HIGH RATE RECONSTRUCTION OF RANDOM SIGNALS FROM GENERALIZED SAMPLES

SampTA 2007 – June 1 - 5, 2007, Thessaloniki, Greece

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Abstract

We address the problem of reconstructing a random signal from samples of its filtered version using a given interpolation kernel. In order to reduce the mean squared error (MSE) when using a non-optimal interpolation kernel we propose using a scheme with reconstruction rate that is greater than the sampling rate. A digital correction system that processes the samples prior to their multiplication with the shifts of the interpolation kernel is developed. This system is constructed such that the reconstructed signal is the linear minimum MSE estimate of the original signal given its samples. Simulations, as well as theoretical arguments, confirm the reduction in MSE with respect to a system with equal rates of sampling and reconstruction. An explicit condition is also derived such that the optimal MSE is achieved with the given kernel.

1. Introduction

The study of sampling random signals was initiated in the late 1950's by Balakrishnan [1]. His well known sampling theorem states that a bandlimited wide sense stationary (WSS) random signal $x(t)$ can be perfectly reconstructed in a mean squared-error (MSE) sense from its ideal samples whenever the sampling rate exceeds twice the signal's bandwidth. The reconstruction is achieved by using the sinc function as an interpolation kernel. In practice, though, the signal is never perfectly bandlimited and the sampling device is not ideal, i.e. it does not produce the exact values of the signal at a uniform set of locations. A common situation is that the sampling device integrates the signal, usually over small neighborhoods around the sampling locations. Moreover, the use of the sinc kernel for reconstruction is usually not feasible as it has a very slow decay.

Balakrishnan's result was later extended by several authors to account for some of its practical limitations. In [2] a sampling theorem for multiband WSS signals was developed. It was shown that under certain conditions on the support of the signal's spectrum $\Lambda_{xx}(\omega)$, perfect reconstruction in an MSE sense is achievable by using an interpolation filter with the same support. This was a first departure from the bandlimited case to more general types of random signals.

A more general setup was considered in [3], where no limitation on the signal's spectrum is imposed and the sampling device produces nonideal samples, i.e. samples of a filtered version of the signal. Clearly this setting does not always allow for perfect reconstruction. The strategy developed in [3] was the minimization of the MSE between the original and reconstructed signals. A similar setup was also treated in [4] where a random signal $x(t)$ is estimated from the samples of another random signal $y(t)$. This result generalizes [3] in that one can set $y(t)$ to be the convolution of $x(t)$ with the impulse response of the sampling device to obtain [3]. We refer to this system as the hybrid Wiener filter as it operates on a discrete-time signal whereas its output is a continuous-time signal. The reconstruction in the hybrid Wiener filter setup is obtained, as in the standard case, by modulating the shifts of a properly designed interpolation kernel with the samples of the signal.

In many practical applications the interpolation kernel is either given in advance or there is limited ability to shape its frequency response (e.g. in the case of analog filters). In these settings a more appropriate approach to reconstruction of signals from their samples is to confine the system to use a predefined interpolation kernel. In order to obtain a "good" reconstruction in this setup, one can employ a digital correction system that processes the samples and produces the expansion coefficients, as depicted in Fig. 1. This scheme was first introduced in [5] where the authors considered a stochastic setting. A rigorous treatment of this system from a deterministic viewpoint was given in [6] and [7]. In [8] several approaches to the design of a digital correction filter were developed and compared, including both deterministic and stochastic formulations.

The constraint to a predefined interpolation kernel may lead to severe degradation in the MSE of the reconstruction. This emphasizes the fundamental trade-off between performance and implementation considerations. An intriguing question that arises, then, is whether one can improve the MSE of such a sampling-reconstruction system by modifying the reconstruction mechanism. In this paper we suggest compensating for the non-ideal behavior of the given interpolation kernel by using a higher reconstruction rate. Specifically, we consider a reconstruction rate that is an integer multiple of the sampling rate. This new setting no longer allows the use of a linear time-invariant (LTI) digital correction system but rather forces the use of a multirate system.

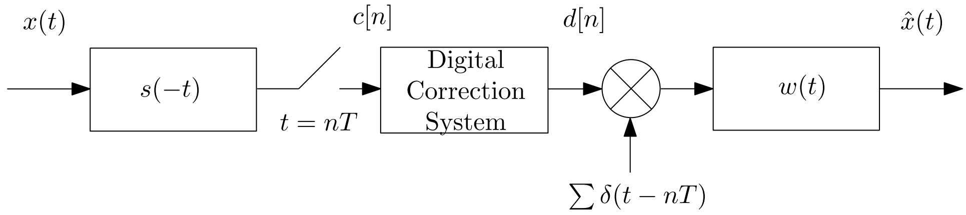


Figure 1: Sampling and reconstruction setup.

Our proposed framework can be viewed as a generalization of the widely practiced methods for sampling rate conversion, known as first and second order approximation [9]. These methods correspond to a rectangular and triangular interpolation filter respectively, and a correction system in the form of a polyphase filter structure. However, besides extending the discussion to general interpolation filters, in this work we also relax the standard assumption that the input signal is bandlimited. Furthermore, as stated above, we take a stochastic viewpoint so that we design a correction system that is best adapted to the input signal's spectrum. This is in contrast to the common deterministic formulation [9].

The paper is organized as follows. In Section 2 we briefly present the hybrid Wiener filtering problem and its solution. We also present the high interpolation rate scheme and discuss its differences with respect to the hybrid Wiener filter approach. In Section 3 we discuss the problematic nature of the MSE as a measure to be minimized in our framework. This motivates the use of an alternative error measure called the average MSE. We further address the well known phenomena of artifacts in the reconstructed signal, caused as a side effect of minimizing the MSE. This is done by studying the statistical properties of the reconstructed signal. In Section 4 an explicit expression for the digital correction system as a function of the sampling filter, the reconstruction filter and the signal's spectrum is derived. This section also includes an investigation of the special case $K = 1$, in which our system is shown to become identical to that developed in [8]. An error analysis of our scheme is presented in Section 5. As a special case we obtain expressions for the MSE in the standard sampling scheme both with a predefined and with the optimal reconstruction kernel. This enables us to obtain necessary and sufficient conditions for perfect recovery of a signal from its nonideal samples. We also show in this section in what cases our system completely compensates for the nonideal interpolation kernel and produces the optimal solution. Finally, in Section 6 we confirm the efficiency of our approach with simulations.

Proofs and detailed derivations are omitted throughout the paper due to lack of space. They can be found in [12].

2. The Hybrid Wiener Filter and the High Rate Interpolation Scheme

2.1 The Hybrid Wiener Filter

We begin by reviewing the hybrid Wiener solution and discuss its application to the recovery of a random signal from its nonideal samples. The hybrid Wiener filtering problem, in its most general form, is the following. We wish to linearly estimate the WSS random signal $x(t)$ given the equidistant samples of another random signal $y(t)$ such that the MSE $E[|x(t) - \widehat{x}(t)|^2]$ is minimized for every t. The spectrum of $y(t)$ and the cross spectrum of $x(t)$ and $y(t)$ are assumed to be known and are denoted by $\Lambda_{yy}(\omega)$ and $\Lambda_{xy}(\omega)$ respectively. The term "hybrid" refers to the fact that the input to the estimator is the discrete-time signal $y(nT)$, $n \in \mathbb{Z}$, whereas the output is a continuous-time signal $\widehat{x}(t)$, $t \in \mathbb{R}$. Throughout the paper we use a normalized interpolation period of $T = 1$ to simplify the exposition.

Interestingly, the solution to this problem highly resembles the standard Wiener filter [10] and is given by [4]

$$\widehat{x}(t) = \sum_{n\in\mathbb{Z}} y(n)w(t - n), \tag{1}$$

where $w(t)$ is an analog filter whose frequency response is

$$W(\omega) = \frac{\Lambda_{xy}(\omega)}{\sum_{l\in\mathbb{Z}} \Lambda_{yy}(\omega_l)} \tag{2}$$

assuming the denominator is nonzero,and

$$\omega_l = \omega + 2\pi l. \tag{3}$$

As can be seen in (1), the hybrid Wiener solution amounts to a shift-invariant interpolation inbetween the samples of $y(t)$ using the kernel (2).

In our setup, a signal $x(t)$ is sampled after pre-filtering by a filter $s(-t)$, which corresponds to the impulse response of the nonideal sampling device. This is described by setting $y(t) = x(t) * s(-t)$. Substituting the appropriate expressions for $\Lambda_{xy}(\omega)$ and $\Lambda_{yy}(\omega)$ in (2), the optimal reconstruction kernel corresponding to our setup is

$$W(\omega) = \frac{S(\omega)\Lambda_{xx}(\omega)}{\sum\limits_{l\in\mathbb{Z}} |S(\omega_l)|^2 \Lambda_{xx}(\omega_l)}. \tag{4}$$

The denominator of (4) is the spectrum $\Lambda_{cc}\left(e^{i\omega}\right)$ of the sequence of samples $c[n] = (x(t) * s(-t))\,|_{t=n}$. Thus it can be shown that $W(\omega)$ can be chosen arbitrarily for frequencies where the denominator vanishes.

Note that the hybrid Wiener interpolation scheme possesses the same structure as the system depicted in Fig. 1, only without the digital correction system. Nevertheless, it can also be represented as a digital correction filter followed by an analog interpolator, as done in [11]. The optimal interpolation filter used in [11] is

$$W_{opt}(\omega) = S(\omega)\Lambda_{xx}(\omega) \tag{5}$$

and the digital filter has frequency response

$$H_{opt}\left(e^{i\omega}\right) = \frac{1}{\sum\limits_{l\in\mathbb{Z}} |S(\omega_l)|^2 \Lambda_{xx}(\omega_l)}, \tag{6}$$

where, again, $H_{opt}\left(e^{i\omega}\right)$ can be chosen arbitrarily for frequencies at which the denominator is zero. The notation $H\left(e^{i\omega}\right)$ denotes the discrete time Fourier transform (DTFT) of a sequence $h[n]$, which is 2π–periodic in ω.

This representation is not unique because the continuous time Fourier transform of $\widehat{x}(t)$ is related to the DTFT of the sequence $c[n]$ through $\widehat{X}(\omega) = C\left(e^{i\omega}\right) H\left(e^{i\omega}\right) W(\omega)$. Therefore a multiplication of $W(\omega)$ by any non vanishing 2π–periodic function can be compensated for by dividing $H\left(e^{i\omega}\right)$ by the same function. It is thus apparent that by inserting the digital correction filter block to the sampling scheme, we effectively create a set of optimal interpolation kernels, instead of just one. Formally stated, an interpolation filter $W(\omega)$ is optimal if there exists a non vanishing 2π–periodic function $\alpha\left(e^{i\omega}\right)$ such that

$$W(\omega) = \alpha\left(e^{i\omega}\right) S(\omega)\Lambda_{xx}(\omega), \quad \forall\omega\in\Omega_c, \tag{7}$$

where Ω_c is defined by

$$\Omega_c \triangleq \left\{\omega : \sum_{l\in\mathbb{Z}} |S(\omega_l)|^2 \Lambda_{xx}(\omega_l) \neq 0\right\}. \tag{8}$$

If this requirement is not fulfilled then either there are frequencies of the input $x(t)$ that are not reproducible by the system, or the reconstructed signal contains frequency components that are not present in $x(t)$.

We emphasize that (7) describes the set of optimal interpolation filters for a system with a digital correction *filter*. However, it can be shown that even if the restriction that the correction system be LTI is removed then (7) is still a necessary condition [12]. In Section 3 we show that when using a high interpolation rate, condition (7) is relaxed, meaning that the set of optimal interpolation kernels is enlarged.

2.2 High Rate Interpolation Scheme

As opposed to the hybrid Wiener filtering problem, where no constraint is imposed on the linear interpolation system, we wish to reconstruct the signal $x(t)$ from its nonideal samples $c[n]$ using a predefined interpolation kernel $w(t)$. Thus our only freedom is in designing the digital correction system that processes the samples $c[n]$ prior to their modulation of the shifts of the interpolation kernel. Since the filter $w(t)$ is not guaranteed to be of the form (7), it is generally not possible to attain the optimal MSE with a system of this type. In order to compensate for the use of a nonideal interpolation filter we use an interpolation rate which is higher than the sampling rate.

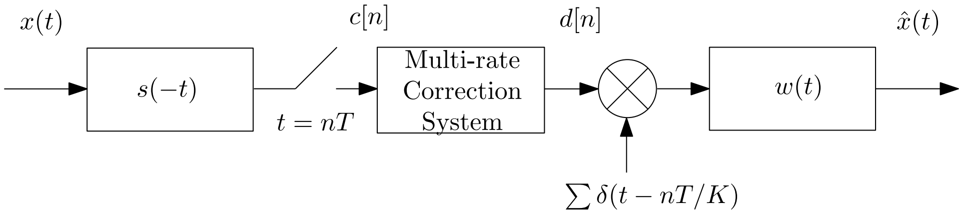


Figure 2: High rate reconstruction setup.

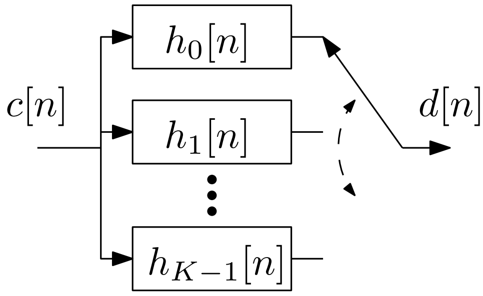


Figure 3: Multi-rate digital correction system.

In our generalization of the standard sampling setup, we cannot impose that the digital correction system be time invariant, as done in [8]. Instead, we are forced to use a multi-rate system. Our sampling-reconstruction setup is depicted in Fig. 2. Note that the rate of the sequence $d[n]$ at the output of the correction system is K times larger than the rate of the sequence $c[n]$ at its input. We seek a linear correction system in the form of a filter bank, as shown in Fig. 3. For every input sample, the commutator in Fig. 3 goes through all K positions, generating K output samples.

3. Definition of an Error Measure

As a first step towards deriving a solution to the K–rate reconstruction problem, we first study the statistical properties of the interpolated signal in the standard case of $K = 1$. This step is crucial in order to pose a proper definition of the error to be minimized. Interestingly, the constraint to a predefined interpolation kernel renders the problem much more challenging than the unconstrained setup. Specifically, it was shown in [8] that it is generally impossible to minimize the pointwise MSE for every t in this setup.

3.1 Average MSE Criterion

The signal $x(t)$ is assumed to be WSS and, as a consequence, the sequence $c[n]$ in Fig. 1 is a discrete WSS random process. Therefore, if the correction system is a digital filter, as used in [5], [8], then $d[n]$ is also WSS.

The reconstructed signal in our system is

$$\widehat{x}(t) = \sum_{n \in \mathbb{Z}} d[n]\, w(t-n). \tag{9}$$

Signals of this type have been studied extensively in the communication literature in the context of pulse amplitude modulation (PAM). It is a known fact that if the sequence $d[n]$ is a WSS process then $\widehat{x}(t)$ is generally not WSS but rather wide sense cyclostationary with period 1 [13]. The non stationary behavior of $\widehat{x}(t)$ is the reason why the pointwise MSE generally cannot be minimized for every t. To overcome this obstacle we can average the pointwise MSE over one sampling period, as done in [14]. Our error measure is thus the sampling-period-average-MSE, which is defined as

$$\mathrm{MSE} = E\left[\int_{t_0}^{t_0+1} |x(t) - \widehat{x}(t)|^2\, dt\right]. \tag{10}$$

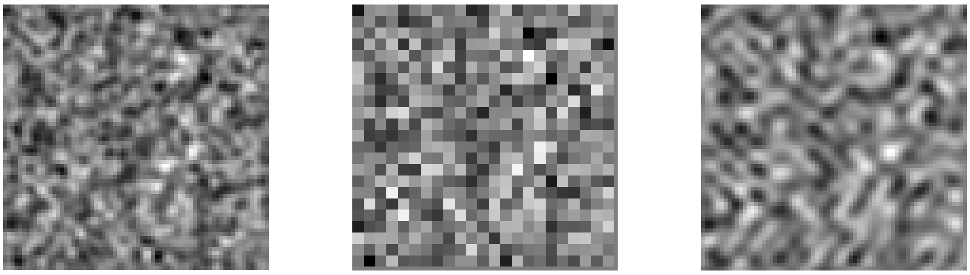


Figure 4: A stationary 2D random process (left) was downsampled by a factor of 3 and then reconstructed using a rectangular kernel (middle) and the sinc kernel (right). The MSE of both reconstructions is the same, however the rectangular kernel introduces block-artifacts.

An important property of the above definition is that in situations where the pointwise MSE *can* be minimized for every t, the minimization of the average MSE (10) leads to the same solution. This follows from the fact that the pointwise MSE is nonnegative for every t. In Section 4 we show that the minimization of (10) leads to a correction system independent of t_0.

We note that when the signals of interest are natural images or audio signals, there is no one-to-one correspondence between the MSE of the reconstruction and its quality, as subjectively perceived by the human visual or auditory system. Despite the fact that natural signals are rarely stationary to begin with, it is still relevant to study how an interpolation algorithm reacts to stationary signals. In fact, if an interpolation scheme outputs a cyclostationary signal when fed with a stationary input, then it will commonly produce reconstructions with degraded subjective quality also when applied to real world signals. We illustrate this in Fig. 4, where a stationary 2D function is downsampled by a factor of 3 and then reconstructed using a rectangular kernel and the sinc kernel. Both interpolation methods lead to the exact same MSE, however the rectangular interpolation filter introduces block structure in the reconstructed image, an artifact which is unpleasant to the human observer. We stress that it is not the scope of this paper to battle these undesired effects. We are merely concerned with the minimization of the MSE. However, it is of interest to study when such artifacts occur. Specifically, we wish to obtain necessary and sufficient conditions on the interpolation kernel and the correction system such that $\widehat{x}(t)$ in (9) is WSS.

3.2 Stationarity of the Reconstruction

One example for construction of a WSS PAM signal (9) is when $d[n]$ is a WSS sequence and $w(t)$ is the bandlimited filter $w(t) = \mathrm{sinc}(t)$ [13]. An important question is whether this is the only case. Specifically, one may suspect that it is possible to counter-balance the periodic correlation induced by the shifts of $w(t)$ by using a non stationary sequence $d[n]$. This would shed doubt on the virtue of using a digital correction filter, as done in [5], [8], as to produce a non stationary sequence $d[n]$ one would have to employ a time-varying digital correction system.

The following theorem gives a necessary and sufficient condition on the random sequence $d[n]$ and on the filter $w(t)$ such that $\widehat{x}(t)$ in (9) is WSS.

Theorem 1 *Consider the signal $\widehat{x}(t)$ in (9). Then $\widehat{x}(t)$ is a continuous time WSS process if and only if*

1. *the sequence $d[n]$ can be written as $d^S[n] + d^N[n]$ where $d^S[n]$ is a WSS sequence whose passband is contained in* $\mathrm{supp}\{W(\omega)\}$ *and $d^N[n]$ is an arbitrary random sequence (not necessarily WSS) with zero energy in* $\mathrm{supp}\{W(\omega)\}$*, and*
2. *the support of the reconstruction filter $W(\omega)$ is contained in the set $[-2\pi + B, 2\pi - B] \cup \Omega_d^c$, where $B \le \pi$ is the bandwidth of $d[n]$ and Ω_d^c is the complementary of the set $\Omega_d \triangleq \mathrm{supp}\left\{\Lambda_{d^S d^S}\left(e^{i\omega}\right)\right\}$.*

We see that the use of a non-stationary sequence $d[n]$ does not aid in imposing that $\widehat{x}(t)$ be stationary. The non-stationary component $d^N[n]$ can contain only frequencies that are suppressed by $W(\omega)$ and thus has no effect on $\widehat{x}(t)$. Furthermore, condition 2 implies that $\widehat{x}(t)$ is B–bandlimited, which means that a PAM signal

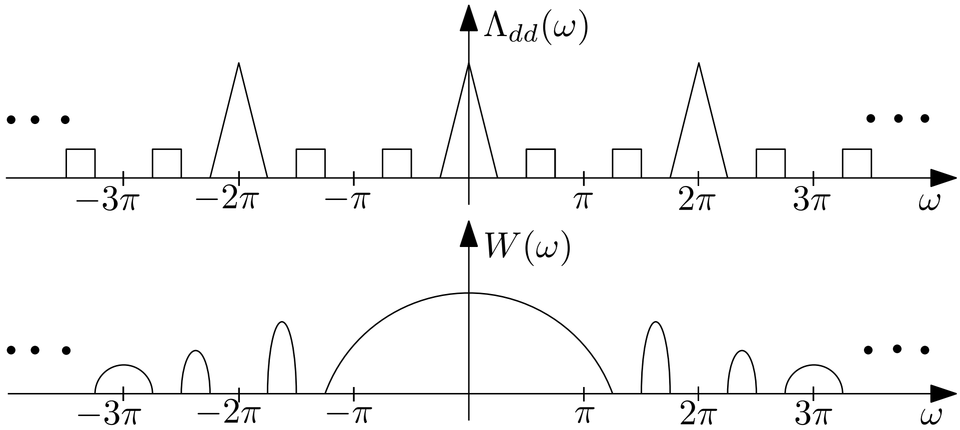


Figure 5: Example of a pair $W(\omega)$, $\Lambda_{dd}\left(e^{i\omega}\right)$ that forms a WSS signal. The support of $\Lambda_{dd}\left(e^{i\omega}\right)$ (top) in the interval $[-\pi,\pi]$ is $\left[-\frac{3}{4}\pi,-\frac{1}{2}\pi\right]\cup\left[-\frac{1}{4}\pi,\frac{1}{4}\pi\right]\cup\left[\frac{1}{2}\pi,\frac{3}{4}\pi\right]$ and hence $B=\frac{3}{4}\pi$. The support of $W(\omega)$ (bottom) then must be contained in the union of $\left[-\frac{5}{4}\pi,\frac{5}{4}\pi\right]$ and 2π translates of $\left[-\frac{1}{2}\pi,-\frac{1}{4}\pi\right]\cup\left[\frac{1}{4}\pi,\frac{1}{2}\pi\right]\cup\left[\frac{3}{4}\pi,\frac{5}{4}\pi\right]$.

cannot be both stationary and nonbandlimited simultaneously. Figure 5 demonstrates a concrete example of a pair $W(\omega)$, $\Lambda_{d^Sd^S}\left(e^{i\omega}\right)$ that forms a WSS signal.

The optimal reconstruction kernel $W_{opt}(\omega)$ of the hybrid Wiener solution (5) generally does not satisfy condition 2. Therefore $\widehat{x}(t)$ is not guaranteed to be stationary when using it. As demonstrated in Fig 4, this can cause undesired effects in the recovered signal.

4. Digital Correction System

In the following theorem we give an analytic expression for the digital correction system that minimizes the MSE in our high rate interpolation scheme.

Theorem 2 *Consider the sampling-reconstruction setup depicted in Fig. 2 with the multirate digital correction system of Fig. 3. Then the correction filters that minimize the average MSE (10) are independent of t_0 and are given by*

$$H_n\left(e^{i\omega}\right)=\sum_{m=0}^{K-1}\frac{\sum\limits_{l\in\mathbb{Z}}S\left(\omega_{m+lK}\right)\Lambda_{xx}\left(\omega_{m+lK}\right)W^*\left(\omega_{m+lK}\right)}{K\sum\limits_{l\in\mathbb{Z}}\left|S\left(\omega_l\right)\right|^2\Lambda_{xx}\left(\omega_l\right)\sum\limits_{l\in\mathbb{Z}}\left|W\left(\omega_{m+lK}\right)\right|^2}e^{\frac{in\omega_m}{K}},\tag{11}$$

where the fraction should be replaced by 0 for frequencies at which the denominator vanishes. Here $H_n\left(e^{i\omega}\right)$ denotes the frequency response of the n'th filter for $n=0,\ldots,K-1$.

We note that there may be frequencies at which there are infinitely many choices of $\left(H_0\left(e^{i\omega}\right),\ldots,H_{K-1}\left(e^{i\omega}\right)\right)$ that lead to the same minimal MSE. Among all possible solutions, the above expression corresponds to the set of filters whose sum of L_2 norms, i.e. $\sum_m\int_{-\pi}^{\pi}\left|H_m\left(e^{i\omega}\right)\right|^2d\omega$, is minimal.

4.1 Equal-Rates of Sampling and Reconstruction

The special case of reconstruction rate that equals the sampling rate can be easily obtained from (11) by setting $K=1$. In this case, the (single) correction filter is given by

$$H\left(e^{i\omega}\right)=\frac{\sum\limits_{l\in\mathbb{Z}}S\left(\omega_l\right)\Lambda_{xx}\left(\omega_l\right)W^*\left(\omega_l\right)}{\sum\limits_{l\in\mathbb{Z}}\left|S\left(\omega_l\right)\right|^2\Lambda_{xx}\left(\omega_l\right)\sum\limits_{l\in\mathbb{Z}}\left|W\left(\omega_l\right)\right|^2}.\tag{12}$$

This filter coincides with the one developed in [8], which minimizes an error measure called the projected MSE.

5. Error Analysis

The average MSE of the reconstruction in our high interpolation rate scheme is given in the next theorem.

Theorem 3 *Consider the setup of Theorem 2 with the digital correction filters given by (11). Then the average MSE (10) of the reconstruction is given by*

$$MSE = R_{xx}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=0}^{K-1}\frac{\left|\sum_{l\in\mathbb{Z}} S(\omega_{m+lK})\Lambda_{xx}(\omega_{m+lK})W^{*}(\omega_{m+lK})\right|^{2}}{\sum_{l\in\mathbb{Z}}|S(\omega_l)|^{2}\Lambda_{xx}(\omega_l)\sum_{l\in\mathbb{Z}}|W(\omega_{m+lK})|^{2}}d\omega, \tag{13}$$

where $R_{xx}(t)$ is the autocorrelation function of the signal $x(t)$. The fraction in (13) should be replaced by 0 *for frequencies at which the denominator vanishes.*

We now study how the interpolation rate K and reconstruction filter $W(\omega)$ affect the MSE. We do so by looking at various special cases.

5.1 The Standard Sampling Setup with a Predefined Kernel

The standard sampling setup corresponding to $K = 1$ was considered in [8] however no explicit formula was given for the resulting MSE. Setting $K = 1$ in (13), the MSE is given by

$$\mathrm{MSE} = R_{xx}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|\sum_{l\in\mathbb{Z}} S(\omega_l)\Lambda_{xx}(\omega_l)W^{*}(\omega_l)\right|^{2}}{\sum_{l\in\mathbb{Z}}|S(\omega_l)|^{2}\Lambda_{xx}(\omega_l)\sum_{l\in\mathbb{Z}}|W(\omega_l)|^{2}}d\omega. \tag{14}$$

In [14, theorem 3] the average MSE of a scheme with equal rates of sampling and interpolation is analyzed. This scheme comprises given sampling and interpolation filters but, unlike our setup, no digital correction system. Formula (14) can be shown to coincide with [14, theorem 3] if we incorporate the effect of the correction filter into the interpolation kernel and define an effective reconstruction filter as $W_{eff}(\omega) = H\left(e^{j\omega}\right)W(\omega)$.

5.2 The Hybrid Wiener Filter and Perfect Reconstruction

The MSE of the hybrid Wiener filter can be calculated from (14) by substituting the optimal reconstruction kernel (5) for $W(\omega)$, resulting in

$$\mathrm{MSE}_{\mathrm{opt}} = R_{xx}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sum_{l\in\mathbb{Z}}|S(\omega_l)|^{2}\Lambda_{xx}^{2}(\omega_l)}{\sum_{l\in\mathbb{Z}}|S(\omega_l)|^{2}\Lambda_{xx}(\omega_l)}d\omega. \tag{15}$$

It should be noted that in [4] an expression for the pointwise MSE $E[|x(t) - \widehat{x}(t)|^2]$ of the hybrid Wiener filter is derived. The formula given in [4] is different than (15) for two reasons. First, recall that (15) gives the average MSE and not the pointwise MSE. Second, the expression given in [4] is wrong. This is because in the derivations of the MSE the author made the implicit assumption that the pointwise MSE is time independent and substituted $t = 0$. Practically, the formula in [4] gives the pointwise MSE at integer times but not for the entire continuum, i.e. $E[|x(n) - \widehat{x}(n)|^2], \quad n \in \mathbb{Z}$.

Equation (15) can be used to obtain necessary and sufficient conditions on the sampling filter $S(\omega)$ and on the spectrum $\Lambda_{xx}(\omega)$ such that the signal $x(t)$ can be perfectly reconstructed from its nonideal samples. This is done by studying in what cases $\mathrm{MSE}_{\mathrm{opt}} = 0$. Not surprisingly, this gives rise to a condition on the passband of $x(t)$, as described in the following corollary.

Corollary 1 *A WSS signal $x(t)$ with spectrum $\Lambda_{xx}(\omega)$ can be linearly perfectly reconstructed from samples of its filtered version $(x(t) * s(-t))|_{t=n}$ if and only if*

1. *$S(\omega) \neq 0$ for every $\omega \in supp\{\Lambda_{xx}(\omega)\}$ and*
2. *distinct 2π–shifted replicas of $\Lambda_{xx}(\omega)$ do not overlap, i.e. $\sum_{l\neq 0}\Lambda_{xx}(\omega_l) = 0, \quad$ for every $\omega \in \mathrm{supp}\{\Lambda_{xx}(\omega)\}$.*

A necessary and sufficient condition that allows to perfectly recover a WSS signal from its *ideal* samples was given in [2]. This condition can be obtained as a special case of Corollary 1 by regarding $S(\omega)$ as an all-pass filter, i.e. $S(\omega) \equiv 1$. In this case the only condition is that 2π–translates of the spectrum $\Lambda_{xx}(\omega)$ are disjoint. When the sampling is not ideal we have the additional requirement that the sampling filter does not zero out any frequency components contained in $x(t)$.

5.3 Optimal Reconstruction Using High Interpolation Rate

An interesting question is when our high rate interpolation scheme (with a pre-specified interpolation filter $W(\omega)$) attains the optimal MSE. In such cases, our scheme allows to bypass the need for designing the analog interpolation filter without any increase in MSE. Clearly, our scheme is optimal when MSE of (13) is equal to $\mathrm{MSE}_{\mathrm{opt}}$ of (15). The next theorem specifies the property that the interpolation filter $W(\omega)$ must possess in order for that to happen.

Theorem 4 *The high rate interpolation scheme depicted in Fig. 2 and Fig. 3 with the correction filters given in (11) attains the minimal average MSE attainable by any linear system if and only if there exists a non vanishing $2\pi K$–periodic function $\alpha\left(e^{i\omega/K}\right)$ such that*

$$W(\omega) = \alpha\left(e^{i\omega/K}\right) S(\omega)\Lambda_{xx}(\omega), \quad \forall\omega \in \Omega_c, \tag{16}$$

where Ω_c is defined by (8).

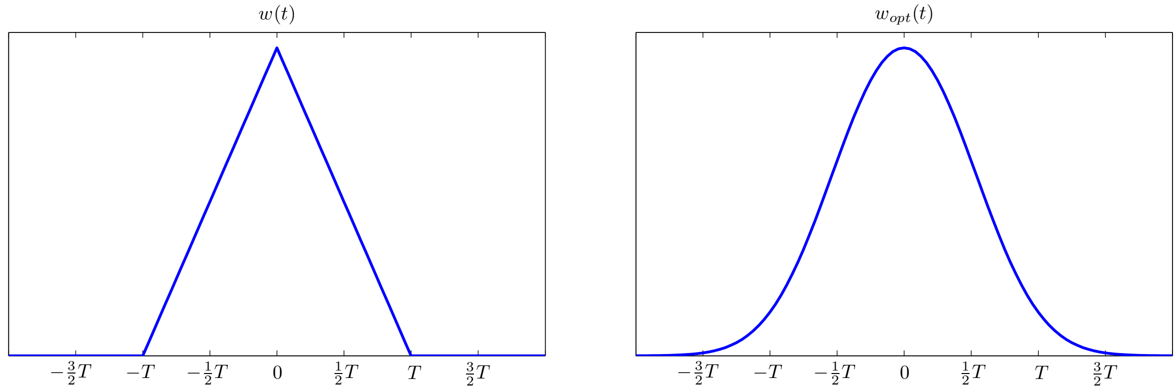
Condition (16) is a generalization of (7), which was developed for $K = 1$. This requirement provides the essential justification for using the high rate reconstruction scheme. Specifically, it states that the set of optimal kernels becomes larger as the interpolation rate is increased. In practice, for a large enough rate one may use almost any reasonable interpolation kernel and attain an MSE which is very close to $\mathrm{MSE}_{\mathrm{opt}}$. We also remark that when the reconstruction filter $W(\omega)$ satisfies condition (16), the high rate interpolation scheme not only minimizes the average MSE but also the pointwise MSE.

To illustrate the strength of our method, let us consider the case where the input signal $x(t)$ is B–bandlimited, i.e. $\Lambda_{xx}(\omega) = 0$ for $|\omega| > B$, where B may be greater than π. In this case the optimal interpolation kernel $W_{opt}(\omega)$ of the hybrid Wiener filter is a lowpass filter with cutoff frequency B. Now, suppose that $W_{opt}(\omega)$ is hard to implement. From (16) we see that any B–bandlimited reconstruction filter $W(\omega)$ can be used to attain the minimal MSE given that it does not vanish in the support of $S(\omega)\Lambda_{xx}(\omega)$ and that the interpolation rate satisfies $K \geq B/\pi$. This is because in this case $2\pi K \geq 2B$ and thus any such $W(\omega)$ can be written as a multiplication of $S(\omega)\Lambda_{xx}(\omega)$ and a non vanishing $2\pi K$ periodic function. We conclude that for bandlimited input signals it is possible to attain the minimal MSE with *any bandlimited reconstruction kernel* that does not vanish in the support of $W_{opt}(\omega)$, simply by increasing the reconstruction rate.

6. Simulations

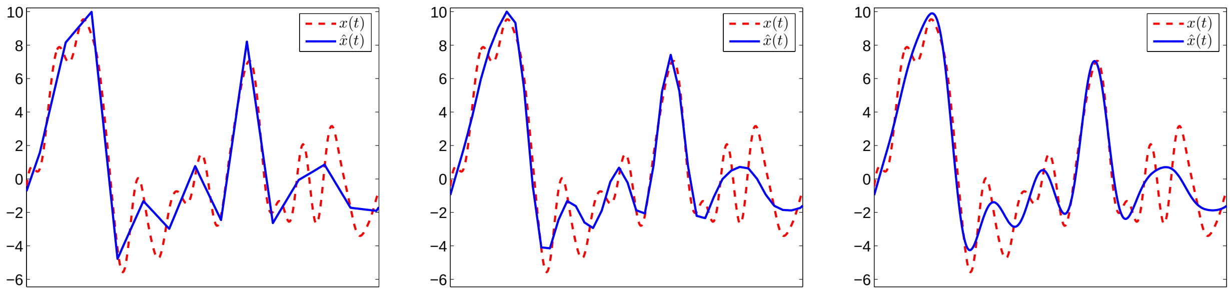
In order to confirm the efficiency of our proposed scheme, we generated a discrete time Gaussian random process $x[n]$, filtered it with a pre-filter $s[-n]$ and then down sampled it with sampling period $T = 24$ to obtain a sequence of samples $c[n]$. Our purpose was to reconstruct the original signal using the pre-specified interpolation kernel $w[n]$ shown in Fig. 6(a). This interpolation kernel corresponds to linear interpolation when using a reconstruction period of T. The filter $w[n]$ has a fast decay with respect to the optimal interpolation kernel, which is depicted in Fig. 6(b). Figure 7(a) shows the MMSE reconstruction with an interpolation period that equals the sampling period T (i.e. $K = 1$) and with the correction filter (7), as proposed in [8]. Graphs (b) and (c) in Fig. 7 depict the reconstructions obtained by the high rate interpolation scheme proposed in this paper for $K = 3$ and $K = 24$ respectively. It can be seen that for low reconstruction rates, the interpolated signal exhibits artifacts in the form of non-continuity of its derivative. As the reconstruction rate increases, these undesired effects become less dominant. The result in Figure 7(c) is exactly identical to the reconstruction that is obtained using the optimal interpolation kernel (with a reconstruction period of T).

Figure 8(a) shows the average MSE attained by the high-rate interpolation scheme as a function of K. The dashed line is $\mathrm{MSE}_{\mathrm{opt}}$ of the hybrid Wiener filter. The MSE of the standard sampling scheme ($K = 1$) is roughly 5% higher than $\mathrm{MSE}_{\mathrm{opt}}$. However, an increase of the interpolation rate by a factor of $K = 4$ is enough to close most of the gap in this case.



(a) (b)

Figure 6: (a) Given interpolation kernel; (b) Optimal interpolation kernel.



(a) (b) (c)

Figure 7: (a) Reconstruction with $K = 1$; (b) Reconstruction with $K = 3$; (c) Reconstruction with $K = 24$. This result is identical to the one obtained with the optimal interpolation kernel shown in Fig. 6(b).

Figure 8(b) shows the pointwise MSE of the hybrid Wiener filter as a function of time. This figure illustrates that even when using the optimal interpolation kernel, the reconstructed signal may be highly non-stationary. In this case the pointwise MSE at times $\{lT\}_{l\in\mathbb{Z}}$ is lower than the pointwise MSE at times $\{(l+1/2)T\}_{l\in\mathbb{Z}}$ by a factor of 6. As explained in Section 3, this can cause undesired artifacts in images or audio signals. One could eliminate this effect by using an interpolation kernel that is π−bandlimited. Nevertheless, while suppressing non-stationarity, this would result in a higher MSE.

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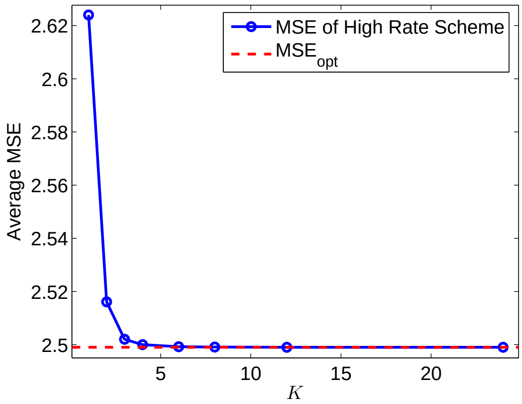
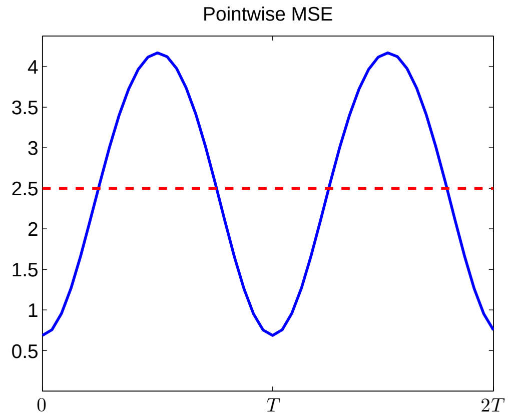


(a)



(b)

Figure 8: (a) Average MSE as a function of K; (b) Pointwise MSE as a function of time for interpolation with the optimal reconstruction kernel.

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