Two-Dimensional Sampling and Representation of Folded Surfaces Embedded in Higher Dimensional Manifolds

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• Two - dimensional flat representation of $3-$ dimensional scanned data is a fundamental task in various applications (i.e. medical - imaging)

• Flattening process should maintain geometric features such as length, angles, area as possible, so that image analysis (medical diagnosis) will be accurate.

• Flattening algorithm is performed on a triangulated reconstruction of assumed well sampled surface.

• In most cases distorting the mentioned geometric measures is inevitable because of presence of curvature.

• It is wished to have an estimate for the measurement error or length distortion.

Mathematically, this is provided by the following definition:

Definition 1 *Let $D \subset \mathbb{R}^3$ be a domain. A homeomorphism $f : D \to \mathbb{R}^3$ is called a* quasi-isometry *(or a* bi-lipschitz mapping*), if there exists $1 \leq C < \infty$, such that*

$$\frac{1}{C}|p_1 - p_2| \leq |f(p_1) - f(p_2)| < C|p_1 - p_2|\,, \text{ for all } p_1, p_2 \in D\,;$$

where $|\cdot|$ denotes distance.

$C(f) = \min\{C \,|\, f \text{ is a quasi} - \text{isometry}\}$ *is called the* minimal distortion *of f (in D).*

• In fact, one can define quasi-isometries between any two *metric spaces* (X, d) and (Y, ρ).

• For the case of a surface in $\mathbb{R}^3$, distances are the *induced intrinsic distances* on the surface.

Definition 2 *A homeomorphism $f : D \to \mathbb{R}^3$ is called* quasi-conformal, *if it almost preserve angles. Formally, it is the same as a* quasi-isometry *while* distances *are replaced by* angles *(i.e.* scalar products of tangent vectors*)*

The minimal distortion, $C(f)$, is replaced by the dilatation of angles, *$K(f)$.*

Lemma 3 *If f is a quasi - isometry then,*

$$K(f) \leq (C(f))^2 .$$

Remark 4 f *quasi-isometry* $\Rightarrow f$ *quasi-conformal;*

but

f *quasi-conformal* $\not\Rightarrow f$ *quasi-isometry.*

• One would wish to have a conformal map yet, for many application it suffices to have quasi-conformality.

Since we are interested in images of 3-dim objects on 2-dim "screens", i.e. projections, we are conducted to ask the following questions:

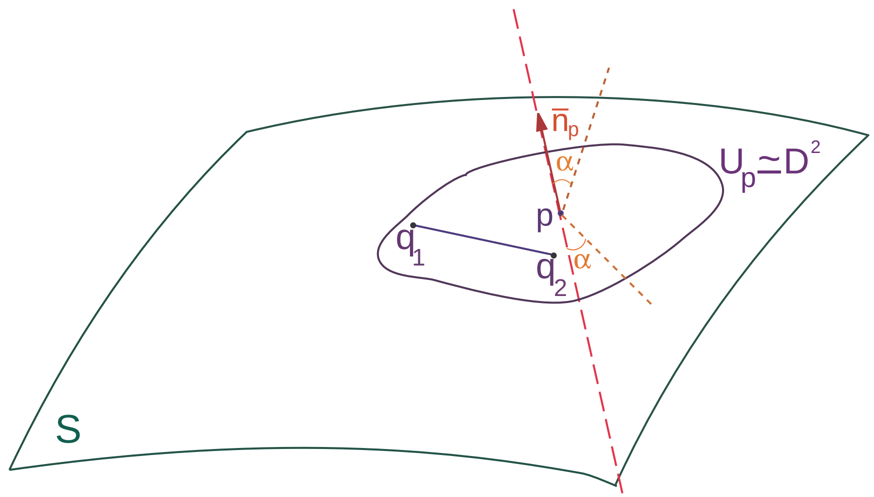
Question 1 When is orthogonal projection a quasi-isomorphism (quasi-conformal mapping)?

Question 2 And if it is, what are its distortion and dilatation?

The answer to these questions is to be found in the classical work of F. Gehring and Y. Väisälä (1965).

The Geometric (Gehring) Condition: We say $S \subset \mathbb{R}^3$ satisfies the *Gehring Condition* if for any $p \in S$ there exists $\vec{n}_p$ such that for any $\varepsilon > 0$, there exists $U_p \simeq D^2$, such that for any $q_1, q_2 \in U_p$ the acute angle $\measuredangle(q_1q_2, \vec{n}_p) \geq \alpha$, where:

$$0 < \inf_{p \in S} \alpha_p < \frac{\pi}{2} - \varepsilon \,.$$



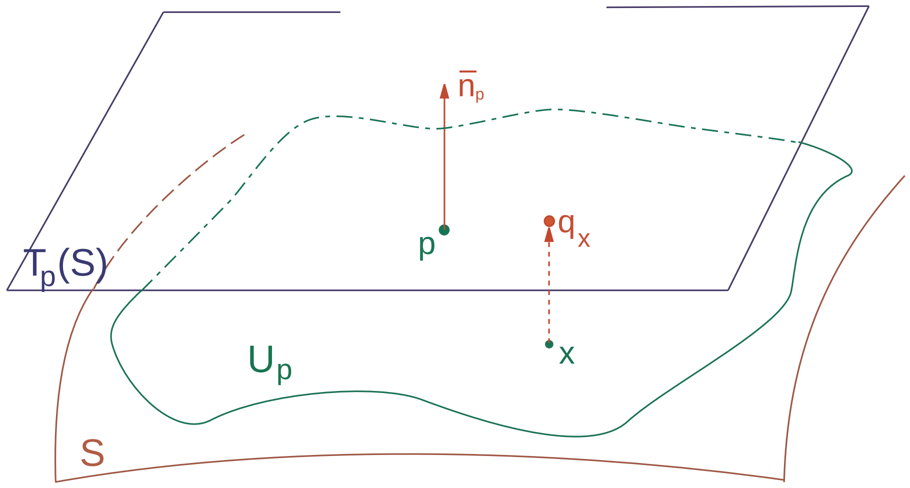
Example 5 *Any surface in $S \subset \mathbb{R}^3$ that admits a well-defined continuous* turning tangent plane *at any point $p \in S$ satisfies the geometric condition.*

Then for any $x \in U_p$ there is a unique representation of the following form:

$$x = q_x + u\vec{n}\,;$$

where q_x lies on the plane through p which is orthogonal to $\vec{n}$ and $u \in \mathbb{R}$. Define:

$$Pr(x) = q_x\,.$$



Remark 6 $\vec{n}$ *need* **not** *be the normal vector to* S *at* p.

Moreover, we can compute bounds for $C(f)$ and $K(f)$, for $f \equiv Pr$. We get ([GV]):

$$C(f) \leq \cot\alpha + 1\,; \tag{1}$$

$$K(f) \leq \left(\left(\frac{1}{2}(\cot\alpha)^2 + 4\right)^{\frac{1}{2}} + \frac{1}{2}\cot\alpha\right)^{\frac{3}{2}} \leq (\cot\alpha + 1)^{\frac{3}{2}}. \tag{2}$$

• The proposed algorithm based on the above (presentation will follow) is the first known, to have such error estimates and control for the distortion and dilatation.

• Naturally, the existence of faithful quasi-conformal/quasi-isometric representations for sampled surface strongly depends on the quality of the sampling.

Theorem 7 *[asz] Given a $\mathcal{C}^2$ surface Σ, with absolute principal curvatures bounded by some bound K_Σ, there exists a sampling scheme of the surface Σ, with a proper density $\mathcal{D}$, corresponding to the maximum absolute curvature K_Σ, i.e. $\mathcal{D} = \mathcal{D}(K_\Sigma)$.*

Theorem 8 *[asz] If Σ is not a $\mathcal{C}^2$ surface, then there exists a smoothing reproducing kernel $\mathcal{H}_\Sigma$, for which $\mathcal{H}_\Sigma * \Sigma$ is of class $\mathcal{C}^\infty$. The smooth surface can be represented by a sampling scheme of density $\mathcal{D}$, according to Theorem 7.*

$*$ Similar ideas but without precise density function and rigorous presentation appear in [AB]

- Reproducing of the sampled surface is given by the Secant Approximation of a manifold [Mun].

- Both sampling theorems apply for higher dimensional manifolds as well. Principal curvatures are replaced by appropriate combinations of the scalar, sectional and Ricci curvatures of Riemannian manifolds.

An algorithm for triangulated surfaces is readily produced from the results above:

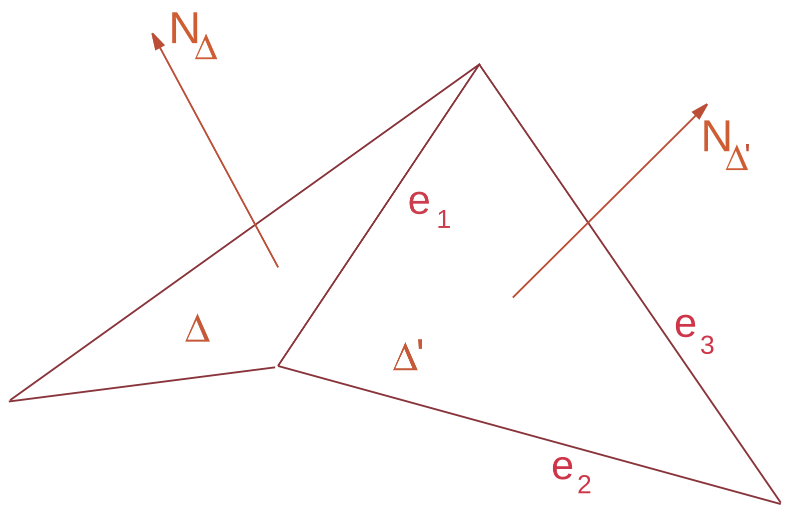
• Let N_p stand for the normal vector to the surface at a point p on the surface.

• Choose a triangle Δ, of the triangulation. There are two possibilities to chose Δ: one is in a random manner and the other is based on curvature considerations. Trivially project Δ onto itself. *

• Suppose Δ' is a neighbor of Δ having edges e_1, e_2, e_3, where e_1 is the edge common to both Δ and Δ'. We will call Δ' *Gehring compatible w.r.t* Δ, if the maximal angle between e_2 or e_3 and N_Δ (the normal vector to Δ), is greater then a predefined measure suited to the desired predefined maximal allowed distortion, i.e. $\max\{\varphi_1, \varphi_2\} \geq \alpha$, where $\varphi_1 = \measuredangle(e_2, N_\Delta)$, $\varphi_2 = \measuredangle(e_3, N_\Delta)$.

*A variant of this algorithm will be discussed.

• Project Δ' *orthogonally* onto the plane included in Δ and insert it to the patch of Δ, iff it is Gehring compatible w.r.t Δ.



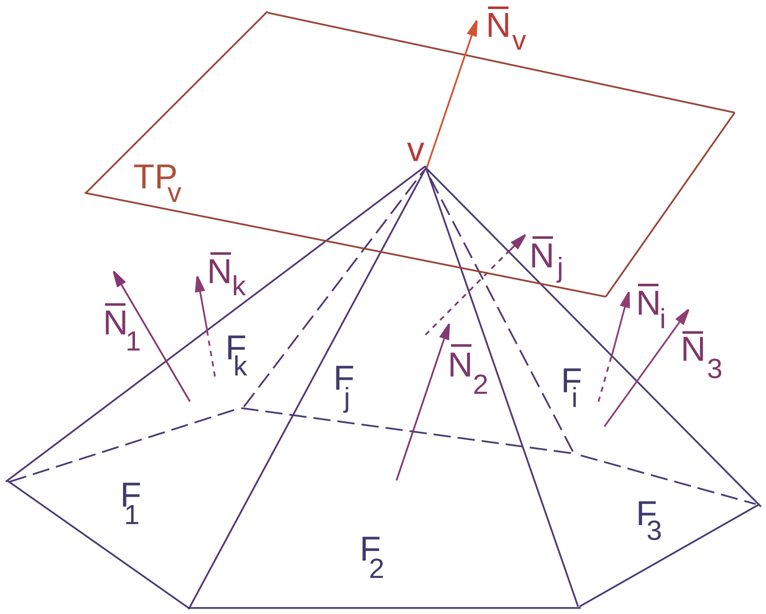
• If by this time all triangles where added to the patch we have completed constructing the mapping. Otherwise, chose a new triangle that has not been projected yet, to be the starting triangle of a new patch.

†Variant of the Basic Algorithm:

• Project the faces adjacent to the vertex v on the plane TP_v through v, orthogonal to the *mean normal*:

$$\overline{N}_v = \frac{1}{k}\sum_{1}^{k}\overline{N}_i\,;$$

where $\overline{N}_i$ is the normal to the face F_i adjacent to v.



- Choose starting vertex using Gauss Curvature K:

For triangulated (PL) surfaces we define* K at every vertex as the *defect* of the sum of angles surrounding it:

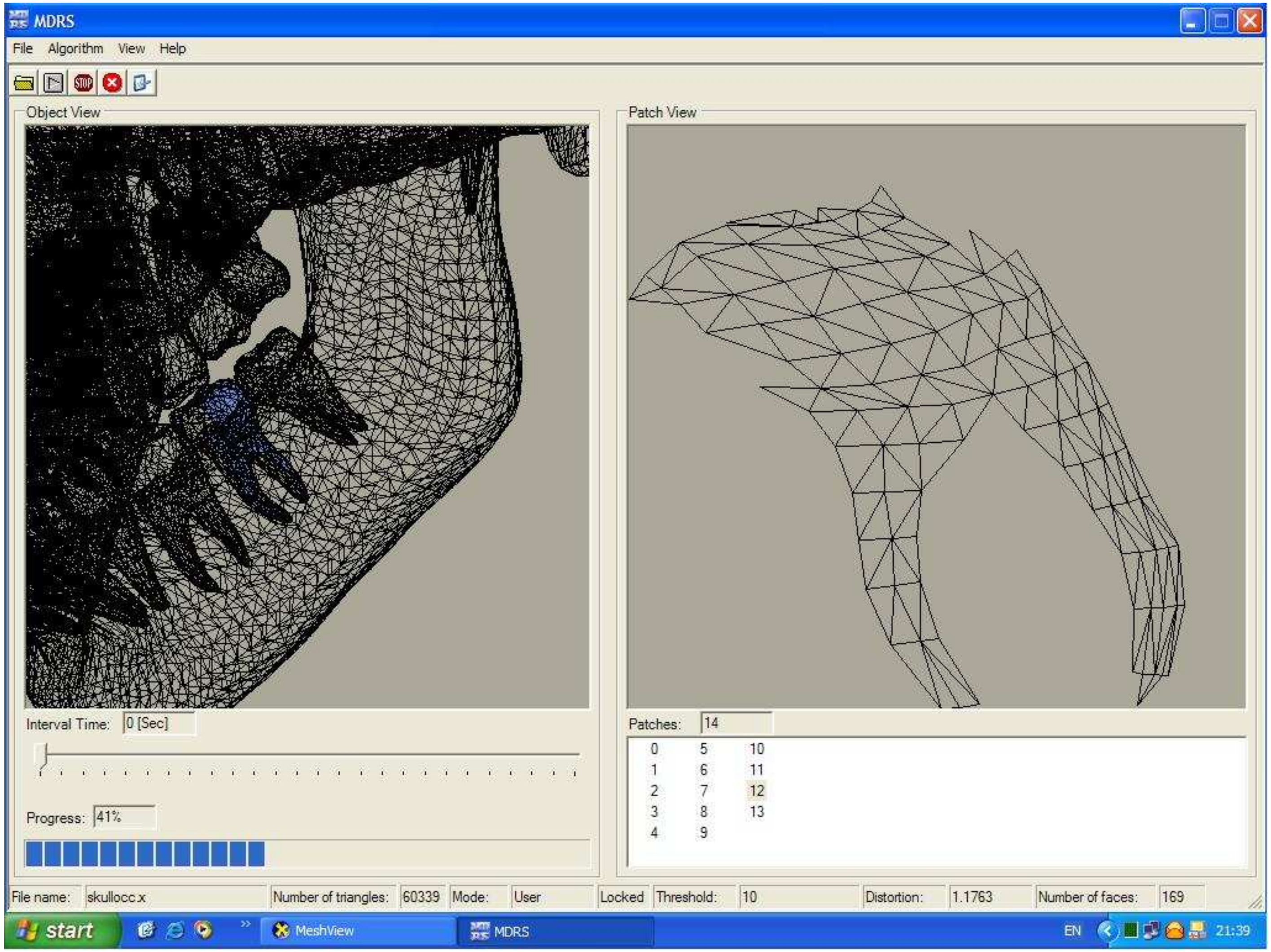
$$K(v) = \delta(v) = 2\pi - \sum_i \alpha_i\,.$$

Remark 9 *The curvature based method is better fitted for:*

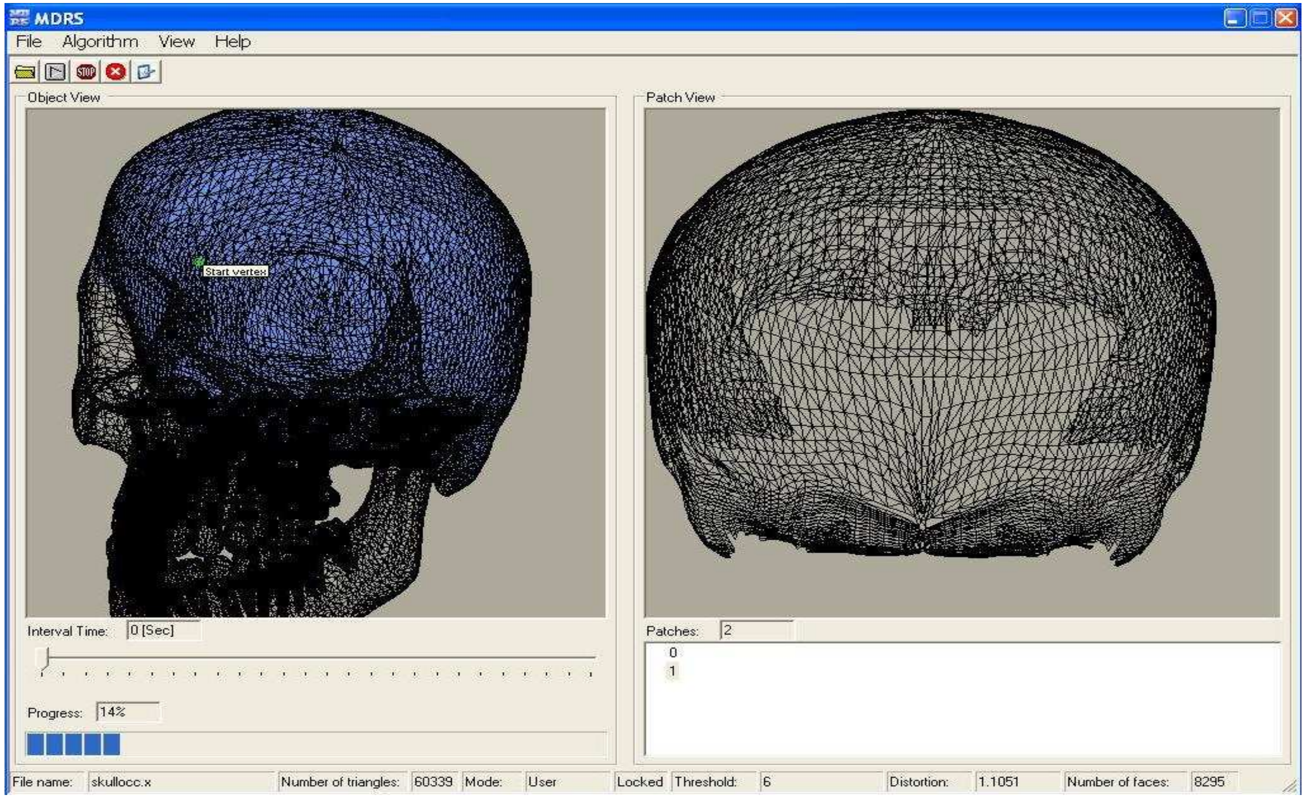
- *Low curvature ("almost flat") surfaces;*
- *High α.*

*following **Descartes** and, in more recent times, **Hilbert–Cohn-Vossen, Pólya, Banchoff,...**

We present some experimental results, both on synthetic surfaces and on data obtained from actual CT scans (of the Human Brain Cortex and Colon):

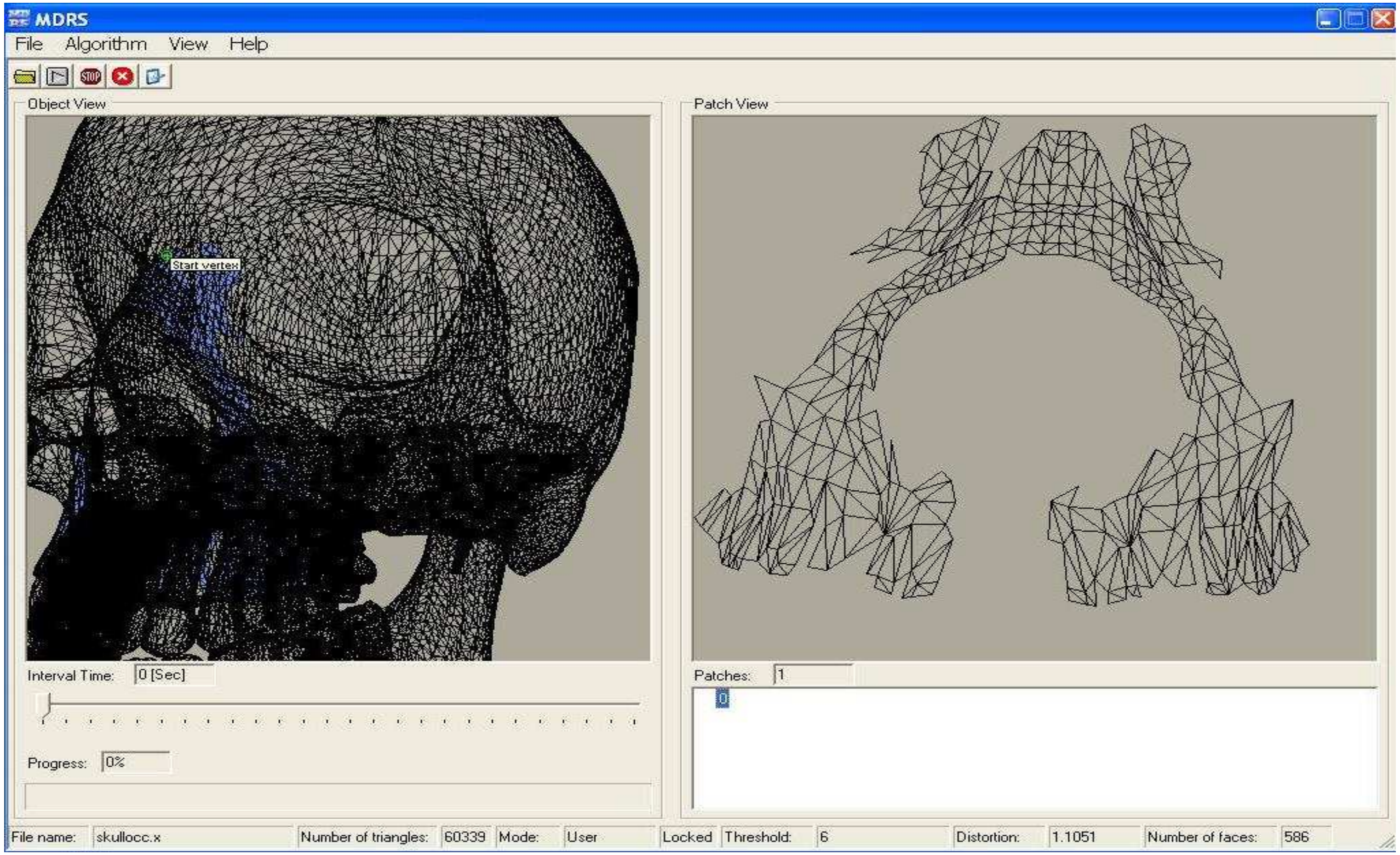


A **lower** curvature produces a **larger** patch (with more triangles)...



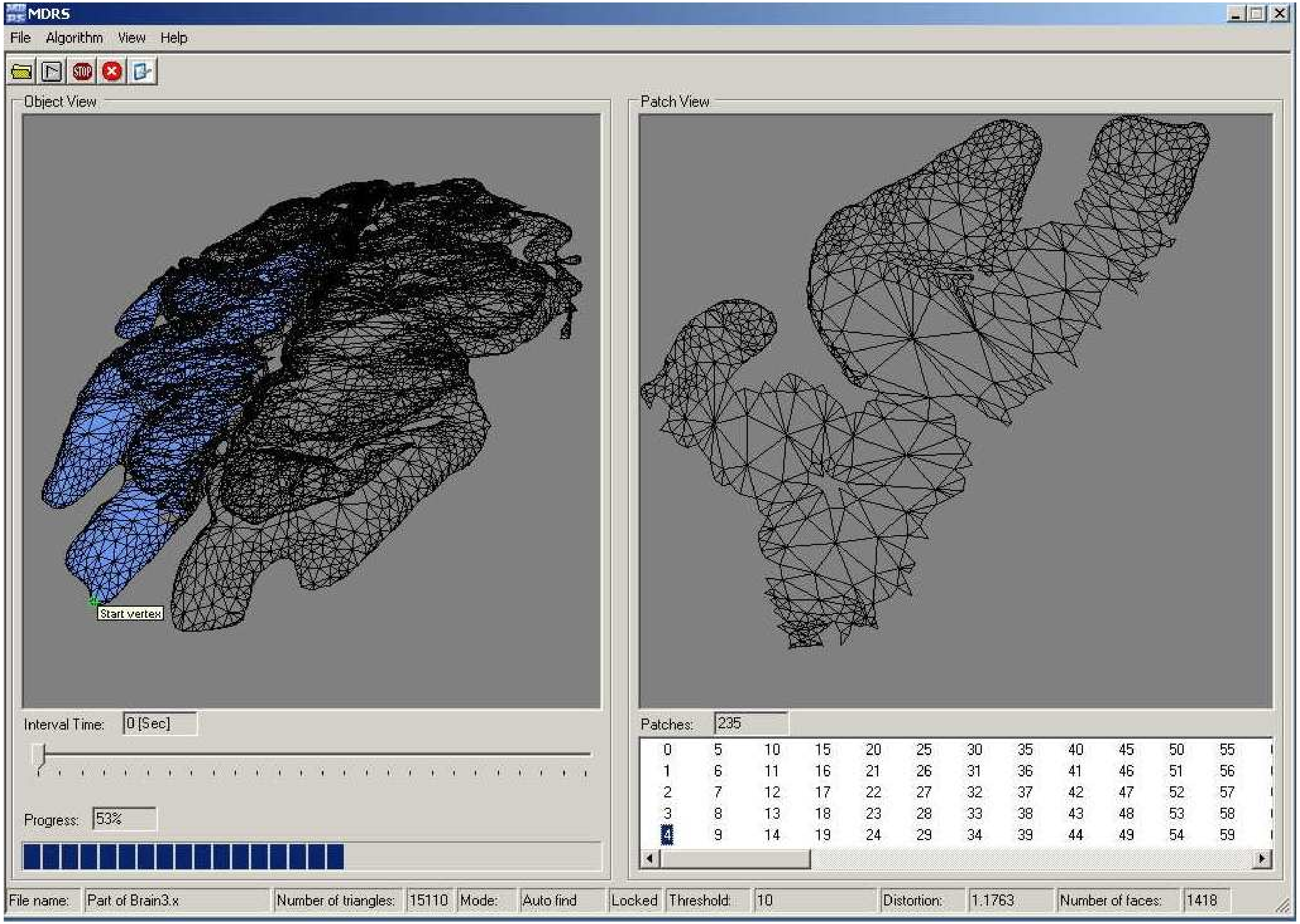
In The Skull Model the resolution is of 60,339 triangles. Here $\alpha = 10^\circ$ and the dilatation is 1.1763.

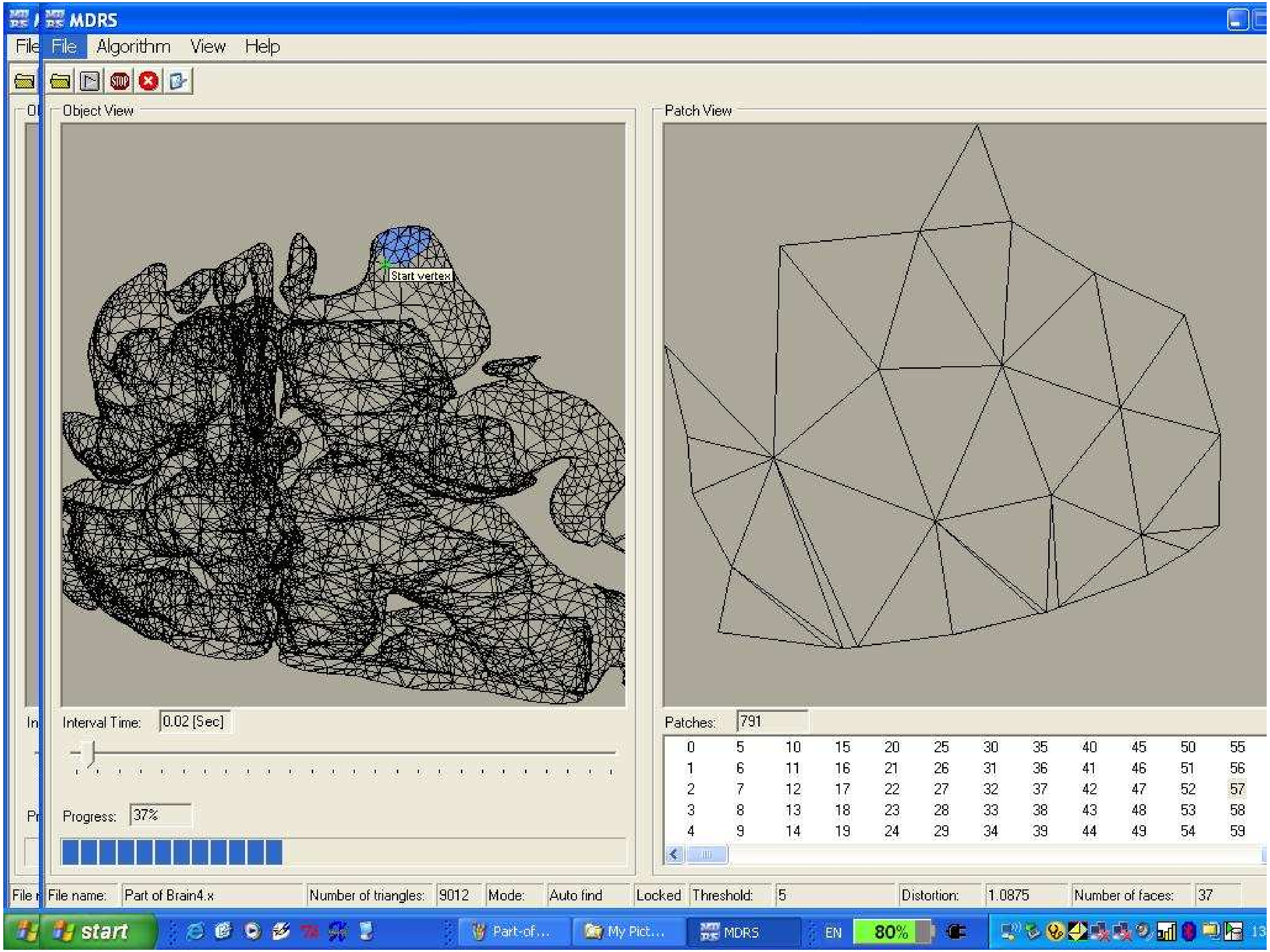
...than when flattening regions of higher curvature:



Here $\alpha = 6^{\circ}$ and the dilatation is 1.1051.

It is also evident in the development of the Human Brain Cortex:

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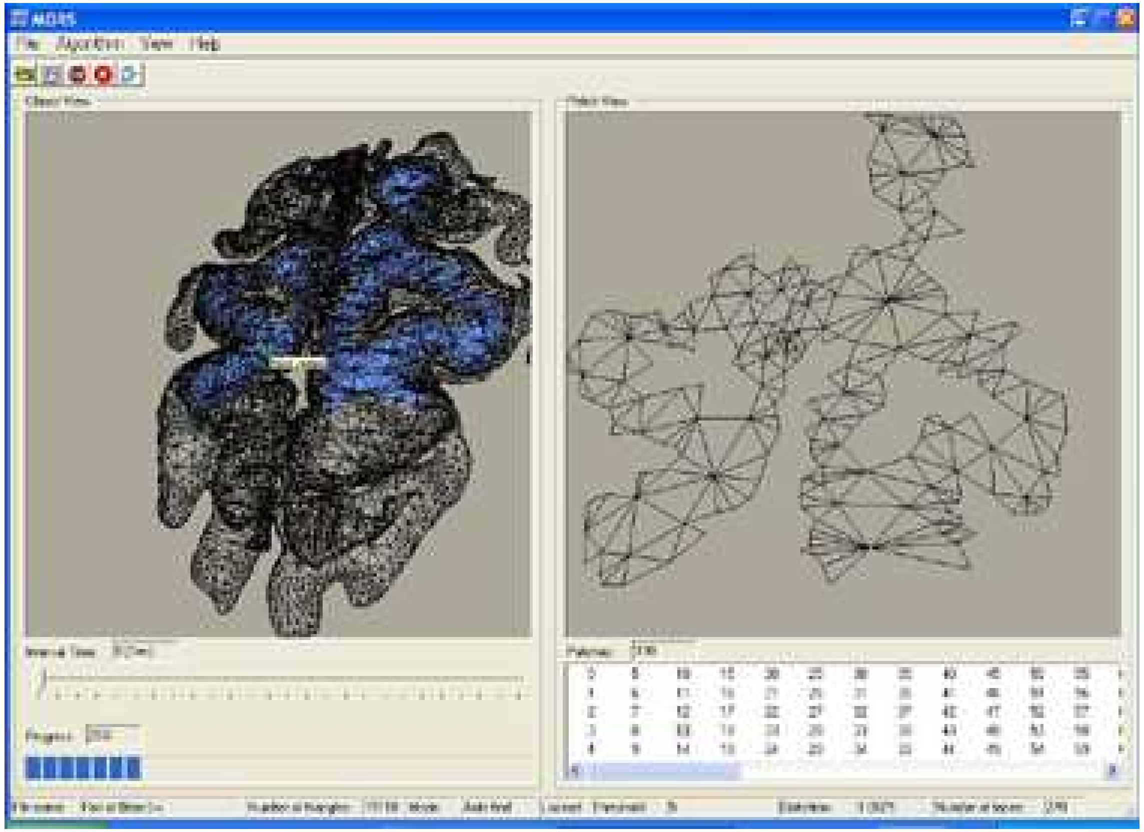
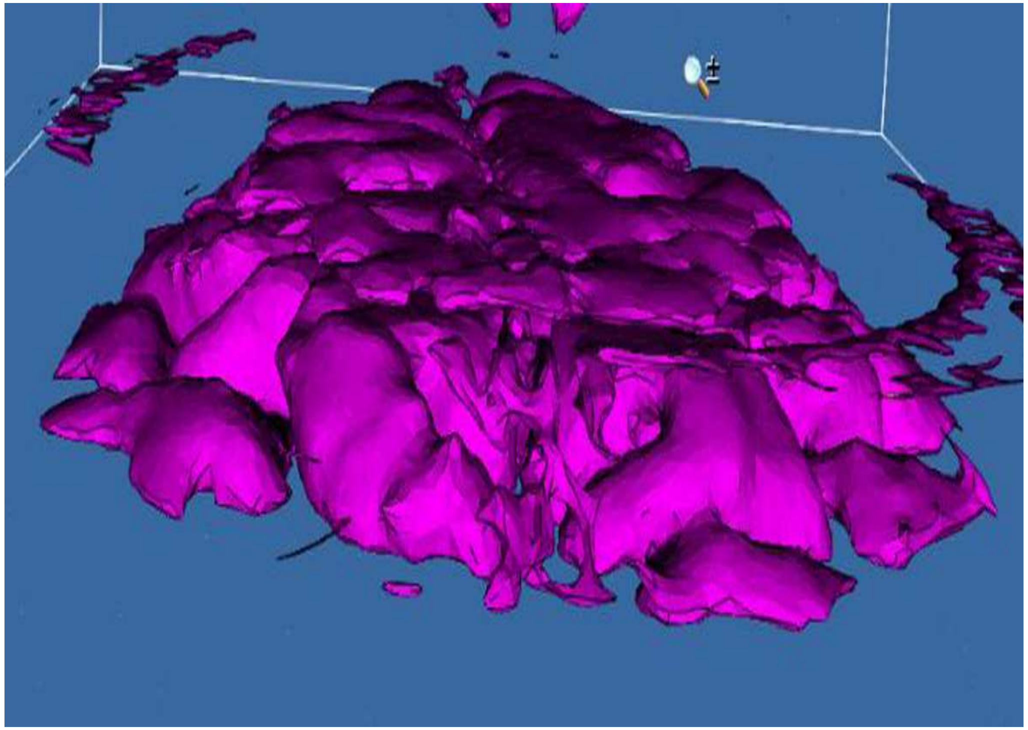
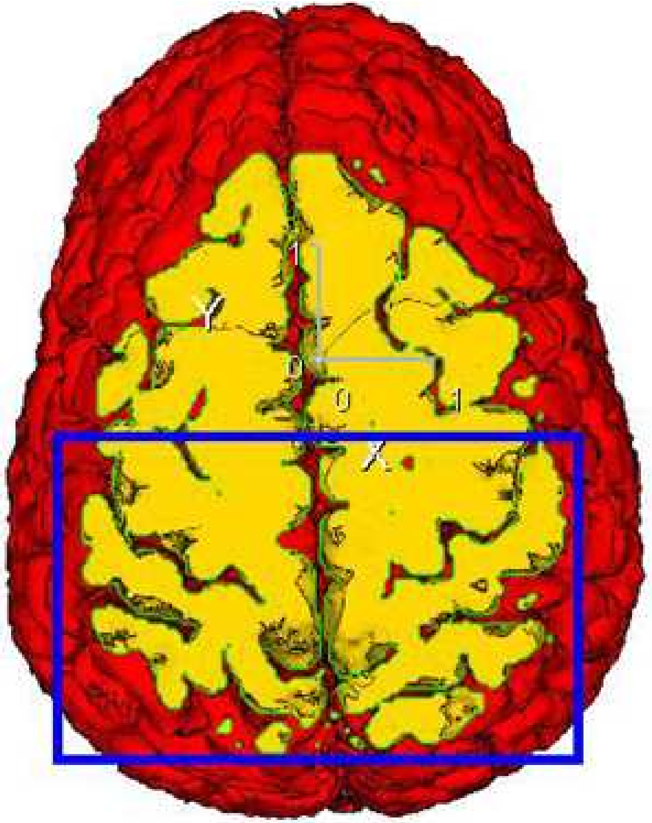
Remark 10 *Note that* **non-simply connected** *patches may be obtained.*



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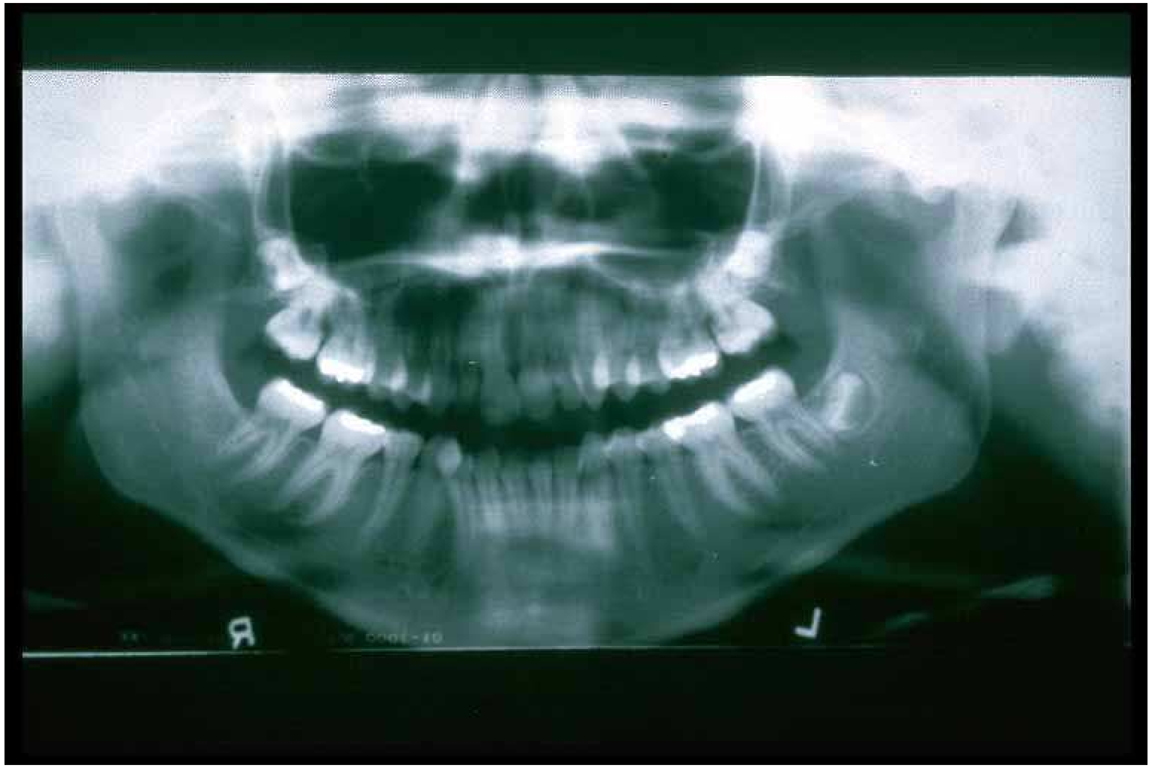
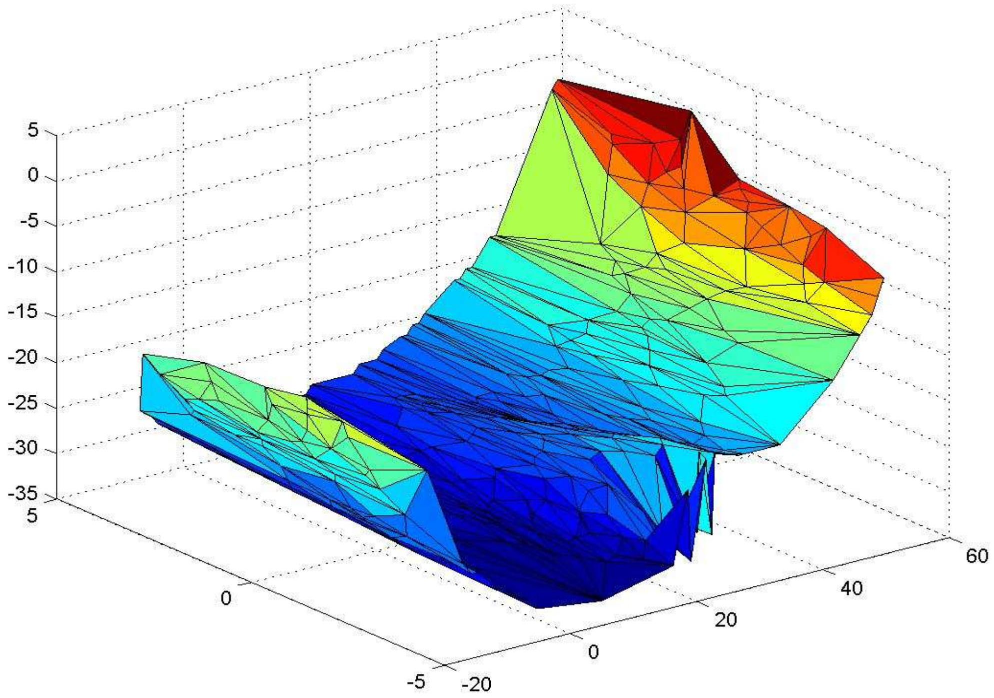
Gluing Different Patches to obtain a Global Flattening:

The need for gluing patches together into a global picture is well known in Radiography as "*pantomograph*"

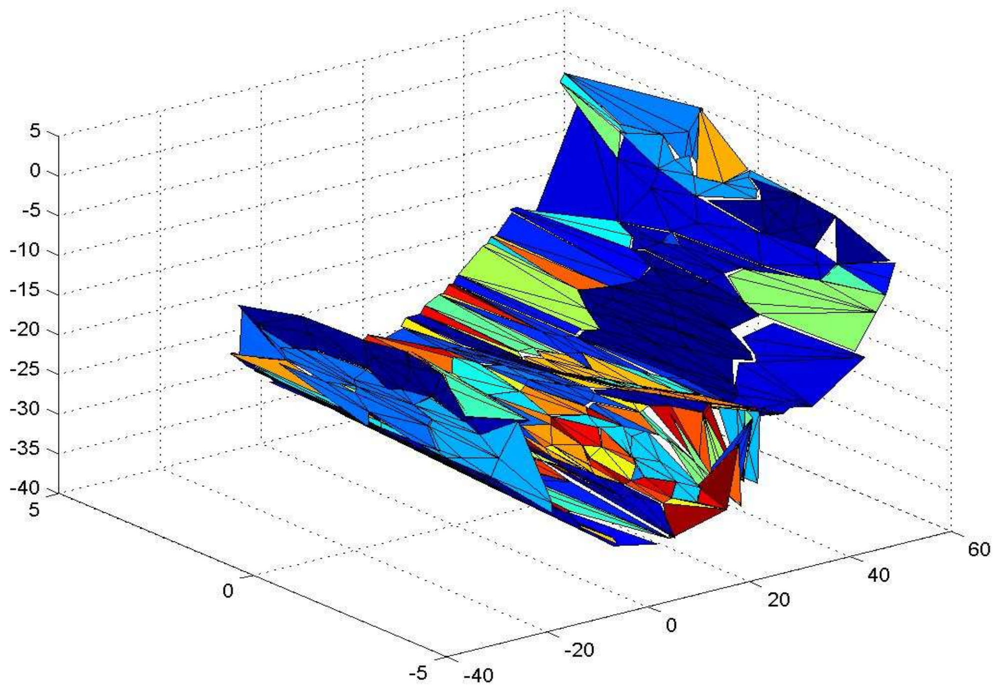


This is done very **approximatively** and with **no control of the dilatation**.

We have applied a “naive” (but **with** dilatation control) gluing process to the triangulated surface obtained from 3 slices of human colon scan:



(a)

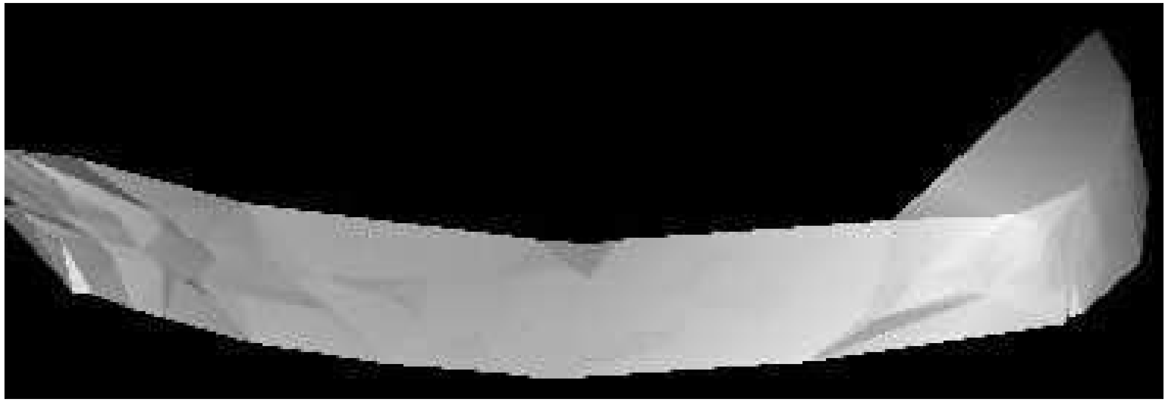


(b)

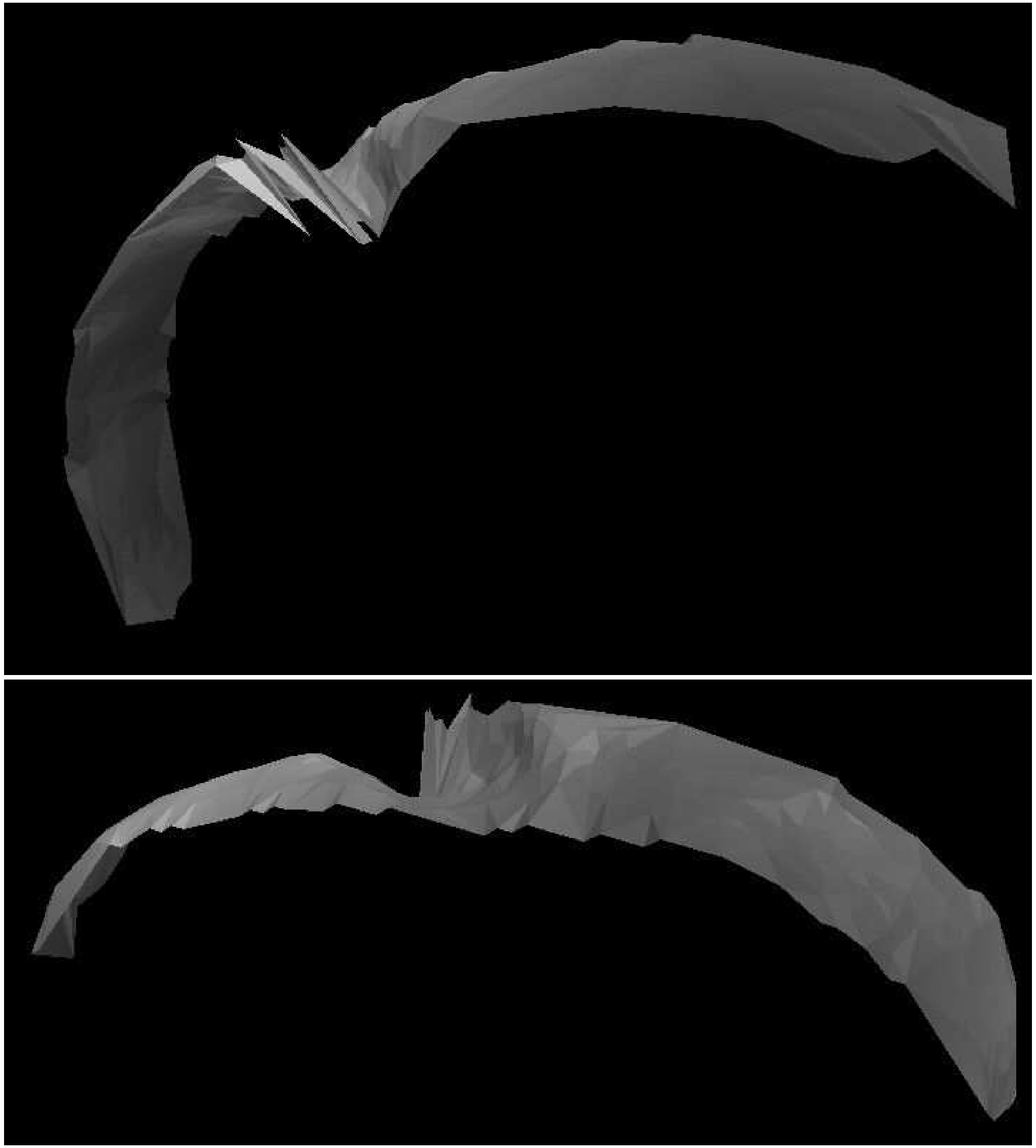
The reason for these "cuts" and "holes" resides in the fact that (evidently) one can have two neighbouring patches, with markedly different dilatations/distorsions, which results in different lengths for the common boundary edges. Therefore, "cuts" and "holes" appear when applying a "naive" gluing – as the colon flattening example shows.

The discontinuities appear at the common boundary of two patches obtained from regions with very different curvature.

Indeed, the "back part" seems close enough to be half of a cylinder (and thus *developable*)...



...but in fact it is highly folded:



Concluding Remarks

- The proposed algorithm is **local** but it gives a *measure of globality*.

- Our algorithm is best suited for flattening of highly folded surfaces.

- The theory and algorithm guarantee minimal (and computable!) metric, angular and area distortion.

- Relatively simple – yet correct(!), robust and computationally efficient, since it does not require computations of derivatives.

- Holds in **any** dimension.

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