R-dimensional vector field $\mathbf{x}(t_1,t_2,\cdots,t_R)$ ($\mathbf{x}[n_1,n_2,\cdots,n_R]$) is a mapping of the continuous (discrete) R-dimensional space $(t_1,t_2,\cdots,t_R)$ $(n_1,n_2,\cdots,n_R)$ into a Hilbert or Hermitian space. The function $R_x(t_1,t_2,\cdots,t_R;s_1,s_2,\cdots,s_R) = <\mathbf{x}(t_1,t_2,\cdots,t_R), \mathbf{x}(s_1,s_2,\cdots,s_R)>$ ($R_x[n_1,n_2,\cdots,n_R;m_1,m_2,\cdots,m_R] = <\mathbf{x}[n_1,n_2,\cdots,n_R],\mathbf{x}[m_1,m_2,\cdots,m_R]>$) is called the autocorrelation function of the vector sequence.

In the rest of this section, we shall restrict our attention to discrete multi-dimensional vector sequences.

Definition 7: The homogeneous (stationary) multidimensional vector sequence is the vector sequence whose autocorrelation function depends only on the difference between the corresponding elements of the first and the second arguments:

$$\begin{aligned} &R[n_1,n_2,\cdots,n_R;m_1,m_2,\cdots,m_R] \\ &\quad = R[n_1-m_1,n_2-m_2,\cdots,n_R-m_R;0,0,\cdots,0]. \end{aligned}$$

Definition 8: The multidimensional set of $N_1\times N_2\times\cdots\times N_R$ vectors $\{\mathbf{x}[n_1,n_2,\cdots,n_R]\}_{n_1=0,n_2=0,\cdots,n_R=0}^{N_1,N_2,\cdots,N_R}$ is called a CS sequence, with period N_i in the ith dimension, if its periodic extension in each of the dimensions generates a stationary sequence. By periodic extension of an $N_1\times N_2\times\cdots\times N_R$ sequence, $\{\mathbf{x}[n_1,n_2,\cdots,n_R]\}_{n_1=0,n_2=0,\cdots,n_R=0}^{N_1,N_2,\cdots,N_R}$ is meant to be a sequence $\{\mathbf{p}[n_1,n_2,\cdots,n_R]\}_{n_1=-\infty,n_2=-\infty,\cdots,n_R=-\infty}^{\infty,\infty,\cdots,\infty}$ of infinite extent in each of the dimensions, with the property that $\mathbf{p}[n_1+a_1N_1,n_2+a_2N_2,\cdots,n_R+a_RN_R] = \mathbf{x}[n_1,n_2,\cdots,n_R]$ for any integers $a_1,a_2,\cdots,a_R$, where, for $0\le n_i<N_i$, $i=1,2,\cdots,R$.

The orthogonalization and decomposition theorems can be applied to multidimensional stationary and CS vector sequences by substitution of multidimensional Fourier transforms for one dimensional ones. In addition, the biorthogonal basis can be obtained in exactly the same way by means of multidimensional Fourier transforms.

B. Group Theory Approach to Multidimensional Circular Stationarity

The properties of multidimensional circular stationary sequences can be readily expressed through group theory. For this purpose, we begin with a basic theorem from Abelian group theory.

Theorem 9: Given a finite Abelian (commutative) group G, we can always find a set of elements of the group $\{P_1,P_2,\cdots,P_R\}$, such that the groups established by the elements of the set are cyclic, with the length of the cycle corresponding to P_i equal to N_i, and every element A of G can be represented as $P_1^{a_1}P_2^{a_R}$, where $a_1,a_2,\cdots,a_R$ are integers. The set $\{P_1,P_2\cdots P_R\}$ is called the basis set of the group. The number R is called the dimension of the group. The basis and the dimension of the group may not be unique.

We state the most pertinent theorem of this subsection.

Theorem 10: An R-dimensional set $\{\mathbf{x}[n_1,n_2,\cdots,n_R]\}_{n_1=0,n_2=0,\cdots,n_R=0}^{N_1,N_2,\cdots,N_R}$ is CS iff there exists a finite Abelian (commutative) group G of linear isometries of a Hermitian (Hilbert) space onto itself, with basis $\{P_1,P_2,\cdots,P_R\}$, with the length of the cycle corresponding to P_i being equal to N_i, such that

$$\begin{aligned} &\mathbf{x}[n_1 \bmod N_1, n_2 \bmod N_2,\cdots,n_R \bmod N_R] \\ &\quad = P_1^{n_1}P_2^{n_2}\cdots P_R^{n_R}\mathbf{x}[0,0,\cdots,0]. \quad (41) \end{aligned}$$

We call $\{P_1,P_2,\cdots,P_R\}$ the set of isometries of the CS sequence $\{\mathbf{x}[n_1,n_2,\cdots,n_R]\}_{n_1=0,n_2=0,\cdots,n_R=0}^{N_1,N_2,\cdots,N_R}$. The proof of the theorem is very similar to that of the analogous theorem for 1-D CS sequences. The theory for the multidimensional case is very similar to that for the 1-D one.

The following theorem is helpful for further considerations:

Theorem 11: The set biorthogonal to a multidimensional CS set is also multidimensional CS and has the same set of isometries.

C. Orthogonalization and Decomposition of a Sequence of Discrete Gabor Functions

Here, we apply results obtained in previous sections to the discrete Gabor decomposition of functions (signals). Gabor decomposition is the representation of a function $f(t) \in \mathcal{L}_2$ by a linear combination of time (or position) and frequency shifts of another function, which is called the kernel of the transform, i.e., the problem is to find a function

$$\hat{f}(t) = \sum_{n,m=-\infty}^{\infty} c_{n,m} g(t - n\Delta t) e^{j\omega_0 m} \tag{42}$$

where $\hat{f}$ is the best approximation of the function $f(t)$ in the mean-squared sense. Here, Δt and ω_0 are time and frequency shifts. The Gabor expansion was first introduced in [2] and has been applied to signal and image decomposition (see [3], [13], and [14]). Gabor decomposition of discrete signals was considered in [8].

Here, we consider the Gabor decomposition for discrete periodic signals. The same problem was also considered in [10]–[12].

Consider a linear space $\mathcal{F}_L$ of discrete L-periodic functions (functions with period L). One can introduce inner product on this space in the following way: For two functions $f, g \in \mathcal{F}_L$, take

$$< f, g >= \sum_{i=0}^{L-1} f[i + i_0]\bar{g}[i + i_0] \tag{43}$$

where $\bar{g}$ denotes the complex conjugate of g, and i_0, which is the starting point of the summation, is some integer. One can see that the inner product defined as above does not depend on i_0. The functions from $\mathcal{F}_L$ can be obtained in the following two ways: Either one can take a discrete square integrable function f and consider the sum of its L-shifts:

$$\sum_{n=-\infty}^{\infty} f(k - Ln) \tag{44}$$

or one can let such a function assume any values for $k = i_0, i_0 + 1, i_0 + L - 1$ and extend it periodically for all L.

Assuming N and M are some divisors of L, let us define two unitary operators on $\mathcal{F}_L$ of time shift and frequency shift T_M and E_N for $f \in \mathcal{F}_L$ in the following way:

$$T_M(f)[k] = f[k - L/M] \tag{45}$$

and

$$E_N(f)[k] = e^{\frac{2\pi}{N}k} f[k]. \tag{46}$$

Definition 9: The periodic $M \times N$ Gabor set for the space $\mathcal{F}_L$ is the set of functions $\{g_{n,m}\}_{n=0,m=0}^{M-1,N-1}$ such that

$$g_{n,m}[k] = (E_N)^m (T_M)^n (g)[k] = e^{\frac{2\pi}{N}km} g[k - nL/M]. \tag{47}$$

The function $g \in \mathcal{F}_L$ is called the kernel of the transform.

The strong point of the methods for the Gabor decomposition presented in this paper is that all of them (including the method considered below) work in case of undersampling (the Gabor functional set $\{g_{n,m}\}_{n=0,m=0}^{M-1,N-1}$ does not span the whole space $\mathcal{F}_L$) and in the case of oversampling (the functional set is linearly dependent) since all of the methods are based on the method described in the decomposition of the CS vector sequences theorem or the biorthogonal functions approach.

One can check that $(T_M)^M = (E_N)^N = I$, where I is the identity operator on the space. We will show that the two operators commute if $M = \frac{L}{Np}$, where p is an integer:

$$\begin{aligned} T_M E_N(f)[k] &= e^{\frac{2\pi}{N}(k-L/M)} f[k - L/M] \\ &= e^{\frac{2\pi}{N}(k-Np)} f[k - Np] = e^{\frac{2\pi}{N}k} f[k - Np] \\ &= E_N T_M(f)[k]. \end{aligned} \tag{48}$$

The conditions of Theorem 10 are hence satisfied, and the Gabor set $\{g_{n,m}\}_{n=0,m=0}^{M-1,N-1}$ is 2-D circular stationary when $M = \frac{L}{Np}$. Therefore, it can be orthogonalized by the 2-D Fourier transform.

Let us find the orthogonalization:

$$\begin{aligned} h_{s,t}[k] &= \sum_{n=0}^{M-1}\sum_{m=0}^{N-1} g[k - Npn] e^{j\frac{2\pi}{N}mk} e^{-j\frac{2\pi}{M}ns} e^{-j\frac{2\pi}{N}mt} \\ &= \sum_{n=0}^{M-1} g[k - Npn] e^{-j\frac{2\pi}{M}ns} \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}m(k-t)} \\ &= N \sum_{n=0}^{M-1} g[k - Npn] e^{-j\frac{2\pi}{M}ns} N \sum_{l=-\infty}^{\infty} \delta[k - t - Nl] \\ &= N \sum_{l=-\infty}^{\infty} \sum_{n=0}^{M-1} g[t + Nl - Npn] e^{-j\frac{2\pi}{M}ns} \delta[k - t - Nl] \end{aligned} \tag{49}$$

where s and t are integers; $0 \le s < M$, and $0 \le t < N$.

Now, using the analog of (8), with the sequence $h_{s,t}[k]$ in place of $\mathbf{y}_i$, $f[k]$ instead of $\mathbf{r}$, and $g_{n,m}[k]$ instead of $\mathbf{x}_i$, we can find the coefficients $c_{n,m}$ of the decomposition of an arbitrary function $f[k] \in F_L$ on the Gabor basis $g_{n,m}[k]$:

$$\begin{aligned} c_{n,m} &= \sum_{s,t;||h_{s,t}||\neq 0} \frac{1}{||h_{s,t}||^2} < f, h_{s,t} > e^{-j\frac{2\pi}{M}sn} e^{-j\frac{2\pi}{N}tm} \\ &= \sum_{s,t;||h_{s,t}||\neq 0} \frac{1}{||h_{s,t}||^2} \Big(\sum_{l=0}^{Mp-1} \Big(\sum_{i=0}^{M-1} \\ &\quad \cdot g[t + Nl - Npi] e^{-j\frac{2\pi}{M}is} \Big) f[t + Nl] \Big) e^{-j\frac{2\pi}{M}sn} e^{-j\frac{2\pi}{N}tm}. \end{aligned} \tag{50}$$

Notice that $||h_{s,t}||^2$ and $h_{s,t}[k]$ can be calculated offline and stored in memory beforehand.

Once the coefficients of the Gabor decomposition have been obtained, we can apply the fact that the $h_{s,t}[k]$ functions are nonzero only on $\frac{L}{N}$ samples of one period and zero on the rest. This offers a method of fast signal reconstruction from the Gabor coefficients. From (50), it is seen that the set $\frac{1}{||h_{s,t}||^2} < f, h_{s,t} >$ can be obtained from the set of the Gabor coefficients $c_{n,m}$ by taking the inverse Fourier transform:

$$\frac{1}{||h_{s,t}||^2} < f, h_{s,t} >= NM \sum_{n,m} c_{n,m} e^{j\frac{2\pi}{M}sn} e^{j\frac{2\pi}{N}tm}. \quad (51)$$

Since the functions $h_{s,t}[k]$ form an orthogonal set of functions, we can write

$$f_{pr}[k] = \sum_{s,t;||h_{s,t}||\neq 0} \frac{1}{||h_{s,t}||^2} < f, h_{s,t} > h_{s,t}[k] \quad (52)$$

where $f_{pr}[k]$ is the projection of the signal on the span of the Gabor functions $span(\{g_{n,m}\}_{n=0,m=0}^{M,N})$. From (51) and (52), we obtain

$$f_{pr}[k] = NM \sum_{s,t} (\sum_{n,m} c_{n,m} e^{j\frac{2\pi}{M}sn} e^{j\frac{2\pi}{N}tm}) h_{s,t}[k]. \quad (53)$$

D. The Biorthogonality Approach

Using tools developed in Section IV, one can find a basis $\{\tilde{g}_{n,m}\}_{n=0,m=0}^{M-1,N-1}$ that is biorthogonal to $\{g_{n,m}\}_{n=0,m=0}^{M-1,N-1}$. Using (24), which is modified for the 2-D case, we have

$$\tilde{g}_{n,m}[k] = \sum_{s=0,t=0,||h_{s,t}||\neq 0}^{M-1,N-1} \frac{h_{s,t}[k]}{||h_{s,t}||^2} e^{j\frac{2\pi}{M}sn} e^{j\frac{2\pi}{M}tm}. \quad (54)$$

Observing that the original and biorthogonal sequences have the same set of linear isometries, we find

$$\tilde{g}_{0,0}[k] = \sum_{s=0,t=0,||h_{s,t}||\neq 0}^{M-1,N-1} \frac{h_{s,t}[k]}{||h_{s,t}||^2}. \quad (55)$$

All the other functions of the biorthogonal set can be obtained by applying linear isometries of the original set $\{g_{n,m}\}_{n=0,m=0}^{M-1,N-1}$ shifted along time and frequency axes, i.e.

$$\tilde{g}_{n,m}[k] = \tilde{g}_{0,0}[k - Npn] e^{j\frac{2\pi}{M}mk}. \quad (56)$$

The biorthogonal function method also works in the cases of oversampling and undersampling.

The biorthogonal functions decomposition overall is slower than the method described in the previous section (h-functions method) and the one of the next section (the Zak transform method) when using serial computation by one processor, but it is faster when using parallel computation. Once the biorthogonal functions are computed and stored in memory, the process of taking the inner products with them can be parallelized and does not require Fourier transforms, as do other methods considered in this paper.

VIII. Restatement by Zak Transform and Comparisons of Methods

The Zak transform was successfully used by Janssen as a tool for the exploration of continuous-time signal Gabor decomposition [9]. Auslander *et al.* [10] and Zeevi and Gertner [11] developed an algorithm for calculation of coefficients of the Gabor decomposition based on the discrete Zak transform. Gertner and Zeevi [12] and Zeevi and Gertner [11] modified this algorithm for the decomposition of images.

Definition 10: Let $g[k]$ be zero outside the interval $[0, L-1]$ or L-periodic, where $L = NM$. The discrete $N \times M$ Zak transform of the function $g[k]$ is

$$Z(g)(\frac{n}{N}, \frac{m}{M}) = \sum_{i=0}^{N-1} g[m + iM] e^{-j\frac{2\pi}{N}in}, \quad 0 \leq n < N, 0 \leq m < M. \quad (57)$$

Our method can be expressed in terms of the Zak transform. Starting from expression (49) for $h_{s,t}[k]$, rewriting the inner sum, and using representation $l = pq + r$, where r is the remainder after division of l by p, yields

$$\begin{aligned} &\sum_{n=0}^{M-1} g[t + Nl - Npn] e^{-j\frac{2\pi}{M}ns} \\ &= \sum_{n=0}^{M-1} g[t + Npq + Nr - Npn] e^{-j\frac{2\pi}{M}ns} \\ &= \sum_{n=0}^{M-1} g[t + Nr + Np(q - n)] e^{-j\frac{2\pi}{M}ns} \\ &= \sum_{n=0}^{M-1} g[t + Nr + Np(q - n)] \\ &\quad \cdot e^{j\frac{2\pi}{M}(q-n)s} e^{-j\frac{2\pi}{M}qs} \\ &= e^{-j\frac{2\pi}{M}qs} \sum_{i=0}^{M-1} g[t + Nr + Npi)] e^{j\frac{2\pi}{M}is} \\ &= e^{-j\frac{2\pi}{M}qs} Z(g)(-\frac{s}{M}, \frac{t + Nr}{Np}). \end{aligned} \quad (58)$$

Substituting the above expression into (49) after some algebraic manipulations yields

$$h_{s,t}[k] = N \sum_{r=0}^{p-1} Z(g)(-\frac{s}{M}, \frac{t + Nr}{Np}) \cdot \sum_{q=-\infty}^{\infty} e^{-j\frac{2\pi}{M}qs} \delta[k - t - Nqp - Nr]. \quad (59)$$

Using (59), we can find $< f, h_{s,t} >$ for any L-periodic function $f[k]$:

$$\begin{aligned} < f, h_{s,t} > &= \sum_{k=0}^{L-1} f[k] \bar{h}_{s,t}[k] \\ &= N \sum_{k=0}^{L-1} \sum_{r=0}^{p-1} \bar{Z}(g)(-\frac{s}{M}, \frac{t + Nr}{Np}) \\ &\quad \cdot \sum_{q=-\infty}^{\infty} e^{j\frac{2\pi}{M}qs} \delta[k - t - Nqp - Nr] f[k] \\ &= N \sum_{r=0}^{p-1} \bar{Z}(g)(-\frac{s}{M}, \frac{t + Nr}{Np}) \\ &\quad \cdot \sum_{q=0}^{M-1} e^{j\frac{2\pi}{M}qs} f[t + Nr + Nqp] \end{aligned}$$

$$= N\sum_{r=0}^{p-1} Z(f)(-\frac{s}{M},\frac{t+Nr}{Np}) \cdot \bar{Z}(g)(-\frac{s}{M},\frac{t+Nr}{Np}) \quad (60)$$

and

$$\|h_{s,t}\|^2 =< h_{s,t},h_{s,t} >= MN^2\sum_{r=0}^{p-1}|Z(g)(-\frac{s}{M},\frac{t+Nr}{Np})|^2. \quad (61)$$

Substitution of (60) and (61) into (50) yields

$$c_{n,m} = \sum_{s,t:\sum_{r=0}^{p-1}|Z(g)(-\frac{s}{M},\frac{t+Nr}{Np})|^2\neq 0} \frac{1}{MN\sum_{r=0}^{p-1}|Z(g)(-\frac{s}{M},\frac{t+Nr}{Np})|^2} \cdot [\sum_{r=0}^{p-1} Z(f)(-\frac{s}{M},\frac{t+Nr}{Np}) \cdot \bar{Z}(g)(-\frac{s}{M},\frac{t+Nr}{Np})]e^{-j\frac{2\pi}{M}sn}e^{-j\frac{2\pi}{N}tm}. \quad (62)$$

The coefficients of the decomposition are expressed according to (62) in terms of the Zak transforms of the decomposed function f and the kernel of the Gabor basis. Let us consider a special case when $p = 1$ and, correspondingly, $MN = L$ (the number of samples in the signal equals the number of basis functions). The orthogonal functions are

$$h_{s,t}[k] = NZ(g)(-\frac{s}{M},\frac{t}{N})\sum_{q=-\infty}^{\infty} e^{-j\frac{2\pi}{M}qs}\delta[k-t-Nq] \quad (63)$$

with

$$\|h_{s,t}\|^2 = MN^2|Z(g)(-\frac{s}{M},\frac{t}{N})|^2. \quad (64)$$

Note that in the case $p = 1$, a zero in the Zak domain means that the corresponding $h_{s,t}$ is zero, and

$$< f,h_{s,t} >= Z(f)(-\frac{s}{M},\frac{t}{N})\bar{Z}(g)(-\frac{s}{M},\frac{t}{N}). \quad (65)$$

We obtain then the following formula for the coefficients:

$$c_{n,m} = \sum_{s,t;|Z(g)(-\frac{s}{M},\frac{t}{N})|\neq 0} \frac{Z(f)(-\frac{s}{M},\frac{t}{N})}{MNZ(g)(-\frac{s}{M},\frac{t}{N})}e^{-j\frac{2\pi}{M}sn}e^{-j\frac{2\pi}{N}tm}. \quad (66)$$

Up to some notational differences, the above formula is the same as the one of the Auslander *et al.* method (for the case when there are no zeros in the Zak domain). Our method offers, however, the following advantages:

The discrete symmetric functions always have a zero in the middle of the Zak plane [12], [17]. To circumvent this problem, Gertner and Zeevi [12], and Khaled *et al.* [17] have translated the Gaussian by a subpixel distance. They showed that for maximum stability, the Gaussian has to be shifted by half a pixel. Further, they took the inner products of the functions from the Gabor set and the decomposed function in the time domain, which prevented them from using the extra advantages of the FFT. By comparison, our method uses the h functions. These functions are nonzero only every Nth sample, and therefore, taking the inner product with them is faster. The corresponding Zak transform approach would be even faster since it would require just one inner product in the Zak transform domain instead of $N \times M$ in case of the time domain or the h functions domain. In addition, because they operate in the time domain, their approach required two FFT's (as in (11)) as opposed to one FFT as in our case.

Moreover, when $p > 1$, our method (which is no longer equivalent to the discrete Zak transform method) gives the projection of the signal on the span of the Gabor set (the best mean-squared approximation for this basis), whereas the discrete finite Zak transform method yields an approximation that is not a projection and, hence, is not as close as ours.

Our algorithm using biorthogonal functions complements the one developed by Wexler and Raz [8]. They considered only the oversampling and critical sampling cases, whereas we developed an algorithm using biorthogonal functions for the case of undersampling as well. Further, using the approach developed by us one does not encounter the problem of inverting possibly noninvertible matrices in order to obtain the biorthogonal basis. Through our method, we are able to obtain the basis that yields the Gabor coefficients producing the best approximation of the decomposed function. In this way, one can use the method for the Gabor bases with symmetric elementary functions.

For large enough periods, in case of the sequential processing, the Zak transform approach is definitely the best among the h functions and the biorthogonal functions since it requires only one inner product (in the Zak domain), one Zak transform of the signal, and one FFT, provided that the Zak transform of the kernel and the divisors for (62) have been precomputed and stored in memory. In the case of parallel processing, the biorthogonal functions approach would have an advantage as shown in Section VII-D.

IX. Conclusion

In this paper, we utilized the theory of CS vector sequences and applied it to the Gabor decomposition of signals in the case of undersampling. Zibulski and Zeevi in [15] and [16] considered the oversampling case of the Gabor decomposition when the number of functions in the set is greater then the number of samples in the signal. Our approach can also be applied to that problem, although in the oversampling case, the Gabor set is not CS. The approach considered herein leads to results that are more general in their applicability than those of Zibulski and Zeevi since there is no requirement that the Gabor basis must constitute a frame.

The theory can be easily extended and applied to the Gabor decomposition of images, in which case, the Gabor functions are 2-D and the corresponding FFT is 4-D. Various other fast decomposition algorithms can be developed, by which we can decompose a signal (image) on time (space) or frequency shifts of any kernel function. The decomposition of a signal on time shifts of a kernel corresponds to $N = 1$, whereas the decomposition of a signal on frequency shifts of a kernel corresponds to $M = 1$ in our Gabor decomposition algorithm. The same approach can be applied to the development of multiresolution algorithms. Such decomposition algorithms may be quite useful in image and signal processing and compression since they have the characteristic of separating coefficients bearing little information from those bearing a lot of information. Moreover, the decomposition schemes can emulate, to some degree, the functions of the human eye and ear and, consequently, find applications in image and speech processing.

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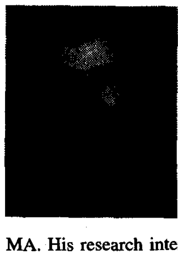

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