Exercise 5: Kernel Density Estimation (Due 20/6/2017)*

Statistical Methods in Image Processing 048926

Density Estimation (30 points)

The multivariate kernel density estimation of a density $f(\mathbf{x})$ is given by

$$\hat{f}(\mathbf{x}) = \frac{1}{N}\frac{1}{|H|}\sum_{i=1}^{N} K(H^{-1}(\mathbf{x}_i - \mathbf{x})),$$

where $\mathbf{H}$ is a bandwidth matrix, $K(\cdot)$ is the kernel and $\{\mathbf{x}_i\}_{i=1}^{N}$ are i.i.d. samples drawn from $f(\mathbf{x})$. Consider the density from HW2:

$$f(\mathbf{x}; \{\mu_i\}) = \sum_{i=1}^{N} \frac{1}{N}\frac{1}{2\pi}\exp\left\{-\frac{1}{2}||\mathbf{x} - \mu_i||^2\right\},$$

with $N = 4$ and $\mu = \{(0,0)^T, (0,2)^T, (2,0)^T, (2,2)^T\}$.

1. Draw $N = 10000$ samples $\mathbf{x}_i$ from $f(\mathbf{x}; \{\mu_i\})$ using the function that was implemented in HW2.
2. Write a function that accepts samples $\{\mathbf{x}_i\}$ and a bandwidth matrix $\mathbf{H}$ and returns $\hat{f}(\mathbf{x})$. Use the two-dimension separable kernel $K(\mathbf{u}) = k(u_1)k(u_2)$ where $k(u) = \frac{1}{\sqrt{2\pi}}\exp\{\frac{-u^2}{2}\}$.
3. Compare between $f(\mathbf{x})$ and $\hat{f}(\mathbf{x})$ using the bandwidth matrices $\mathbf{H} = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$ with $h = 0.1, 0.5, 1$. Discuss the trade-off of the choice of h.

*Please send your solutions to Tamar

Denoising (70 points)

In this section you will implement external and internal image denoising algorithms. Given a noisy image $y = x + n$, where $n \sim \mathcal{N}(0, \sigma^2 I)$, these algorithms aim to estimate a clean image $\hat{x}$.

External Denoising:

1. Given a patch y_j from the noisy image, and a dataset of clean patches $\{x_i\}_{i=1}^N$, the MMSE estimator of the clean patch $\hat{x}_j$ can be approximated as

$$\hat{x}_j = \sum_{i=1}^{N} x_i w_i \,, \quad w_i = \frac{\exp\{-\frac{1}{2\sigma^2}\|y_j - x_i\|^2\}}{\sum_{m=1}^{N} \exp\{-\frac{1}{2\sigma^2}\|y_j - x_m\|^2\}}. \tag{1}$$

   Write a function that accepts a noisy patch y_j and a data-set of clean patches $\{x_i\}$ and returns the MMSE estimator of the patch $\hat{x}_j$.

2. Write a function that accepts a set of overlapping patches and the size of the original image and constructs a full image by averaging all the overlapping patches (you can use the same function you implemented for EPLL in HW4).

3. Implement the full Denoising scheme. Use the image 'camaraman.tif' as x and construct y by adding white gaussian noise with $\sigma = 25$. Generates the patch dataset from all 5×5 overlapping patches in the remaining four images. For each of the 5×5 overlapping patches of y, find its estimator $\hat{x}_j$ using 1. Construct the full image $\hat{x}$ using 2. Display the original image, the noisy image and the denoised image. Compare the PSNR before and after Denoising.

Internal Denoising:

1. Repeat 3, this time using all overlapping patches from the noisy image y as the patch dataset. Use $\tilde{\sigma} = \sqrt{2}\sigma$ instead of σ in (1) (why?) and compute the sum only over $i \neq j$. Compare the results of external and internal denoising. Discuss the differences between the two methods.

2. Repeat 3, this time using all overlapping patches from the $\times 2$ scaled-down version of the noisy image y as the patch dataset. Use $\tilde{\sigma} = \sqrt{1.25}\sigma$ instead of σ in (1) (why?). Compare the results of internal denoising in different scales. Discuss the differences between the two methods.

3. The *nonlocal means* algorithm works slightly differently. It is based on the observation that the MMSE estimate $\hat{x}_n$ of **pixel** (not patch!) x_n based on the neighborhood pixels y_{N_n} of the corresponding noisy pixel y_n, can be approximated as

$$\hat{x}_n = \sum_{m=1, m\neq n}^{M} y_m w_m(y) \,, \quad w_m = \frac{\exp\{-\frac{1}{2h^2}\|y_{N_m} - y_{N_n}\|^2\}}{\sum_{l=1}^{L} \exp\{-\frac{1}{2h^2}\|y_{N_n} - y_{N_l}\|^2\}},$$

where M is the number of pixels within y and h is a kernel bandwidth. Note that y_{N_k} are all the neighboring pixels of y_k, *not including* y_k, in a neighborhood of size $N \times N$. In other words, the clean pixel $\hat{x}_n$ is estimated by a weighted average of all the pixels in y, where the weights are calculated according to the similarities between neighborhoods.

(a) Write a function that accepts a noisy pixel y_n and the noisy image y and returns the clean pixel estimate $\hat{x}_n$.

(b) Implement the full Denoising scheme. Use the image 'camaraman.tif' as x and construct y by adding white gaussian noise with $\sigma = 25$. For each of the pixels in y, find the estimator $\hat{x}_n$ using 3a with $N = 5$ and $h = \sqrt{2}\sigma$. Construct the full image $\hat{x}$. Display the noisy image and the denoised image. Compare the PSNR before and after Denoising. Compare the results of using center pixels (non-local means denoising) and patch-averaging.