ANISOTROPIC SELECTIVE INVERSE DIFFUSION FOR SIGNAL ENHANCEMENT IN THE PRESENCE OF NOISE

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ABSTRACT

Signal and image enhancement in the presence of noise is considered in the context of the scale-space approach. A modified dynamic process, based on the action of a nonlinear diffusion equation, is presented. The diffusion coefficient is adjusted according to the local gradient, intensity and other image properties, and as such also reverses its sign, i.e. switches from a forward to a backward (inverse) diffusion process according to a given criterion. This results in enhancement of transients and singularities in the one-dimensional case, and of edges in images, while locally denoising smoother segments of the signal or image. Regularization of ill-posed inverse diffusion problem is discussed. Examples of both one-dimensional signals and images are presented.

1. INTRODUCTION

The scale-space approach and partial differential equations (PDE) techniques have been extensively applied to signal and image processing over the last decade. As Witkin [1] had pointed out, the diffusion process (or heat equation) is equivalent to a smoothing process with a Gaussian kernel. However, a major drawback of the linear scale space framework is its uniform filtering of local signal features and noise. This was addressed by Perona and Malik (P-M) [2], who proposed a process known as nonlinear anisotropic diffusion, where diffusion can take place with a variable conductance in order to control the smoothing effect. A close inspection [13] reveals that the Perona-Malik diffusion process is isotropic but non homogenous. Genuine anisotropic processes were suggested recently by Sochen, Kimmel, Malladi [10].

The conductance coefficient in the Perona-Malik process was chosen to be a decreasing function of the gradient of the signal. This operation selectively lowpass filters regions that do not contain large gradients (singularities as a step jump or an edge in the case of an image). The P-M results stimulated a wide range of application of their approach (for a review of further results see [3]). Some drawbacks and limitations of the original model have been mentioned in the literature [4-6]. Catte et al. [4] have proved the ill-posedness of the diffusion equation when using the P-M conductance coefficient and proposed a regularized version, where the coefficient is a function of a smoothed gradient. Weickert et al. [7] showed how the stability of the P-M equation could be explained by spatial discretization, and proposed [8] a generalized regularization formula in the continuous domain.

The aim of this work is to extend the nonlinear PDE-based filtering methods, and to apply them to signal and image enhancement and restoration. We focus on enhancing blurry signals, while still allowing a considerable amount of additive noise to interfere in the process. We try to avoid amplification of noise, which is inherently a byproduct of signal enhancement, by combining backward and forward diffusion processes.

In section 2 we discuss the possibility of enhancing through diffusion and describe the rationale that led to a new type of diffusion process. In section 3 we present the new conductance coefficient. Examples are presented in section 4 demonstrating that the process although not yet fully stabilized, yields promising results, even in very noisy cases.

2. ENHANCEMENT BY DIFFUSION

Most of the PDE-based studies have been devoted so far to denoising, attempting trying to preserve the edges. Both forward linear and nonlinear diffusion processes converge ($t \to \infty$) to a trivial constant solution (i.e. the average value of signal, asuming constant boundary conditions). To preserve singularities, previous studies relied primarily on slower diffusion in the vicinity of singularities. The P-M nonlinear diffusion equation is of the form:

$$(1)\quad u_t = \mathrm{div}(c(|\nabla u|)\nabla u) \qquad , c > 0$$

where C is a decreasing function of the gradient.

According to the "Minimum-Maximum" principle no new local minima or maxima should be created at any time in the 1D case, in order not to produce new artifacts in the diffused signal. Moreover, the values of the global minimum and maximum along the evolution of the signal in time are bounded by that of the initial state U(at t=0) in any dimension. These conditions were obeyed by the P-M and most other aniosotropic diffusion processes that were subsequently introduced in image processing. This guarantied the stability of the PDE and an explosion of the nonlinear diffusion process was avoided.

In signal enhancement/restoration, we do not want to restrict ourselves to the global minimum and maximum of the initial signal. On the contrary, we would like the points of extrema to be emphasized and "stretched" (if they indeed represent singularities and do not come as a result of noise). Therefore, a different approach should be taken.

As we want to emphasize large gradients, we would like to move “mass” from the lower part of a “slope” upwards. This process can be viewed as moving back in time along the scale space, or reversing the diffusion process [11]. Mathematically we can simply change the sign of the conductance coefficient to negative:

$$(2)\quad u_t = \mathrm{div}(-c(\cdot)\nabla u) \qquad , c > 0$$

Note that this is different than what was defined as "inverse diffusion" in previous studies (e.g. [4],[9]). There, in places where the derivative of the flux c*grad(u) was negative, it was defined as inverse diffusion, because one can write the diffusion equation near that point as:

$$(3)\quad u_t = -du_{xx} \qquad , d > 0$$

Although it has the form of an inverse diffusion process, it is weaker since it does not have the important inverse diffusion property of moving signal or image “particles” upward along the slope of the gradient. With positive conductance coefficient C, this could never happen, and therefore the minimum-maximum principal could be kept, for instance. Thus, signal enhancement requires further modification of the diffusion process. Specifically, to deblur and enhance singularities, negative conductance coefficient must be incorporated into the process.

The question is, can we simply use a linear inverse diffusion? The problem is that linear inverse diffusion is a highly unstable process. As mentioned earlier, the linear forward diffusion is analogous to convolution with a Gaussian kernel. Hence, the linear backward (inverse) diffusion is analogous to a Gaussian deconvolution, where the noise amplification explodes with frequency. Application of such a deconvolution process results in oscillations that grow with time until they reach minimum and maximum saturation values and the original signal is completely lost.

Three major problems associated with the linear backward diffusion process must be addressed: The explosive instability, noise amplification and oscillations.

One way to avoid explosion is by deminishing the value of the inverse diffusion coefficient at high gradients. In this way, after the singularity exceeds a certain gradient threshold it does not continue to affect the process any longer. We can also terminate the diffusion process after a limited time, before reaching saturation.

In order not to amplify noise, which after some pre-smoothing, can be regarded as having mainly medium to low gradients, we should also eliminate the inverse diffusion force at low gradients.

To reduce oscillations, we should try to suppress them the minute they are introduced. For this we can combine a forward diffusion force, that smoothes low gradients. This force also smoothes some of the original noise that is in the signal from the beginning. Unfortunately, low gradients which are not due to noise, like those that are characteristic of certain textures in images, are also affected and smoothed out by this force.

The result of this intuitive analysis is that we basically need two forces of diffusion working simultaneously on the signal - one is a backward force (at medium gradients, where singularities are expected), and the other is a forward one, used for stabilizing oscillations and reducing noise. Actually, we can combine those two forces to one complex backward-and-forward diffusion force with a conductance coefficient (which is a function of the gradient) that has both positive and negative values.

3. NEW CONDUCTANCE COEFFICIENT

We suggest a general formula for the conductance coefficient in the form of:

$$(4)\quad c(s) = \begin{cases} 1-(s/k_f)^n & ,0 \le s < k_f \\ \alpha\{((s-k_b)/w)^{2m}-1\} & ,k_b - w < s < k_b + w \\ 0 & ,\text{otherwise} \end{cases}$$

and its smoothed version:

$$(5)\quad c_\sigma(s) = c(s) \otimes G_\sigma(s)$$

where $\otimes$ denotes convolution. [exponent parameters n,m , were chosen to be 4 and 1 respectively, and Kf is smaller than Kb.]

The P-M conductance coefficient, in comparison is:

$$(6)\quad c_{P\text{-}M}(s) = 1/(1+(s/k)^2)$$

C has to be continuous and differentiable. In the discrete domain, (4) could suffice (although it is only piece-wise differentiable). (5) can fit the general continuous case, but raises the problem of non-zero values at high gradients (that diminish fast). Other formulas with similar nature may also be proposed.

As compared with the P-M equation (6), where an "edge threshold" K is the sole parameter, we now have a parameter for the forward force Kf, two parameters for the backward force (we defined them by the center Kb and width W), and the relations between the strength of the backward and forward forces (ratio we termed alpha). We therefore discuss some rules for determining these parameters.

Essentially Kf - is the limit of gradients to be smoothed out, and is similar in nature to the role of the K parameter of the P-M conductance equation.

Kb and W define the backward diffusion range, and should take values of gradients that we want to emphasize. In our formula the range is symmetric, and we usually restrain the width from overlapping the forward diffusion area.

One way of choosing those parameters in the discrete case, without having any previous knowledge about the signal, is by calculating the mean absolute gradient ("mag"), similar to Total Variation analysis. For instance, [Kf,Kb,W] = [2,4,1]*mag.

Local adjustment of the parametrs, can be done by calculating the "mag" value in a window. The parameters change gradually along the signal, and enhancement is accomplished by different thresholds in different locations. This is indeed required in the cases of natural signals or images, due to their nonstationary structure. Usually a minimum value of forward diffusion should be kept, so smooth large areas would not get noisy. A good example where we used the local parameter adjustment is depicted by the parrot image (Fig. 4).

The last parameter, alpha, controls the ratio between the backward and forward diffusion. If the backward diffusion force is too dominant, the stabilizing forward force is not strong enough to avoid oscillations. One can avoid the developing of new singularities in very smooth areas by bounding the maximum flux permissible in the backward diffusion to be less than the maximum of the forward one [for a proof see [14]]. Formally we say:

$$(7)\quad \max_{s<kf}\{s \cdot c(s)\} > \max_{kb-w<s<kb+w}\{s \cdot c(s)\}$$

In the case of our proposed C, we get a simple formula for alpha, which just obeys this inequality by:

$$(8)\quad \alpha = k_f / 2k_b \quad , \text{for any} \quad 0 < w < k_b - k_f$$

In practical applications, this bound can usually be increased up to a double value without experiencing large instabilities.

There are a few ways to increase regularity in this PDE based approach. We can replace the proposed conductance coefficient Eq. (4) by the regularized one, Eq. (5), similar in a way to Catte et al. Given an *a priory* information on the smallest scale of interest, one can smooth smaller scales in a noisy signal by preprocessing. As we enhance the signal afterwards, this smoothing process does not affect the end result that much and enables us to operate in an originaly much noisier environment. Finally, operating in extremely noisy areas, when we know of the type of singularity, we can apply more pre-smoothing, and consider only the largest gradient within the backward diffusion range, (see Fig. 3).

4. EXAMPLES

We used the explicit Euler scheme with forward difference scheme for the time derivative and the central difference scheme with 3x3 stencil for the spatial derivatives.

A few examples of blurry and noisy signal restoration using the selective inverse diffusion are shown below. In the 1-d cases the process was stopped after the process observed only very slight changes between iterations. In the image case the process was stoped by visual inspection.

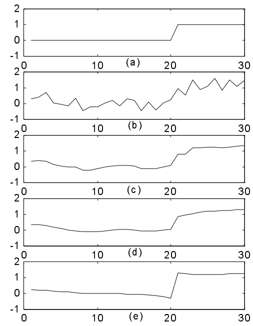

Figure 1. (a) Orginal step, (b) Blurred signal contaminated by white gaussian noise (SNR=5dB), (c-e) Diffusion process after iterations: 20, 40, 160, respectively.

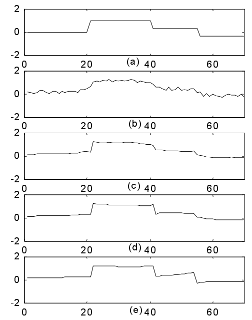

Figure 2. (a) Orginal signal (with both positive and negative discontinuities, (b) Blurred signal contaminated by white uniform noise (SNR=8dB), (c-e) Diffusion process after iterations: 40, 80, 320, respectively.

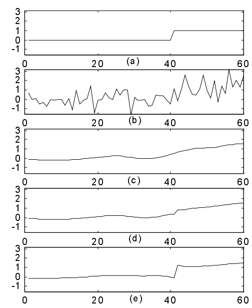

Figure 3. An example of processing an extremely noisy signal: (a) Orginal step, (b) Blurred signal contaminated by white gaussian noise (SNR=-4dB), (c) After pre-smoothing (d-e) Diffusion process after iterations: 40, 240, respectively.

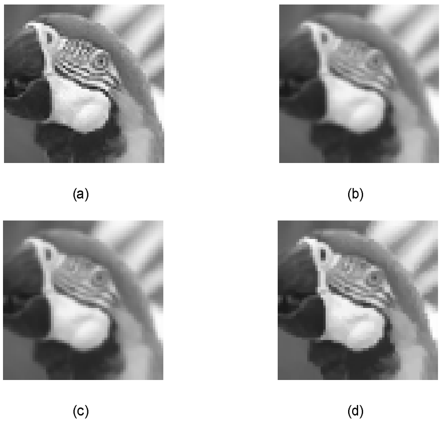

Figure 4. (a) Orginal parrot image, (b) Blurred image, (c-d) Diffusion process after iterations: 16, 128, respectively.

5. CONCLUSIONS

In this study we address the outstanding issue of how can the conflicting requirements of signal and image lowpass filtering, thereby smoothing the signal, and sharpening and enhancement of such signals, even beyond what their original spectrum permits, be incorporated into a diffusion-type PDE approach. Our approach is a generalization of the diffusion process into a forward-and-backward process. This is demonstrated by the specific example of generalization of the Peron-Malik equation. Thus, to reiterate, the novely in our approach lies in the fact that the enhancement process includes both smoothing and sharpening. This is accomplished by assigning positive and negative values to a conductance coefficient in the diffusion equation that ranges from a certain negative to a positive value. Various parameters control the exact shape of this local conductance function and an analysis of their meaning, relations and suitable ranges were presented. The proposed algorithm was tested on one- and two-dimensional signals with promising results.

Similar generalizations of the diffusion-type processes, such as the one associated with the Beltrami equation, and a rigorous treatment of the issues of stability and regularization, that address the problem of oscilations and the creation of singularities will be published elsewhere [14].

6. ACKNOWLEDGEMENTS

This research has been supported in part by the Ollendorff Center, The Fund for Promotion of Research at the Technion, and the Ministry of Science.

7. REFERENCES

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