Correspondence

Frame Analysis of the Discrete Gabor-Scheme

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***Abstract*—The properties of the discrete Gabor scheme are considered in the context of oversampling. The approach is based on the concept of frames and utilizes the piecewise finite Zak transform (PFZT). The frame operator is represented as a matrix-valued function in the PFZT domain, and its properties are examined in relation to this function. The frame bounds are calculated by means of the eigenvalues of the matrix-valued function, and the dual frame, which is used in calculation of the expansion coefficients, is expressed by means of the inverse matrix. DFT-based algorithms for computation of the expansion coefficients, and for the reconstruction of signals from these coefficients, are generalized for the case of oversampling of the Gabor space. The algorithms are implemented in an example of representation of a nonstationary signal.**

I. Introduction

Whereas the optimal discrete Gabor-scheme, i.e., critical sampling (of the combined space), has been thoroughly analyzed [1], [2], undersampling and oversampling are less understood. We address the issue of oversampling of the discrete Gabor space by combining the finite Zak transform (FZT) with the concept of frames. Our analysis is based on the analysis of the continuous-time Gabor scheme presented in [3], where the relation between the two schemes is derived as well.

Consider L-periodic discrete signals, that is, signals that satisfy $f(i+L)=f(i)$, $i \in \boldsymbol{Z}$. For such signals, given two divisors M, N of L, the discrete (finite) Gabor scheme is [4]

$$f(i)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} c_{m,n}g_{m,n}(i) \tag{1}$$

where for the window function $g(i)$, we define

$$g_{m,n}(i)=g(i-mN')\exp\left(2\pi\imath\frac{in}{N}\right),\ L=N'M. \tag{2}$$

The following three categories of the discrete Gabor (combined) space sampling are identified. *Undersampling–*$NM < L$: The number of representation functions is smaller than the length of the signal. *Critical sampling–*$NM = L$: The number of representation functions is equal to the length of the signal. *Oversampling–*$NM > L$: The number of representation functions is larger than the length of the signal.

II. Preliminaries

A. Frames [5]

Definition 1: A sequence $\{\psi_n\}$ in a Hilbert Space H constitutes a frame if there exist numbers $0 < A \le B < \infty$ such that for all $f \in H$, we have $A\|f\|^2 \le \sum_n |\langle f, \psi_n\rangle|^2 \le B\|f\|^2$, where $\langle\cdot,\cdot\rangle$ denotes the inner product corresponding to the Hilbert space H.

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Definition 2: Given a frame $\{\psi_n\}$ in a Hilbert space H, the frame operator $\mathcal{S}$ is defined by $\mathcal{S}f \triangleq \sum_n\langle f, \psi_n\rangle\psi_n$.

Corollary 1:

1. $\{\mathcal{S}^{-1}\psi_n\}$ is a frame with bounds B^{-1}, A^{-1}, which is called the dual frame of $\{\psi_n\}$.
2. Every $f \in H$ can be represented by either the frame or the dual frame in the following manner: $f=\sum_n\langle f, \mathcal{S}^{-1}\psi_n\rangle\psi_n = \sum_n\langle f, \psi_n\rangle\mathcal{S}^{-1}\psi_n$.

Note that in general, the reconstruction coefficients $\langle f, \mathcal{S}^{-1}\psi_n\rangle$ are not unique (unless the frame is a basis). The choice of the dual frame for computing the reconstruction coefficients yields the minimal solution in l^2 sense [5]. For a finite-dimensional Hilbert space (as is our case), this solution corresponds to the so-called pseudoinverse.

We address the problem of finding conditions for the sequence $\{g_{m,n}\}$ to constitute a frame by examining the frame operator associated with $\{g_{m,n}\}$.

B. The Finite Zak Transform

The Zak transform (ZT) is a fundamental tool in the analysis of the Gabor expansion. We utilize the discrete finite version of the transform as introduced by Auslander *et al.* [6] for 1-D signals and by Zeevi and Gertner [2] for images. Since a detailed presentation of the FZT is available in [2], we only review its important relevant properties and introduce the piecewise finite Zak transform (PFZT).

The DFT-based FZT of an L-periodic 1-D signal is defined by

$$(Zf)(i,v)=\sum_{l=0}^{M'-1} f(i+Nl)\exp\left(-2\pi\imath\frac{lv}{M'}\right),\quad (i,v)\in\boldsymbol{Z}^2 \tag{3}$$

where $L=M'N$, and its inverse is given by

$$f(i)=\frac{1}{M'}\sum_{v=0}^{M'-1}(Zf)(i,v),\qquad i\in\boldsymbol{Z}. \tag{4}$$

The FZT satisfies the following periodic and quasiperiodic properties [2]:

$$\begin{aligned}(Zf)(i,v+M')&=(Zf)(i,v),\\ (Zf)(i+N,v)&=\exp\left(2\pi\imath\frac{v}{M'}\right)(Zf)(i,v).\end{aligned} \tag{5}$$

Denote by $l^2(Z/L)$ the Hilbert space of L–periodic, square summable, 1-D signals with the following inner-product

$$\langle f, g\rangle=\sum_{i=0}^{L-1} f(i)\overline{g(i)}$$

where $f, g \in l^2(Z/L)$. In this context, the FZT (3) defines a unitary mapping of $l^2(Z/L)$ onto $l^2(N\times M')$. The latter is a Hilbert space of square summable 2-D functions with the inner produce

$$\langle Zf, Zg\rangle=\frac{1}{M'}\sum_{i=0}^{N-1}\sum_{v=0}^{M'-1}(Zf)(i,v)\overline{(Zg)(i,v)}.$$

As a consequence, we obtain the inner product preserving property $\langle f, g\rangle = \langle Zf, Zg\rangle$.

The ZT is useful in the analysis of the critical sampling case. For the case of oversampling, the piecewise Zak transform (PZT) introduced in [3] generalizes the role of the ZT in the analysis of the Gabor scheme. In this case of oversampling $L = MN' = NM'$. Let $L/(MN) = p/q$, where p, q are relatively prime integers, then $M'/p = M/q$ is an integer. Based on the definition of the FZT (3), we define the PFZT as a vector-valued function of size p:

$$F(i, v) = [F_0(i, v), \cdots, F_{p-1}(i, v)]^T \tag{6}$$

where

$$F_r(i, v) \triangleq (Zf)\left(i, v + r\,\frac{M'}{p}\right), \qquad 0 \le r \le p-1. \tag{7}$$

The vector-valued function F belongs to a Hilbert space of vector-valued functions with the inner product

$$\langle F, G\rangle = \frac{1}{M'} \sum_{i=0}^{N-1} \sum_{v=0}^{(M'/p)-1} \sum_{r=0}^{p-1} F_r(i, v) \le \overline{G_r(i, v)}.$$

Here, we also obtain the inner-product preserving property $\langle f, g\rangle = \langle Zf, Zg\rangle = \langle F, G\rangle$.

III. Analysis of the Discrete Gabor Scheme

A. Discrete Gabor Frames

In order to characterize the frame properties of the sequence $\{g_{m,n}\}$, we examine the operator:

$$\mathcal{S}f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle f, g_{m,n}\rangle g_{m,n}. \tag{8}$$

If the sequence $\{g_{m,n}\}$ constitutes a frame, this is clearly the frame operator. The action of the frame operator in the PFZT domain can be expressed in terms of matrix algebra as [3]

$$(\mathcal{S}F)(i, v) = \boldsymbol{S}(i, v)F(i, v) \tag{9}$$

where $\boldsymbol{S}(i, v)$ is a $p \times p$ matrix-valued function with $\mathcal{S}_{k,j}(i, v)$ elements given by

$$\mathcal{S}_{k,j}(i, v) = \frac{N}{p} \sum_{r=0}^{q-1} (Zg)(i - rN', v + kM'/p) \cdot \overline{(Zg)(i - rN', v + jM'/p)} \tag{10}$$

and F is a vector-valued function of size p, which is defined by (6). The matrix $\boldsymbol{S}(i, v)$ is self-adjoint and positive semi-definite for each (i, v), since by defining a $q \times p$ matrix-valued function $\boldsymbol{G}(i, v)$ with elements $\boldsymbol{G}_{r,j}(i, v) = \overline{(Zg)(i - rN', v + jM'/p)}$, we obtain $\boldsymbol{S}(i, v) = (N/p)\boldsymbol{G}^*(i, v)\boldsymbol{G}(i, v)$, where * denotes the conjugate transpose.

Considering the frame property of the sequence $\{g_{m,n}\}$ in the cases of critical sampling and oversampling, based on representation (9) of the frame operator and on the unitary property of the PFZT, Theorem 1 follows:

Theorem 1: Given $g \in l^2(Z/L), MN \ge L$ and a matrix-valued function $\boldsymbol{S}(i, v)$ as in (10), the sequence $\{g_{m,n}\}$ associated with g constitutes a frame if and only if $\det(\boldsymbol{S})(i, v) \ne 0$ for all $(i, v) \in \boldsymbol{Z}^2$.

Note that (i, v) can be restricted to a rectangle of size $N \times M'/p$ in $\boldsymbol{Z}^2$. We also note that for finite-dimensional spaces, as in this case, frame property and completeness are identical. Therefore, if the sequence $\{g_{m,n}\}$ is not a frame, it is impossible to expand any desired signal by means of $\{g_{m,n}\}$.

In the case of critical sampling, i.e., for $L = MN$, $\boldsymbol{S}$ is a scalar-valued function (a 1×1 matrix), $\boldsymbol{S} = N|Zg|^2$, in which case, $\det(\boldsymbol{S}) = N|Zg|^2$. Therefore, the sequence $\{g_{m,n}\}$ constitutes a basis (a frame is equivalent to a basis in this case) if and only if $Zg(i, v)$ does not vanish. Theorem 1 generalizes this known fact to the case of oversampling. Note that, in general, $\boldsymbol{S}$ is scalar-valued if $L/(MN) = 1/q$, $q \in \boldsymbol{N}$, in which case, we obtain $\boldsymbol{S}(i, v) = N \sum_{r=0}^{q-1} |(Zg)(i - rN', v)|^2$.

B. Stability, Frame Bounds, and Tight Frames

Utilizing the frame bounds A, B, the ratio B/A, which is the so-called condition number, expresses the stability of the representation. Maximum stability is achieved if $B/A = 1$. Based on representation (9) of the frame operator and on the unitary property of the PFZT, the frame bounds can be derived by calculating the eigenvalues of the matrix-valued function $\boldsymbol{S}$, where a total number of $N(M'/p)$ matrices should be examined:

$$A = \min_{(i,v)\in Z^2, 1\le j\le p} \lambda_j(\boldsymbol{S})(i, v),$$
$$B = \max_{(i,v)\in Z^2, 1\le j\le p} \lambda_j(\boldsymbol{S})(i, v) \tag{11}$$

where $\lambda_j(\boldsymbol{S})(i, v)$ are the eigenvalues of the matrix $\boldsymbol{S}(i, v)$, and we can again restrict (i, v) to a rectangle of size $N \times M'/p$ in $\boldsymbol{Z}^2$. Note that in the case of critical sampling (and oversampling with an integer ratio), the matrix $\boldsymbol{S}$ is a scalar-valued function. Therefore, in order to find the frame bounds, the minimum and maximum values of the function itself should be examined.

Special types of frames for which $A = B$ are called tight frames. The frame operator for tight frames is $\mathcal{S} = A\mathcal{I}$, where $\mathcal{I}$ is the identity operator. One of the advantages of having a tight frame $\{\psi_n\}$ in H is the simple reconstruction formula associated with it: $f = A^{-1} \sum_n \langle f, \psi_n\rangle \psi_n$ for $f \in H$. For the Gabor scheme, the condition for tight frames in the PFZT domain is $\boldsymbol{S}(i, v) = A\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix, and $A = \|g\|^2(MN/L)$. An interesting fact is that for the maximal oversampling rate, i.e., for $M = N = L$, the sequence $\{g_{m,n}\}$ is always a tight frame for any window function with $A = L\|g\|^2$ (see the Appendix). This property is called resolution of identity.

C. Expansion Coefficients and the Dual Frame

Assuming $\{g_{m,n}\}$ constitutes a frame, according to Corollary 1, the expansion coefficients $c_{m,n}$, as in (1), can be calculated utilizing the dual frame. The dual frame $\{\gamma_{m,n}\}$ has the same structure as the frame $\{g_{m,n}\}$ [3], i.e., it is generated by a single window function $\gamma(i)$

$$\gamma_{m,n}(i) = \gamma(i - mN') \exp\left(2\pi\imath\,\frac{in}{N}\right). \tag{12}$$

The function γ can be found by using the inverse of the frame operator: $\gamma = \mathcal{S}^{-1}g$. Utilizing the matrix approach, the dual frame window can be expressed by the inverse of the matrix $\boldsymbol{S}$:

$$\Gamma(i, v) = \boldsymbol{S}^{-1}(i, v)G(i, v) \tag{13}$$

where Γ, G are the PFZT of γ, g. Generally, $L/(MN) = p/q$, $p, q \in \boldsymbol{N}$. If $p = 1$, we obtain

$$(Z\Gamma)(i, v) = \frac{(Zg)(i, v)}{N \sum_{r=0}^{q-1} |(Zg)(i - rN', v)|^2}. \tag{14}$$

Moreover, an explicit solution in the ZT domain can be found for any given p [7]. Note that (13) can be used in order to prove that the dual frame is generated by a single window function.

We can find the expansion coefficients by calculating the inner products of the signal and the dual frame: $c_{m,n} = \langle f, \gamma_{m,n} \rangle$. This calculation can be done in either the signal or the transform space [2]. In the case of oversampling, the expansion coefficients are not unique. As was pointed out in [4], these coefficients can be calculated by projecting the signal on sequences that are generated by different dual window functions. The dual window that corresponds to the unique dual frame provides the coefficients that are of minimal norm in l^2 sense.

For the case of critical sampling, as was shown in [6] and [2], the FZT lends itself to efficient algorithms for computation of the expansion coefficients and for reconstruction of the signal from these coefficients. For the case of oversampling, the PFZT generalizes these algorithms. Since $(Z\gamma_{m,n})(i, v) = (Z\gamma)(i - mN', v) \exp(2\pi\imath(in/N))$, for the expansion coefficients in the FZT domain we obtain

$$c_{m,n} = \frac{1}{M'} \sum_{i=0}^{N-1} \sum_{v=0}^{M'-1} (Zf)(i, v) \overline{(Z\gamma)(i - mN', v) \exp\left(2\pi\imath \frac{in}{N}\right)}. \qquad (15)$$

Utilizing the PFZT domain, for $m = m'q+r$, $0 \le q-1$, $0 \le m' \le (M'/p) - 1$, $(M'/p = M/q)$, by reordering $v = v' + jM'/p$, $0 \le v' \le M'/p - 1$, $0 \le j \le p - 1$, (15) can be rewritten as

$$c_{m'q+r,n} = \frac{1}{M'} \sum_{i=0}^{N-1} \sum_{v=0}^{(M'/p)-1} \exp\left(-2\pi\imath \frac{in}{N}\right) \cdot \exp\left(2\pi\imath \frac{vm'}{M'/p}\right) \sum_{j=0}^{p-1} F_j(i, v) \overline{\Gamma_j(i - rN', v)} \qquad (16)$$

which yields the following DFT-based algorithm for calculating the expansion coefficients:

1. Precompute the FZT of the dual frame window function $(Z\gamma)(i, v)$.
2. Compute the FZT of the signal $(Zf)(i, v)$.
3. Compute q times the sum: $\sum_{j=0}^{p-1} F_j(i, v)\overline{\Gamma_j(i - rN', v)}$ as a function of r.
4. Compute q times a 2-D DFT of size $N \times M'/p$ of the sum obtained in the previous stage.

This algorithm can be also used by interchanging the roles of the frame and the dual frame.

The reconstruction of the function from its expansion coefficients is given by

$$(Zf)(i, v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{m,n} (Zg)(i - mN', v) \exp\left(2\pi\imath \frac{in}{N}\right). \qquad (17)$$

Introducing m', r as above, for $0 \le j \le p - 1$, we obtain

$$F_j(i, v) = \sum_{r=0}^{q-1} G_j(i - rN', v) \sum_{n=0}^{N-1} \sum_{m'=0}^{M'/p} \cdot c_{m'q+r,n} \exp\left(2\pi\imath \frac{in}{N}\right) \exp\left(-2\pi\imath \frac{vm'}{M'/p}\right). \qquad (18)$$

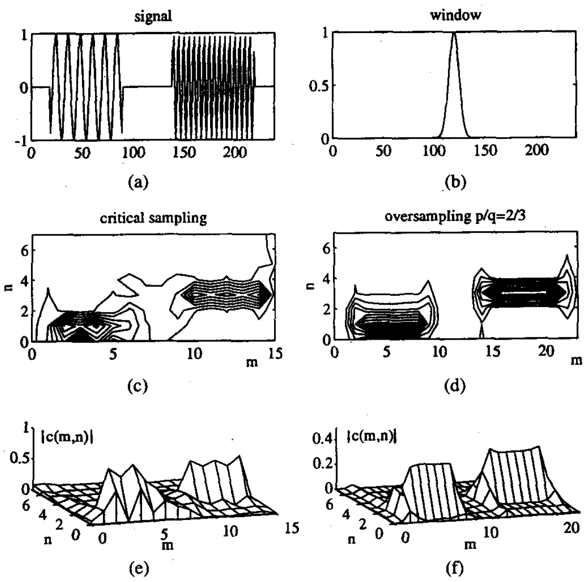


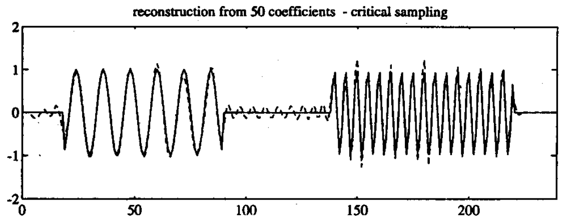
Fig. 1. Example of nonstationary signal representation by a discrete Gabor scheme. The expansion coefficients obtained in the critical sampling and oversampling cases are compared: (a) Original signal; (b) Gaussian window function; (c), (d) level crossings of smoothed distribution of the absolute values of the coefficients obtained in the cases of critical and overcritical sampling, respectively; (e), (f) absolute values of the expansion coefficients obtained in the cases of critical sampling and oversampling, respectively. Note the better temporal clustering (separation) of the Gabor expansion coefficients obtained in the case of oversampling.

The reconstruction algorithm consists of the following stages:

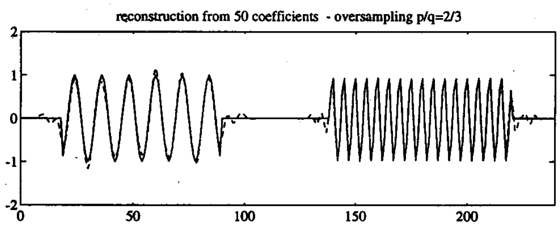
1. Precompute the FZT of the window function $(Zg)(i, v)$.
2. Compute q times a 2-D DFT of size $N \times M'/p$ of the expansion coefficients $c_{n'q+r,n}$ as a function of r.
3. Compute p times the sum $\sum_{r=0}^{q-1} G_j(i - rN', v)$ DFT$[c_{m'q+r,n}]$.
4. Compute the inverse FZT of $(Zf)(i, v)$.

D. Example of Implementation

In various cases of signal (and image) representation by partial information, it is desirable to oversample the Gabor space. Previously, oversampling was applied by implementing the (unstable) biorthogonal function, which corresponds to critical sampling [1], or by derivation of the dual window by minimum energy constraint [4], where the computation of the representation coefficients is accomplished in the signal space. In the following example, we illustrate the application of a discrete (finite) Gabor scheme in the PFZT domain. The analyzed signal of length 240 (Fig. 1(a)) is comprised of two temporally separated tones. The cases of critical sampling, where $M = 16, N = 15$, and oversampling with $p/q = 2/3$, where $M = 24, N = 15$ are compared. The absolute values of the expansion coefficients are shown in Fig. 1(e) and (f) for the critical sampling and oversampling cases, respectively, and the level crossings of the smoothed absolute values of the coefficients are shown in Figs. 1(c) and (d) for the cases of critical sampling and oversampling, respectively. It appears that in the oversampling case better temporal clustering (separation) than in the critical sampling case is obtained.



(a)



(b)

Fig. 2. Signal reconstruction from partial information of the Gabor expansion coefficients. Reconstruction (approximations) of the signal shown in Fig. 1(a) are compared by using the 50 coefficients of the highest magnitude: (a) Reconstruction in the case of critical sampling; (b) reconstruction in the case of oversampling. Note the better reconstruction and separation of the two tones obtained from the same number o fcoefficients in the case of oversampling.

Obviously, the temporal separation depends significantly on the type of signal and width of the Gaussian window, but for a given window-width, the temporal separation can be improved by the choice of proper rate of oversampling. To further stress the advantage offered by oversampling, we also compare the reconstructions (approximations) of the signal from partial information in the above two cases of critical sampling and oversampling. The signal is reconstructed by using the 50 coefficients of the highest magnitude. Fig. 2(a) and (b) show the reconstructed signals (dashed lines) superimposed on the original ones (continuous lines) in the cases of critical sampling and oversampling, respectively. Again, the reconstructed signal appears to be better in the case of oversampling.

IV. Conclusion

Representation in the PFZT domain provides new results concerning frame properties of the sequence $\{g_{m,n}\}$ in relation to the matrix-valued function $\boldsymbol{S}(i, v)$. In case $\{g_{m,n}\}$ constitutes a frame, this matrix-valued function represents the frame operator in the PFZT domain. The minimum and maximum eigenvalues of $\boldsymbol{S}$ are the frame lower and upper bounds, respectively. In cases of tight frames, the matrix $\boldsymbol{S}$ is simply the identity matrix. In the context of matrix algebra, finding the dual frame (which is important for finding the expansion coefficients) is also possible by calculating the inverse of the matrix $\boldsymbol{S}$. Utilizing the matrix algebra approach in the PFZT domain, DFT-based algorithms for computation of the expansion coefficients and for the reconstruction of signals from these coefficients are generalized for the case of oversampling of the Gabor space.

Finally, we conclude that the application of the approach presented in this correspondence to analysis of Gabor-type frames highlights the conditions required of a set of functions to constitute a frame, a tight frame, or a complete sequence. This approach lends itself also to the development of algorithms for construction of frames and bases.

Appendix
Proof of the Resolution of Identity

For $M = N = L$, we get $g_{m,n}(i) = g(i-m)\exp(2\pi\imath(in/L))$. A direct calculation yields

$$\begin{aligned}(Sf)(i) &= \sum_{m=0}^{L-1}\sum_{n=0}^{L-1} g_{m,n}(i) \sum_{j=0}^{L-1} \overline{\cdot f(j)g(j-m)\exp\left(2\pi\imath\frac{jn}{L}\right)}\\ &= \sum_{m=0}^{L-1}\sum_{n=0}^{L-1}\sum_{j=0}^{L-1} f(j)g(i-m) \cdot \overline{g(j-m)}\exp\left(2\pi\imath\frac{(i-j)n}{L}\right)\\ &= \sum_{m=0}^{L-1}\sum_{j=0}^{L-1} f(j)g(i-m)\overline{g(j-m)}L\delta_{(i-j)\bmod L}\\ &= f(i)L\|g\|^2\end{aligned}$$

where δ_j denotes the Kronecker delta function. It therefore follows that $\{g_{m,n}\}$ is a tight frame.

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