Reconstruction of Nonuniformly Sampled Bandlimited Signals by Means of Digital Fractional Delay Filters

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***Abstract*—This paper considers the problem of reconstructing a class of nonuniformly sampled bandlimited signals of which a special case occurs in, e.g., time-interleaved analog-to-digital converter (ADC) systems due to time-skew errors. To this end, we propose a synthesis system composed of digital fractional delay filters. The overall system (i.e., nonuniform sampling and the proposed synthesis system) can be viewed as a generalization of time-interleaved ADC systems to which the former reduces as a special case. Compared with existing reconstruction techniques, our method has major advantages from an implementation point of view. To be precise, 1) we can perform the reconstruction as well as desired (in a certain sense) by properly designing the digital fractional delay filters, and 2) if properly implemented, the fractional delay filters need not be redesigned in case the time skews are changed. The price to pay for these attractive features is that we need to use a slight oversampling. It should be stressed, however, that the oversampling factor is less than two as compared with the Nyquist rate. The paper includes error and quantization noise analysis. The former is useful in the analysis of the quantization noise and when designing practical fractional delay filters approximating the ideal filters.**

***Index Terms*—Bandlimited signals, digital filters, fractional delay filters, nonuniform sampling, time-interleaved analog-to-digital converters, time-skew errors.**

I. INTRODUCTION

In uniform sampling, a sequence $x(n)$ is obtained from an analog[1] function $x_a(t)$ by sampling the latter equidistantly at $t = nT$, i.e., $x(n) = x_a(nT)$, as illustrated in Fig. 1(a). In this case, the time between two consecutive sampling instances is always T. In nonuniform sampling, on the other hand, the time between two consecutive sample instances is dependent on the sampling instances. In this paper, we deal with the situation where the samples can be separated into N subsequences $x_k(m)$, $k = 0, 1, \ldots, N-1$, where $x_k(m)$ are obtained by sampling $x_a(t)$ with the sampling rate $1/(MT)$ at $t = mMT + t_k$, i.e., $x_k(m) = x_a(mMT + t_k)$, with t_k being referred to as time skews. This sampling scheme is illustrated in Fig. 1(b) for $M = N = 2$. Such nonuniformly sampled signals occur in, e.g., time-interleaved analog-to-digital converter (ADC) systems (where $N = M$) due to time-skew errors [1]; see also Section V.

Manuscript received March 20, 2001; revised June 27, 2002. The associate editor coordinating the review of this paper and approving it for publication was Dr. Helmut Bölcskei.

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Digital Object Identifier 10.1109/TSP.2002.804089.

[1] In this paper, it is understood that analog signals are continuous-time signals, whereas digital signals are discrete-time signals.

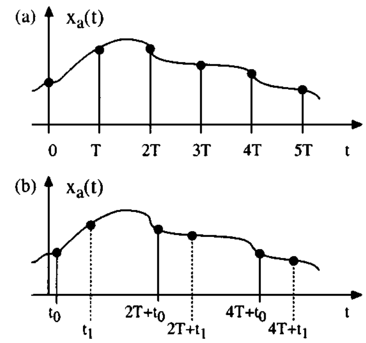

Fig. 1. (a) Uniform sampling. (b) Nonuniform sampling.

A question that arises is how to recover $x_a(t)$ from $x_k(m)$. This can, in principle, be done in two different ways. The first is to reconstruct $x_a(t)$ directly from $x_k(m)$ through analog reconstruction functions. The second way is to first recover $x(n)$ and then employ reconstruction methods used for uniformly sampled signals, e.g., a conventional digital-to-analog converter (DAC) followed by an analog reconstruction filter. It is well known that if $x_a(t)$ is bandlimited to $|\omega| < \pi/T$, then $x_a(t)$ can be recovered from $x_k(m)$ as long as the average sampling rate equals or exceeds the Nyquist rate $1/T$. In other words, $x_a(t)$ can be retained when $N = M$, provided that t_k are distinct. Methods for retaining $x_a(t)$ directly via analog interpolation functions can be found in [2]–[6] and references therein. Techniques for recovering $x(n)$ have been treated in [6]–[8]. It should also be noted that there exist methods that recover $x(n)$ in the special case where $t_k = d_kT$, d_k integers (see e.g., [9]–[11]), but those methods are not applicable here since we allow t_k to be arbitrary (distinct) real numbers.

It is thus well known how to, in principle, retain $x_a(t)$, or $x(n)$ from $x_k(m)$. However, when it comes to practical implementations, we are facing new problems. For instance, it is very difficult to practically implement analog functions with high precision. It is therefore desired to do the reconstruction in the digital domain, i.e., to first recover $x(n)$. We then need only one conventional DAC and one analog filter to obtain $x_a(t)$, which are much easier to implement than M analog functions. In [6], it was shown that $x(n)$ can be retained using a digital synthesis filterbank with ideal noncausal multilevel filters. The issue of using practical causal filters approximating the ideal ones was, however, not treated. That is, it is not known how well a "practical version" of that solution will behave. Another method, which was introduced in [7], employs causal interpolation functions, but it is not clear how to select them so that the output $y(n)$ from the reconstruction system approximates (in some sense) $x(n)$ as close as desired. Yet another approach in the digital domain was proposed in [8], but that is a frequency domain approach that only recovers the spectrum at a finite number of frequencies. It should also be noted that the above-mentioned techniques have a disadvantage in that the interpolation functions (filters) involved need to be changed (redesigned) when the time skews t_k are altered.

In this paper, we introduce a new synthesis system for recovering $x(n)$ from $x_k(m)$. This system makes use of a filterbank composed of digital fractional delay filters with, in general, different gain constants. The overall system (i.e., nonuniform sampling and the proposed system) can be viewed as a generalization of time-interleaved ADC systems, to which the former reduces as a special case. At first sight, it may appear to be a severe limitation to fix the filters to fractional delay filters. There is, however, a major advantage of this approach, whereas the disadvantage is tolerable. The advantage of the proposed system is that it is very attractive from an implementation point of view. To be precise, it has the following features: 1) We can make $y(n)$ approximate $x(n)$ as close as desired by properly designing the digital fractional delay filters, and 2) if properly implemented, the fractional delay filters need not be redesigned in case the time skews t_k are changed. It suffices to adjust some multiplier coefficient values that are uniquely determined by the time skews t_k. The price to pay for these attractive features is that we need to use a slight oversampling. [In other words, we must use $N > M$ to achieve perfect reconstruction (PR). From a practical implementation point of view, it is convenient to handle this situation by making use of what we call regionally perfect reconstruction (RPR) systems.] It should be stressed, however, that the oversampling factor is less than two as compared with the Nyquist rate; it is thus not a large oversampling factor. It should also be noted that using our system to correct errors in time-interleaved ADC systems, the individual ADCs in each channel will still work at a lower sampling rate compared with a single ADC working at the Nyquist rate (except for the simplest case where $M = 2$). In our case, we have a reduction of the sampling rate requirements of some $M/2$ instead of M.

The outline of the paper is as follows. In Section II, we briefly recapitulate uniform sampling and hybrid analog/digital filterbanks, the latter of which is convenient to use when analyzing the class of nonuniformly sampled signals that we are concerned with in this paper. In Section III, we consider PR and RPR systems in the case of bandlimited input signals. Section IV deals with nonuniform sampling and introduces the synthesis system of digital fractional delay filters for obtaining PR. Section V points out the relation between the proposed and time-interleaved ADC systems. Sections VI and VII are concerned with error analysis and quantization noise, respectively. Section VIII provides an example, whereas Section IX discusses fractional delay filter structures that are suitable for real-time applications. Finally, some concluding remarks are given in Section X.

II. UNIFORM SAMPLING AND HYBRID ANALOG/DIGITAL FILTERBANKS

A. Uniform Sampling

Uniform sampling and quantization are represented by the uniform sampler and quantizer in Fig. 2. Ignoring the quantization, the output sequence $x(n)$ is obtained by sampling the analog input signal $x_a(t)$ uniformly at the time instances nT, for all integers n, i.e.,

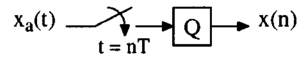

Fig. 2. Uniform sampler and quantizer.

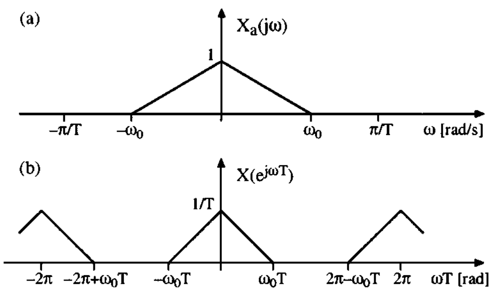

Fig. 3. Spectra of a bandlimited signal $x_a(t)$ and the sequence $x(n) = x_a(nT)$. (Uniform sampling).

$$x(n) = x_a(nT), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{1}$$

where T is the sampling period, and $f_{\text{sample}} = 1/T$ is the sampling frequency. The Fourier transforms of $x(n)$ and $x_a(t)$ are related according to Poisson's summation formula as

$$X(e^{j\omega T}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_a\left(j\omega - j\frac{2\pi r}{T}\right). \tag{2}$$

Since the spectrum of $x(n)$ is periodic with a period of 2π (2π-periodic) with respect to ωT, it suffices to consider $X(e^{j\omega T})$ in the region $|\omega T| \le \pi$. Throughout this paper, it is assumed that $x_a(t)$ is bandlimited according to

$$X_a(j\omega) = 0, \quad 0 < \omega_0 \le |\omega|, \quad \omega_0 \le \frac{\pi}{T} \tag{3}$$

[see also Fig. 3(a)]. That is, the Nyquist criterion for sampling with a sampling frequency of $1/T$ without aliasing is fulfilled. Thus, we have

$$X(e^{j\omega T}) = \frac{1}{T} X_a(j\omega), \quad |\omega T| \le \pi \tag{4}$$

[see also Fig. 3(b)]. Equation (4) implies that $x_a(t)$ can be retained from $x(n)$. We also note that $x_a(t)$ is oversampled unless $\omega_0 = \pi/T$.

B. Hybrid Analog/Digital Filterbanks

Consider the system in Fig. 4, which we refer to as a hybrid analog/digital filterbank, or simply a filterbank ADC. This system makes use of an analog analysis filterbank, uniform samplers and quantizers, upsamplers to retain the desired sampling rate $1/T$, and a digital synthesis filterbank. The sampling and quantization take place at the output of the analysis filters with the lower sampling frequency $1/T_1 = f_{\text{sample}}/M$ since $T_1 = MT$. Ignoring the quantizations, it can be shown that the Fourier transform of the output sequence $y(n)$ can be written as [12]

$$Y(e^{j\omega T}) = \frac{1}{T} \sum_{p=-\infty}^{\infty} V_p(j\omega) X_a\left(j\omega - j\frac{2\pi p}{MT}\right) \tag{5}$$

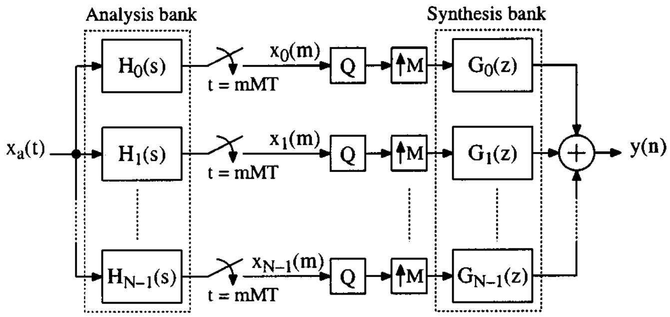

Fig. 4. Hybrid analog/digital filterbank.

where

$$V_p(j\omega) = \frac{1}{M}\sum_{k=0}^{N-1} G_k(e^{j\omega T})H_k\left(j\omega - j\frac{2\pi p}{MT}\right). \quad (6)$$

As for $X(e^{j\omega T})$ it suffices to consider $Y(e^{j\omega T})$ in the region $|\omega T| \leq \pi$, provided that $G_k(e^{j\omega T})$ are 2π-periodic. In Section III, we will treat two different types of reconstruction.

III. Perfect Reconstruction and Regionally Perfect Reconstruction Systems for Bandlimited Signals

A. Perfect Reconstruction Systems

The system in Fig. 4 is a *perfect reconstruction (PR) system* if

$$Y(e^{j\omega T}) = ce^{-jd\omega T}X(e^{j\omega T}), \quad |\omega T| \leq \pi \quad (7)$$

for some nonzero constant c and integer constant d. In the time domain, we have, in the PR case, $y(n) = cx(n-d)$. That is, with $c = 1$, $y(n)$ is simply a shifted version of $x(n)$. Ignoring the delay d, we can thus retain $x_a(t)$ from $y(n)$, provided that the system in Fig. 4 is a PR system. In order to achieve PR, aliasing into the region $|\omega T| \leq \pi$ must be avoided. From (2), (3), (5), and (7), it can be concluded that PR is obtained if

$$V_p(j\omega) = \begin{cases} ce^{-jd\omega T}, & p = 0,\ |\omega| < \frac{\pi}{T} \\ 0, & |p| = 1, 2, \ldots, K_0,\ |\omega| \leq \frac{\pi}{T} \end{cases} \quad (8)$$

where

$$K_0 = \left\lceil \frac{M(\pi + \omega_0 T)}{2\pi} \right\rceil - 1. \quad (9)$$

When $x_a(t)$ is bandlimited according to (3), it thus suffices to consider $2K_0 + 1$ terms in (5).[2] With $\omega_0 = \pi/T$, (9) reduces to

$$K_0 = M - 1. \quad (10)$$

Equation (10) is illustrated in Fig. 5 for $M = 3$. We see that in the region $|\omega T| \leq \pi$, we can discard terms in (5) for which $2\pi p/M + \pi \leq -\pi$ and $2\pi p/M - \pi \geq \pi$, which gives us $K_0 = M - 1$. Similarly, we can deduce (9) by observing that we can discard terms in (5) for which $2\pi p/M + \omega_0 T \leq -\pi$ and $2\pi p/M - \omega_0 T \geq \pi$.

[2] In general, more aliasing terms need to be handled in hybrid analog/digital filterbanks compared with maximally decimated (by M) digital filterbanks, where it suffices to consider M terms [12]. However, for each value of ωT, it suffices to consider M terms. This explains why it is, in principle, possible to reconstruct $x(n)$, and thus $x_a(t)$, with $N = M$.

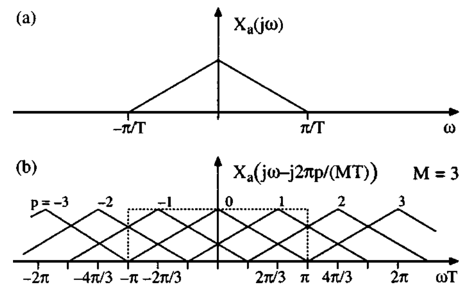

Fig. 5. Illustration of (10) [and (9) with $\omega_0 T = \pi$].

B. Regionally PR (RPR) Systems

The system in Fig. 4 is a *regionally perfect reconstruction (RPR) system* if

$$Y(e^{j\omega T}) = ce^{-jd\omega T}X(e^{j\omega T}), \quad |\omega T| \leq \omega_0 T < \pi \quad (11)$$

for some nonzero constant c and integer constant d. In the RPR case, $y(n)$ is generally not a shifted version of $x(n)$, i.e., $y(n) \neq cx(n-d)$. However, $y(n)$ and $x(n)$ carry the same information in the low-frequency region $|\omega T| < \omega_0 T$. In order to achieve RPR, aliasing into this region must be avoided. From (2), (3), (5), and (7), it can be concluded that RPR is obtained if

$$V_p(j\omega) = \begin{cases} ce^{-jd\omega T}, & p = 0,\ |\omega| < \omega_0 \\ 0, & |p| = 1, 2, \ldots, K_0,\ |\omega| < \omega_0 \\ A_p(j\omega), & |p| = 0, 1, \ldots, K_0,\ |\omega_0| \leq |\omega| \leq \frac{\pi}{T} \end{cases} \quad (12)$$

where $A_p(j\omega)$ are some arbitrary complex-valued functions, and

$$K_0 = \left\lceil \frac{M\omega_0 T}{\pi} \right\rceil - 1. \quad (13)$$

Equation (13) is deduced by noting that we can discard terms in (5) for which $2\pi p/M + \omega_0 T \leq -\omega_0 T$ and $2\pi p/M - \omega_0 T \geq \omega_0 T$ (compare with Section III-A). Finally, we note that the PR system is obtained from the RPR system by simply filtering the output from the RPR system through an ideal lowpass filter with passband region $|\omega T| < \omega_0 T$.

IV. Nonuniform Sampling and Proposed Synthesis System of Fractional Delay Filters

Let $x_k(m), k = 0, 1, \ldots, N-1$ be N subsequences obtained through sampling of $x_a(t)$ at the time instances $t = mMT + t_k$, i.e.,

$$x_k(m) = x_a(mMT + t_k), \quad k = 0, 1, \ldots, N-1. \quad (14)$$

For $M = N = 2$, $x_a(t)$ is sampled according to Fig. 1(b).

The subsequences $x_k(m)$ can be obtained by sampling the output signals from the analysis filters in Fig. 4 if these filters are selected according to

$$H_k(j\omega) = e^{j\omega t_k}, \quad k = 0, 1, \ldots, N-1. \quad (15)$$

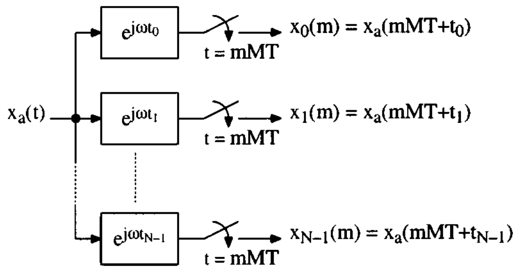

Fig. 6. Filters producing $x_k(m)$ as given by (14).

The analysis filterbank is in this case as shown in Fig. 6. Combining (6) and (15) gives us

$$V_p(j\omega) = \frac{1}{M}\sum_{k=0}^{N-1} G_k(e^{j\omega T})e^{j(\omega-(2\pi p/MT))t_k}. \tag{16}$$

A. Proposed Synthesis System of Fractional Delay Filters

Let $G_k(e^{j\omega T})$ be 2π-periodic filters given by

$$G_k(e^{j\omega T}) = \begin{cases} a_k c e^{-j\omega(t_k+dT)}, & |\omega T| < \omega_0 T \\ 0, & \omega_0 T \le |\omega T| \le \pi. \end{cases} \tag{17}$$

From (16) and (17) we obtain,

$$V_p(j\omega) = \begin{cases} \frac{1}{M} c e^{-jd\omega T} \sum_{k=0}^{N-1} a_k e^{-j(2\pi p/MT)t_k}, & |\omega| < \omega_0 \\ 0, & \omega_0 \le |\omega| \le \frac{\pi}{T}. \end{cases} \tag{18}$$

For PR, it is required that $V_p(j\omega)$, as given by (18), fulfills (8). That is, PR is obtained if

$$\sum_{k=0}^{N-1} a_k e^{-j(2\pi p/MT)t_k} = \begin{cases} M, & p=0 \\ 0, & |p| = 1,2,\ldots,K_0. \end{cases} \tag{19}$$

In general, $G_k(z)$ in (17) are fractional delay filters with different gain constants. (Fractional delay filters delay an input signal by a fraction of the sampling period [13]). However, for some values of k (depending on t_k), the delays may here be integers.

All $G_k(z)$ should be zero in the high-frequency region. In practice, it may therefore be convenient to do the reconstruction in two steps. In the first step, an RPR system is used. In the second step, the output from the RPR system is filtered through an ideal lowpass filter producing an overall PR system. This is achieved by selecting $G_k(e^{j\omega T})$ according to

$$G_k(e^{j\omega T}) = E_k(e^{j\omega T})F(e^{j\omega T}) \tag{20}$$

where

$$E_k(e^{j\omega T}) = \begin{cases} a_k c e^{-j\omega(t_k+d_1 T)}, & |\omega T| < \omega_0 T \\ a_k \hat{A}_k(e^{j\omega T}), & \omega_0 T \le |\omega T| \le \pi \end{cases} \tag{21}$$

with $\hat{A}_k(e^{j\omega T})$ being arbitrary complex-valued functions, and

$$F(e^{j\omega T}) = \begin{cases} e^{-jd_2\omega T}, & |\omega T| < \omega_0 T \\ 0, & \omega_0 T \le |\omega T| \le \pi \end{cases} \tag{22}$$

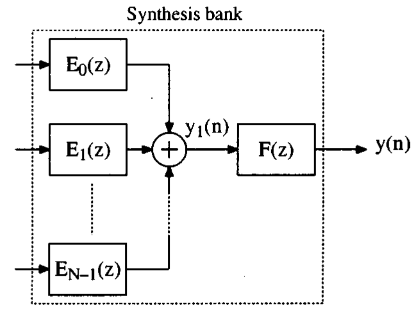

Fig. 7. Proposed synthesis system when $G_k(z)$ are in the form of (20).

where $d = d_1 + d_2$. The synthesis system is in this case realized according to Fig. 7. Selecting a_k in (21) so that (19) is fulfilled ensures that the system as seen from $x_a(t)$ to $y_1(n)$ is an RPR system. Choosing $F(e^{j\omega T})$ as in (22) produces the desired overall PR system (with $c = 1$). Indeed, for the system as seen from $x_a(t)$ to $y_1(n)$, we obtain, using (16) and (20) with $F(e^{j\omega T}) = 1$ and (21)

$$V_p(j\omega) = \begin{cases} \frac{1}{M} c e^{-jd_1\omega T} \sum_{k=0}^{N-1} a_k e^{-j(2\pi p/MT)t_k}, & |\omega| < \omega_0 \\ A_p(j\omega), & \omega_0 \le |\omega| \le \frac{\pi}{T} \end{cases} \tag{23}$$

where

$$A_p(j\omega) = \frac{1}{M}\sum_{k=0}^{N-1} a_k \hat{A}_k(e^{j\omega T})e^{j(\omega-(2\pi p/MT))t_k}. \tag{24}$$

For RPR, it is required that $V_p(j\omega)$, as given by (23), fulfils (11), i.e., that (19) is again satisfied. As explained in Section III, a PR system is then obtained by filtering the output from the RPR system through an ideal lowpass filter, i.e., here, by replacing in (20), $F(e^{j\omega T}) = 1$ by $F(e^{j\omega T})$ given in (22).

B. Computing the Coefficients a_k

Equation (19) can be written in matrix form as

$$\mathbf{Ba} = \mathbf{c} \tag{25}$$

where $\mathbf{B}$ is a $(2K_0+1) \times N$ matrix according to

$$\mathbf{B} = \begin{bmatrix} u_0^{-K_0} & u_1^{-K_0} & \cdots & u_{N-1}^{-K_0} \\ u_0^{-(K_0-1)} & u_1^{-(K_0-1)} & \cdots & u_{N-1}^{-(K_0-1)} \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{K_0} & u_1^{K_0} & \cdots & u_{N-1}^{K_0} \end{bmatrix} \tag{26}$$

with

$$u_k = e^{-j(2\pi/MT)t_k}. \tag{27}$$

Further, $\mathbf{a}$ is a column vector with N elements, and $\mathbf{c}$ is a column vector with $2K_0 + 1$ elements according to

$$\mathbf{a} = [a_0 \quad a_1 \quad \ldots \quad a_{N-1}]^T \tag{28}$$
$$\mathbf{c} = [c_0 \quad c_1 \quad \ldots \quad c_{2K_0}]^T \tag{29}$$

where T stands for the transpose (without complex conjugate). The coefficients a_k are the N unknowns, whereas c_q are

$$c_q = \begin{cases} M, & q = K_0 \\ 0, & q = 0, 1, \ldots, 2K_0, \; q \neq K_0. \end{cases} \tag{30}$$

Equation (25) is a system of $2K_0 + 1$ linear equations with N unknown parameters a_k. We consider two different cases.

Case 1—$2K_0 + 1 = N$: In this case, the number of unknowns equals the number of equations. The coefficients a_k can, in this case, be uniquely determined under the conditions stated by the following theorem.

Theorem 1: If $\mathbf{B}$ and $\mathbf{c}$ are as given by (26) and (29), respectively, $2K_0+1 = N$, and $t_k \neq t_m + MTr, k \neq m, r \in Z$, then there exists a unique $\mathbf{a}$ satisfying (25) and, therefore, unique a_k satisfying (19) as well. Further, all a_k in $\mathbf{a}$ are real-valued constants.

Proof: We first prove that there exists a unique solution. Since $2K_0 + 1 = N$, $\mathbf{B}$ is a square $N \times N$ matrix. If $\mathbf{B}$ is nonsingular, then $\mathbf{a}$ is uniquely determined by

$$\mathbf{a} = \mathbf{B}^{-1}\mathbf{c} \tag{31}$$

where $\mathbf{B}^{-1}$ is the inverse of $\mathbf{B}$. It thus suffices to show that $\mathbf{B}$ is nonsingular under the stated conditions. To this end, we first observe that $\mathbf{B}$, as given by (26), can be written as

$$\mathbf{B} = \mathbf{A}\mathbf{C} \tag{32}$$

where $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_0 & u_1 & \cdots & u_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{2K_0} & u_1^{2K_0} & \cdots & u_{N-1}^{2K_0} \end{bmatrix} \tag{33}$$

and $\mathbf{C}$ is a diagonal matrix according to

$$\mathbf{C} = \text{diag}\left[u_0^{-K_0} \quad u_1^{-K_0} \quad \ldots \quad u_{N-1}^{-K_0} \right]. \tag{34}$$

The matrix $\mathbf{A}$ is a Vandermonde matrix [14].[3] The necessary and sufficient condition for nonsingularity of $\mathbf{A}$ is therefore that u_k be distinct, i.e., $u_k \neq u_m$, $k \neq m$, which is the same condition as $t_k \neq t_m + MTr$, $k \neq m$, $r \in Z$ due to (27). Further, since $\det \mathbf{B} = \det \mathbf{A} \det \mathbf{C}$ and $|\det \mathbf{C}| = 1$, we have

$$\begin{aligned} \det \mathbf{A} \neq 0 &\Leftrightarrow \det \mathbf{B} \neq 0 \\ \det \mathbf{A} = 0 &\Leftrightarrow \det \mathbf{B} = 0. \end{aligned} \tag{35}$$

That is, $\mathbf{B}$ is nonsingular if and only if $\mathbf{A}$ is nonsingular. This proves that $\mathbf{B}$ is nonsingular and that a unique solution $\mathbf{a}$ always exists under the stated conditions.

To prove that a_k in $\mathbf{a}$ are real-valued, we proceed as follows. Assume that we have the unique values a_k satisfying (19). Using (27), (19) can equivalently be written as

$$\begin{aligned} \sum_{k=0}^{N-1} a_k &= M, \quad p = 0 \\ \sum_{k=0}^{N-1} a_k \left[u_k^p\right]^* = \sum_{k=0}^{N-1} a_k u_k^p &= 0, \quad p = 1, 2, \ldots, K_0 \end{aligned} \tag{36}$$

[3]Since $\mathbf{A}$ is a Vandermonde matrix here, the coefficients a_k can be computed using simple formulas [14].

with x^* denoting the complex conjugate of x. By (36), we get

$$\begin{aligned} \sum_{k=0}^{N-1} a_k^* &= M, \quad p = 0 \\ \sum_{k=0}^{N-1} a_k^* u_k^p = \sum_{k=0}^{N-1} a_k^* \left[u_k^p\right]^* &= 0, \quad p = 1, 2, \ldots, K_0. \end{aligned} \tag{37}$$

This shows that the values a_k^* satisfy (19) as well. However, since a_k are unique, it follows that they must be real-valued.

Case 2—$2K_0 + 1 < N$: In this case, the number of unknowns exceeds the number of equations. We can therefore impose $L = N - 2K_0 - 1$ additional linear constraints among the coefficients a_k and still satisfy (19). Here, we restrict ourselves to the case in which L of the N coefficients a_k, for $k = N - L$, $N - L + 1, \ldots, N - 1$, are fixed to some constants. This case covers time-interleaved ADC systems with an even number of channels. Since L coefficients are free, we could, of course, set them to zero, in which case, the corresponding channels would be removed. In that sense, there is no need to consider the cases having an even number of channels. However, as we will see in Section VII, it may be worth considering these cases in order to reduce the quantization noise at the output of the overall system.

The system of linear equations to be solved can here be written in matrix form as

$$\widehat{\mathbf{B}}\mathbf{a} = \hat{\mathbf{c}} \tag{38}$$

with $\widehat{\mathbf{B}}$ being an $N \times N$ matrix, and $\mathbf{a}$ and $\hat{\mathbf{c}}$ being column vectors with N elements according to

$$\widehat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{S} \end{bmatrix} \tag{39}$$

$$\mathbf{a} = \left[\mathbf{a}_u \quad \mathbf{a}_{\text{fix}} \right]^T \tag{40}$$

$$\hat{\mathbf{c}} = \left[\mathbf{c} \quad \mathbf{a}_{\text{fix}} \right]^T \tag{41}$$

where $\mathbf{B}$ is the $(2K_0 + 1) \times N$ matrix as given by (26), $\mathbf{a}_u$ and $\mathbf{a}_{\text{fix}}$ contain the $2K_0 + 1$ unknowns and L fixed constants of $\mathbf{a}$, respectively, $\mathbf{c}$ is the column vector with $2K_0 + 1$ elements as given by (30), $\mathbf{S}$ is an $L \times N$ matrix given by

$$\mathbf{S} = \left[\mathbf{0} \quad \mathbf{I} \right] \tag{42}$$

where $\mathbf{0}$ is an $L \times (2K_0+1)$ null matrix, and $\mathbf{I}$ is an $L \times L$ identity matrix. As in Case 1, a_k can, in Case 2, be uniquely determined under the conditions stated by the following theorem.

Theorem 2: If $\widehat{\mathbf{B}}$ and $\hat{\mathbf{c}}$ are as given by (39) and (41), respectively, $\mathbf{a}_{\text{fix}}$ in (40) contains L real-valued fixed constants, $2K_0 + 1 < N$, and $t_k \neq t_m + MTr$, $k \neq m$, $r \in Z$, then there exists a unique $\mathbf{a}$ satisfying (38) and, therefore, unique a_k satisfying (19) as well. Further, all a_k in $\mathbf{a}$ are real-valued constants.

Proof: The proof follows that of Theorem 1. To prove the existence and uniqueness, it thus suffices to show that $\widehat{\mathbf{B}}$ is nonsingular under the stated conditions since $\mathbf{a}$ then is uniquely determined by

$$\mathbf{a} = \widehat{\mathbf{B}}^{-1}\hat{\mathbf{c}}. \tag{43}$$

To prove nonsingularity of $\widehat{\mathbf{B}}$, we first observe that

$$\det \widehat{\mathbf{B}} = \det \widetilde{\mathbf{B}} \prod_{l=0}^{L-1} \mathbf{S}_{d,ll} = \det \widetilde{\mathbf{B}} \tag{44}$$

where $\widetilde{\mathbf{B}}$ is a $(2K_0+1)\times(2K_0+1)$ submatrix obtained from $\mathbf{B}$ by deleting L columns for $k = N-L, N-L+1, \ldots, N-1$, i.e.,

$$\widetilde{\mathbf{B}} = \begin{bmatrix} u_0^{-K_0} & u_1^{-K_0} & \cdots & u_{2K_0}^{-K_0} \\ u_0^{-(K_0-1)} & u_1^{-(K_0-1)} & \cdots & u_{2K_0}^{-(K_0-1)} \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{K_0} & u_1^{K_0} & \cdots & u_{2K_0}^{K_0} \end{bmatrix}. \tag{45}$$

We know from the proof of Theorem 1 that $\det \widetilde{\mathbf{B}} \neq 0$, and thus, $\det \widehat{\mathbf{B}} \neq 0$ under the stated conditions. This proves that $\widehat{\mathbf{B}}$ is nonsingular and that a unique solution always exists. The proof that a_k in $\mathbf{a}$ are real-valued is done in the same manner as that of Theorem 1.

C. Number of Channels N Versus Bandwidth $\omega_0 T$

From (13), we get

$$\omega_0 T \leq \frac{\pi(K_0+1)}{M}. \tag{46}$$

Further, for $N = 2K_0+1$ and $N = 2K_0+2$, we obtain[4]

$$\omega_0 T \leq \begin{cases} \frac{\pi(N+1)}{2M}, & N = 2K_0+1 \\ \frac{\pi N}{2M}, & N = 2K_0+2. \end{cases} \tag{47}$$

In particular, with $N = M$, we get, from the above expressions

$$\omega_0 T \leq \begin{cases} \frac{\pi}{2} + \frac{\pi}{2M}, & N = M = \text{odd} \\ \frac{\pi}{2}, & N = M = \text{even}. \end{cases} \tag{48}$$

For a Nyquist rate sampling and reconstruction system, $\omega_0 T = \pi$. From (48), we see that, with $N = M$, $\omega_0 T < \pi$. Thus, we need to use a slight oversampling in our system compared with the reconstruction systems referred to in the introduction. However, as pointed out in the introduction, the main reason for introducing our system is that it has attractive features when it comes to practical implementations. This will become clear in Sections V–IX. We also note that

$$\text{oversampling factor} = \begin{cases} \frac{2M}{M+1}, & N = M = \text{odd} \\ 2, & N = M = \text{even} \end{cases} \tag{49}$$

i.e., the oversampling factor never exceeds two. From (47), we also see that we can achieve PR in the whole frequency region (from $-\pi$ to π) by increasing the number of channels N. This is, however, just another way of saying that we need to use oversampling.

V. Time-Interleaved ADC Systems and Their Generalizations

This section points out the relation between the overall system (i.e., nonuniform sampling together with the proposed synthesis system) and time-interleaved ADC systems. As we will see, the former can be viewed as a generalization of the latter.

Consider first the case where $N = M$, with t_k being

$$t_k = kT + \Delta t_k, \quad k = 0, 1, \ldots, M-1 \tag{50}$$

where

$$\Delta t_k = 0, \quad k = 0, 1, \ldots, M-1. \tag{51}$$

Further, let $G_k(e^{j\omega T})$ correspond to integer delays according to $G_k(e^{j\omega T}) = e^{-jk\omega T}$. In this case, $V_p(j\omega)$ in (16) becomes

$$V_p(j\omega) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j2\pi pk/M} = \begin{cases} 1, & p = rM \\ 0, & p \neq rM \end{cases} \tag{52}$$

for all ω and all integers r; thus, PR is obtained. We have, in this case, a time-interleaved ADC system [1]. The output sequence $y(n)$ is here obtained by interleaving the sequences $x_k(m)$. In practice, Δt_k will, however, no longer be exactly zero. In time-interleaved ADC systems, this introduces aliasing components in the output sequence. If Δt_k are known, the aliasing components can be removed by employing the proposed system with $G_k(e^{j\omega T})$ taking the form of (17) with a_k being determined by (31) if N is odd and $2K_0+1 = N$ or (43) if $2K_0+1 < N$. In this case, we must, however, assume that $x_a(t)$ is bandlimited to $\omega_0 T < \pi$ since $N = M$ (see Section III-C).

Consider next the case where $N \neq M$, with t_k being

$$t_k = \frac{kTM}{N} + \Delta t_k, \quad k = 0, 1, \ldots, N-1 \tag{53}$$

where, again, Δt_k are given by (51), i.e., they are zero. Further, let $G_k(e^{j\omega T})$ be given by (17) with $a_k = M/N$, $k = 0, 1, \ldots, N-1$, $c = 1$, and $d = 0$. In this case, $V_p(j\omega)$ in (16) becomes

$$V_p(j\omega) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-j2\pi pk/N} = \begin{cases} 1, & p = rN \\ 0, & p \neq rN \end{cases} \tag{54}$$

for $|\omega| < \omega_0$ and all integers r, and $V_p(j\omega) = 0$ for $\omega_0 \leq |\omega| \leq \pi/T$; thus, PR is obtained. However, in practice, Δt_k will no longer be exactly zero, which, again, introduces aliasing components in the output sequence. If Δt_k are known, the aliasing components can again be removed by employing the proposed system. The difference from the case above where $N = M$ is that, here, with $N \neq M$, we can increase the bandwidth $\omega_0 T$ by increasing the number of channels N.

VI. Error Analysis

This section provides error analysis. More precisely, we derive bounds on the errors in $\mathbf{a}$ and $\mathbf{c}$ when $\mathbf{B}$ and $\mathbf{a}$ are replaced with $\mathbf{B} + \Delta\mathbf{B}$ and $\mathbf{a} + \Delta\mathbf{a}$, respectively. The errors in $\mathbf{a}$ are of interest as far as the quantization noise is concerned, as will become clear in Section VII. The errors in $\mathbf{c}$ tell us how close to the ideal fractional delay filters any practical filters must be in order to meet some prescribed allowable errors in $\mathbf{c}$. Bounds on the errors in $\mathbf{c}$ when $\mathbf{B}$ is replaced with $\mathbf{B} + \Delta\mathbf{B}$ are also given. These are of interest in practical implementations because the estimated time-skew errors inevitably will be represented with finite precision.

We will make use of the L_∞-norms defined as

$$||\mathbf{x}||_\infty = \max |x_i|, \quad 0 \leq i \leq N-1 \tag{55}$$

[4]As is clear from Section IV-B, $\omega_0 T$ will not change when we increase the number of channels from N to $N+1$ for N odd and a fixed M. This explains why $N = 2K_0+1$ and $N = 2K_0+2$ yield different expressions in (47).

for an $N \times 1$ $(1 \times N)$ vector $\mathbf{x}$ with elements x_i, and

$$||\mathbf{X}||_\infty = \max \sum_{k=0}^{N-1} |x_{ik}|, \quad 0 \le i \le N-1 \tag{56}$$

for an $N \times N$ matrix $\mathbf{X}$ with elements x_{ik}.

A. Errors in $\mathbf{a}$

Consider first Case 1, with $2K_0 + 1 = N$. First, assume that we have $\mathbf{Ba} = \mathbf{c}$ for $t_k = kT$ and $a_k = 1$ (as in the first case in Section V). Assume next that t_k and a_k are replaced with $t_k + \Delta t_k$ and $a_k + \Delta a_k$, respectively, whereas $\mathbf{c}$ is kept fixed. This amounts to

$$(\mathbf{B} + \Delta\mathbf{B})(\mathbf{a} + \Delta\mathbf{a}) = \mathbf{c}. \tag{57}$$

The matrix $\Delta\mathbf{B}$ is an $N \times N$ matrix according to

$$\Delta\mathbf{B} = \begin{bmatrix} \Delta b_{-K_0,0} & \Delta b_{-K_0,1} & \cdots & \Delta b_{-K_0,N-1} \\ \Delta b_{-(K_0-1),0} & \Delta b_{-(K_0-1),1} & \cdots & \Delta b_{-(K_0-1),N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta b_{K_0,0} & \Delta b_{K_0,1} & \cdots & \Delta b_{K_0,N-1} \end{bmatrix} \tag{58}$$

where

$$\Delta b_{pk} = e^{-j(2\pi p t_k/MT)}(e^{-j\Delta t_{pk}} - 1) \tag{59}$$

with

$$\Delta t_{pk} = \frac{2\pi p}{MT}\Delta t_k. \tag{60}$$

Now, if

$$||\Delta\mathbf{B}||_\infty ||\mathbf{B}^{-1}||_\infty < 1 \tag{61}$$

then we have the well-known inequality

$$\frac{||\Delta\mathbf{a}||_\infty}{||\mathbf{a}||_\infty} \le \frac{||\Delta\mathbf{B}||_\infty ||\mathbf{B}^{-1}||_\infty}{1 - ||\Delta\mathbf{B}||_\infty ||\mathbf{B}^{-1}||_\infty}. \tag{62}$$

From (58)–(60), we get

$$\begin{aligned} ||\Delta\mathbf{B}||_\infty &= \max \sum_{k=0}^{N-1} |\Delta b_{pk}| \approx \max \sum_{k=0}^{N-1} |\Delta t_{pk}| \\ &\le \frac{2NK_0\pi}{M} \max\left\{\frac{|\Delta t_k|}{T}\right\}. \end{aligned} \tag{63}$$

We have $\mathbf{B} = \mathbf{AC}$, and consequently, $\mathbf{B}^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$. Further, since $\mathbf{A}$ is a DFT matrix (due to $t_k = kT$), its inverse $\mathbf{A}^{-1}$ is an IDFT matrix; hence, $||\mathbf{A}^{-1}||_\infty = 1$. We also have $||\mathbf{C}^{-1}||_\infty = 1$ because $\mathbf{C}^{-1}$ apparently is a diagonal matrix with diagonal elements $u_k^{K_0}$, where u_k are given by (27). We thus have

$$||\mathbf{B}^{-1}||_\infty \le ||\mathbf{C}^{-1}||_\infty ||\mathbf{A}^{-1}||_\infty = 1 \tag{64}$$

which, together with (63), results in

$$||\Delta\mathbf{B}||_\infty ||\mathbf{B}^{-1}||_\infty \lesssim \frac{2NK_0\pi}{M} \max\left\{\frac{|\Delta t_k|}{T}\right\}. \tag{65}$$

By using (62)–(65) and assuming $||\Delta\mathbf{B}||_\infty ||\mathbf{B}^{-1}||_\infty \ll 1$, we finally obtain

$$||\Delta\mathbf{a}||_\infty \lesssim ||\mathbf{a}||_\infty \frac{2NK_0\pi}{M} \max \frac{|\Delta t_k|}{T} \tag{66}$$

with $|\mathbf{a}||_\infty = 1$ since we have assumed that $a_k = 1$. Equation (66) is of interest when computing the quantization noise, as we will see in Section VII.

Consider next Case 2, with $2K_0 + 1 < N$. We assume here that we, ideally, have $t_k = kTM/N$ and $a_k = M/N$ (as in the second case in Section V). Now, assuming for simplicity that $c = 1$ and $d = 0$, and utilizing (54), we see that we can rewrite (38) as

$$\bar{\mathbf{B}}\mathbf{a} = \bar{\mathbf{c}} \tag{67}$$

where $\bar{\mathbf{B}}$ is an $N \times N$ matrix according to

$$\bar{\mathbf{B}} = \begin{bmatrix} u_0^{-K_0} & u_1^{-K_0} & \cdots & u_{N-1}^{-K_0} \\ u_0^{-(K_0-1)} & u_1^{-(K_0-1)} & \cdots & u_{N-1}^{-(K_0-1)} \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{N-K_0-1} & u_1^{N-K_0-1} & \cdots & u_{N-1}^{N-K_0-1} \end{bmatrix} \tag{68}$$

with u_k being given by (27), and $\bar{\mathbf{c}}$ being a column vector with N elements c_q according to

$$c_q = \begin{cases} M, & q = K_0 \\ 0, & q = 0, 1, \ldots, N-1, q \neq K_0. \end{cases} \tag{69}$$

Clearly, we can express $\bar{\mathbf{B}}$ as a product between a DFT matrix (due to $t_k = kTM/N$) and a diagonal matrix. We can therefore proceed in the same way as Case 1, which, again, will give us the bound in (66) but with $||\mathbf{a}||_\infty = M/N$ since we have assumed that $a_k = M/N$.

B. Errors in $\mathbf{c}$ and $V_p(j\omega)$ Due to Errors in $G_k(z)$

Assume that we have $\mathbf{Ba} = \mathbf{c}$ for t_k and a_k. Assume now that t_k, a_k, and c_q are replaced with $t_k + \Delta t_k$, $a_k + \Delta a_k$, and $c_q + \Delta c_q$, respectively. This amounts to

$$(\mathbf{B} + \Delta\mathbf{B})(\mathbf{a} + \Delta\mathbf{a}) = (\mathbf{c} + \Delta\mathbf{c}) \tag{70}$$

where $\Delta\mathbf{B}$ is given by (58). Using (70), we get

$$\Delta\mathbf{c} = \mathbf{B}\Delta\mathbf{a} + \Delta\mathbf{B}\mathbf{a} + \Delta\mathbf{B}\Delta\mathbf{a}. \tag{71}$$

From (71), we obtain

$$\begin{aligned} ||\Delta\mathbf{c}||_\infty \le {} & ||\mathbf{B}||_\infty ||\Delta\mathbf{a}||_\infty + ||\Delta\mathbf{B}||_\infty ||\mathbf{a}||_\infty \\ & + ||\Delta\mathbf{B}||_\infty ||\Delta\mathbf{a}||_\infty. \end{aligned} \tag{72}$$

Utilizing (26) and (58)–(60), we finally end up with

$$\begin{aligned} ||\Delta\mathbf{c}||_\infty \lesssim & N||\Delta\mathbf{a}||_\infty + N\max\{|\Delta t_{pk}|\}||\mathbf{a}||_\infty \\ & + N\max\{|\Delta t_{pk}|\}||\Delta\mathbf{a}||_\infty \\ \approx & N(||\Delta\mathbf{a}||_\infty + \max\{|\Delta t_{pk}|\}||\mathbf{a}||_\infty) \end{aligned} \tag{73}$$

where Δt_{pk} is related to Δt_k through (60).

Equation (73) is useful when designing $G_k(z)$. Recall from Section IV that the ideal filters should have the frequency responses $a_k e^{-j\omega t_k}$ for $|\omega T| < \omega_0 T$ [assuming here, for simplicity, that $c = 1$ and $d = 0$ in (17)]. In practice, $G_k(z)$ can, of course, only approximate these responses. In this frequency region, we can express the frequency responses of $G_k(z)$ as

$$G_k(e^{j\omega T}) = [a_k + \Delta a_k(\omega T)]e^{j[\omega t_k + \Delta t_{pk}(\omega T)]} \tag{74}$$

where $\Delta a_k(\omega T)$ and $\Delta t_{pk}(\omega T)$ are the deviations from the ideal magnitude and phase responses, respectively. Given the allowable errors in $\mathbf{c}$ and (73) and (74), it is thus easy to design $G_k(z)$ so that the requirements are satisfied. The allowable errors in $\mathbf{c}$ are given by some prescribed permitted errors in $V_p(j\omega)$. We have

$$|\Delta V_p(j\omega)| \leq \frac{c\Delta c_p}{M} \tag{75}$$

for $|p| = 0, 1, \ldots, K_0$, which gives us

$$\left|\frac{\Delta V_p(j\omega)}{V_0(j\omega)}\right| \lesssim \frac{||\Delta \mathbf{c}||_\infty}{M}. \tag{76}$$

The above expressions hold in the low-frequency region $|\omega T| < \omega_0 T$. In the high-frequency region $\omega_0 T \leq |\omega T| \leq \pi$, the magnitude of the aliasing terms are governed by the stopband attenuation of $G_k(z)$. If $G_k(z)$ are in the form of (20), i.e., $G_k(z) = E_k(z)F(z)$, then $E_k(z)$ are fractional delay filters, whereas $F(z)$ is a conventional lowpass filter. In this case, the expressions given previously for the low-frequency region are used for $E_k(z)$, whereas the aliasing terms in the high-frequency region are controlled by $F(z)$. (Except for small passband ripples that can be made arbitrarily small, and a certain delay, $F(z)$ does not affect the low-frequency region, provided that it is a linear-phase filter.)

C. *Errors in* $\mathbf{c}$ *and* $V_p(j\omega)$ *Due to Errors in Measurement of* t_k

Assume that we have $\mathbf{Ba} = \mathbf{c}$ for t_k and a_k. Assume now that t_k and c_q are replaced with $t_k + \Delta t_k$, and $c_q + \Delta c_q$, respectively, whereas $\mathbf{a}$ is kept fixed. This amounts to

$$(\mathbf{B} + \Delta\mathbf{B})\mathbf{a} = \mathbf{c} + \Delta\mathbf{c} \tag{77}$$

where $\Delta\mathbf{B}$ is given by (58) with

$$\Delta b_{pk} = e^{-j(2\pi p t_k/MT)}\left(e^{-j(\Delta t_{pk} - \omega\Delta t_k)} - 1\right) \tag{78}$$

where Δt_{pk} is related to Δt_k via (60). From (77), we have

$$\Delta\mathbf{c} = \Delta\mathbf{Ba} \tag{79}$$

from which we obtain

$$||\Delta\mathbf{c}||_\infty \leq ||\Delta\mathbf{B}||_\infty ||\mathbf{a}||_\infty. \tag{80}$$

Combining (58), (60), (78), and (80) gives us

$$\begin{aligned} ||\Delta\mathbf{c}||_\infty &\lesssim N||\mathbf{a}||_\infty \max\{|\Delta t_{pk} - \omega\Delta t_k|\} \\ &\leq N||\mathbf{a}||_\infty \pi\left(\frac{2K_0}{M} + 1\right)\max\left\{\frac{|\Delta t_k|}{T}\right\}. \end{aligned} \tag{81}$$

Equation (81) is useful for analyzing practical implementations where the estimated time-skew errors Δt_k are represented with finite precision. The errors in $\mathbf{c}$ are again related to the errors in $V_p(j\omega)$ through (75).

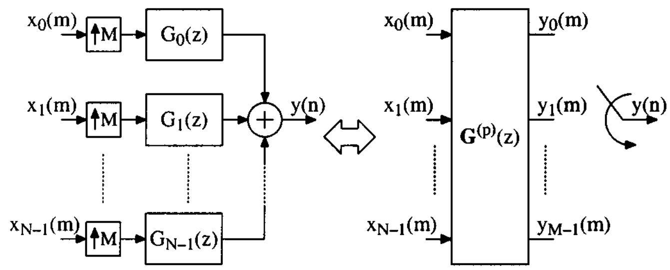

Fig. 8. Polyphase realization.

VII. Quantization Noise Analysis

An important measure of the performance of an ADC is the signal-to-noise ratio (SNR), which relates the average signal power to the average noise power [15]. To compute the noise power, it is customary to model the quantization errors as stationary white noise with zero mean. The noise power is then given by the noise variance.

To analyze the noise variance at the output of the system in Fig. 4, it is convenient to represent the synthesis bank with its so-called polyphase realization [10], as shown in Fig. 8. The output sequence $y(n)$ is obtained by interleaving the sequences $y_i(m)$, $i = 0, 1, \ldots, M-1$. The transfer function of $y(n)$ is given by

$$Y(z) = \sum_{i=0}^{M-1} z^{-i} Y_i(z^M) \tag{82}$$

where

$$\mathbf{Y}(z) = \mathbf{G}^{(p)}(z)\mathbf{X}(z) \tag{83}$$

with

$$\mathbf{X}(z) = [X_0(z) \quad X_1(z) \quad \ldots \quad X_{N-1}(z)]^T \tag{84}$$

$$\mathbf{Y}(z) = [Y_0(z) \quad Y_1(z) \quad \ldots \quad Y_{M-1}(z)]^T \tag{85}$$

$$\mathbf{G}^{(p)}(z) = \begin{bmatrix} G_{00}(z) & G_{01}(z) & \ldots & G_{0,N-1}(z) \\ G_{10}(z) & G_{11}(z) & \ldots & G_{1,N-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ G_{M-1,0}(z) & G_{M-1,1}(z) & \ldots & G_{M-1,N-1}(z) \end{bmatrix}. \tag{86}$$

Here, $G_{ik}(z)$ are the polyphase components of $G_k(z)$ according to

$$G_k(z) = \sum_{i=0}^{M-1} z^{-i} G_{ik}(z^M). \tag{87}$$

Let $x_k(m)$, $k = 0, 1, \ldots, N-1$ be uncorrelated white noise sources with zero mean and variances $\sigma_{x_k}^2$. Since $\mathbf{G}^{(p)}(z)$ describes a linear and time-invariant system, the outputs $y_i(m)$, $i = 0, 1, \ldots, M-1$ are also stationary noise with zero mean. However, the variances of $y_i(m)$, which are denoted here by $\sigma_{y_i}^2$, are, in general, different, even when $\sigma_{x_k}^2$ are equal. The outputs

$y_i(m)$ may also be correlated. The output noise $y(n)$ will therefore generally not be stationary. Its variance, which is denoted here by $\sigma_y^2(n)$, is thus time-varying and periodic with period M since, obviously

$$\sigma_y^2(nM+i) = \sigma_{y_i}^2. \tag{88}$$

We define the average quantization noise at the output as

$$\left(\sigma_y^2\right)_{av} = \frac{1}{M}\sum_{i=0}^{M-1}\sigma_{y_i}^2. \tag{89}$$

Given the synthesis filters $G_k(z)$ and its polyphase components $G_{ik}(z)$, $(\sigma_y^2)_{av}$ is easily computed as

$$\begin{aligned}\left(\sigma_y^2\right)_{av} =& \frac{1}{M}\sum_{i=0}^{M-1}\sigma_{y_i}^2 = \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{N-1}\sigma_{x_k}^2\sum_{n=-\infty}^{\infty}|g_{ik}(n)|^2\\ =& \frac{1}{M}\sum_{k=0}^{N-1}\sigma_{x_k}^2\sum_{i=0}^{M-1}\sum_{n=-\infty}^{\infty}|g_{ik}(n)|^2\\ =& \frac{1}{M}\sum_{k=0}^{N-1}\sigma_{x_k}^2\sum_{n=-\infty}^{\infty}|g_k(n)|^2\\ =& \frac{1}{M}\sum_{k=0}^{N-1}\sigma_{x_k}^2\frac{1}{2\pi}\int_{-\pi}^{\pi}|G_k(e^{j\omega T})|^2 d\omega T.\end{aligned} \tag{90}$$

Now, let the synthesis filters be as in (17) with $c=1$, and let all input variances $\sigma_{x_k}^2$ be equal according to

$$\sigma_{x_k}^2 = \sigma_x^2. \tag{91}$$

Combining (17) with (90) and (91) gives us

$$\left(\sigma_y^2\right)_{av} = \frac{\omega_0 T}{M\pi}\sigma_x^2\sum_{k=0}^{N-1}a_k^2. \tag{92}$$

A question that arises now is for which values of a_k the quantity $(\sigma_y^2)_{av}$ as given by (92) is minimized subject to the constraint that PR is simultaneously achieved. To answer this question, we consider the following problem:

$$\text{minimize } \sum_{k=0}^{N-1}a_k^2 \text{ subject to } \sum_{k=0}^{N-1}a_k = M. \tag{93}$$

The constraint in (93) is one of those that must be satisfied to obtain PR. The well-known solution to (93) is readily obtained by noting that the objective function to be minimized can be rewritten as

$$\begin{aligned}\sum_{k=0}^{N-1}a_k^2 =& \frac{1}{N}\left(\sum_{k=0}^{N-1}a_k\right)^2 + \frac{1}{N}\sum_{k=0}^{N-2}\sum_{q=k+1}^{N-1}(a_k-a_q)^2\\ =& \frac{M^2}{N} + \frac{1}{N}\sum_{k=0}^{N-2}\sum_{q=k+1}^{N-1}(a_k-a_q)^2.\end{aligned} \tag{94}$$

The second equality in (94) holds because the sum of a_k is M according to (93). Hence, the solution to (93) is obtained for $a_k = M/N$, $k=0,1,\ldots,N-1$, giving the minimum value of $(\sigma_y^2)_{av}$ as

$$\left(\sigma_y^2\right)_{av,\min} = \frac{M\omega_0 T}{N\pi}\sigma_x^2. \tag{95}$$

This shows that the average quantization noise at the output is minimized for $t_k = kTM/N$ and $a_k = M/N$. That is, for fixed M and N, both the time-interleaved ADC systems and their generalizations in Section V have minimum noise. Further, we see that for a fixed M, the noise can be reduced by increasing the number of channels N. It should also be noted that for a single ADC, the quantization noise is σ_x^2.

In practice, Δt_k will no longer be exactly zero, which implies that a_k are replaced with $a_k+\Delta a_k$. If Δa_k are small (and $a_k > 0$), the average quantization noise becomes, in this case

$$\begin{aligned}\left(\sigma_y^2\right)_{av} =& \frac{\omega_0 T}{M\pi}\sigma_x^2\sum_{k=0}^{N-1}(a_k+\Delta a_k)^2\\ \approx& \frac{\omega_0 T}{M\pi}\sigma_x^2\sum_{k=0}^{N-1}\left(a_k^2+2a_k\Delta a_k\right)\\ \leq& \frac{\omega_0 T}{M\pi}\sigma_x^2\sum_{k=0}^{N-1}\left(a_k^2+2a_k\|\Delta\mathbf{a}\|_\infty\right).\end{aligned} \tag{96}$$

With $a_k = M/N$ in (96), we obtain

$$\begin{aligned}\left(\sigma_y^2\right)_{av} \lesssim& \frac{\omega_0 T}{M\pi}\sigma_x^2\left(\frac{M^2}{N}+2M\|\Delta\mathbf{a}\|_\infty\right)\\ =& \frac{M\omega_0 T}{N\pi}\sigma_x^2\left(1+\frac{2N\|\Delta\mathbf{a}\|_\infty}{M}\right)\\ =& \left(\sigma_y^2\right)_{av,\min}\left(1+\frac{2N\|\Delta\mathbf{a}\|_\infty}{M}\right).\end{aligned} \tag{97}$$

The quantity $\|\Delta\mathbf{a}\|_\infty$ is bounded according to (66). Equation (97) shows that small changes in t_k, and thus small changes in a_k, have a minor affect on $(\sigma_y^2)_{av}$. That is

$$\left(\sigma_y^2\right)_{av} \approx \left(\sigma_y^2\right)_{av,\min} \tag{98}$$

for small $\|\Delta\mathbf{a}\|_\infty$. The above analysis shows that using the proposed system for correcting the time-skew errors in the time-interleaved ADC systems and their generalizations in Section V, the system has, practically, minimum noise. Furthermore, we can simultaneously achieve arbitrarily small aliasing terms by properly designing the fractional delay filters, as discussed in Section VI. The prerequisite for achieving these two performances is, of course, that we know the time-skew errors and that these errors are reasonably small.

VIII. Example

To illustrate the usefulness of the proposed system, we provide an example. For simplicity, we let the sampling period be $T=1$. We use as analog input signal a sum of four sinusoidals with angular frequencies 0.125π, 0.25π, 0.375π, and 0.5π, respectively. The spectrum of this input signal is plotted in Fig. 9. First, we consider a time-interleaved ADC system with $M=5$. The time-skew errors Δt_k, $k=0,1,\ldots,4$ in (50) are assumed to be 0, $-0.04T$, $0.02T$, $-0.01T$, and $0.03T$, respectively. The obtained sequence has a spectrum according to Fig. 10. Apparently, several undesired frequency components with large amplitudes have been introduced due to the time-skew errors. The largest amplitude is -32.9 dB, assuming that 0 dB is the level of the desired components.

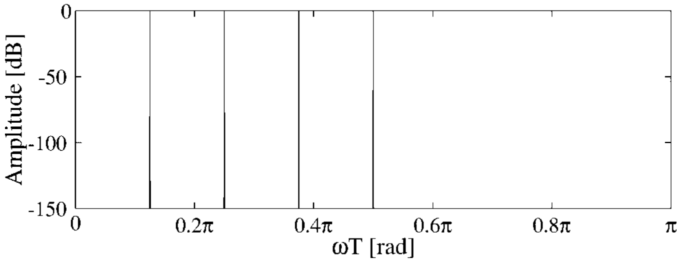

Fig. 9. Spectrum of the input signal.

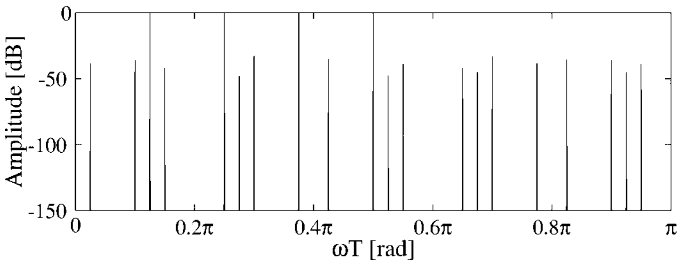

Fig. 10. Spectrum of the sequence obtained from a time-interleaved ADC system with $M = 5$.

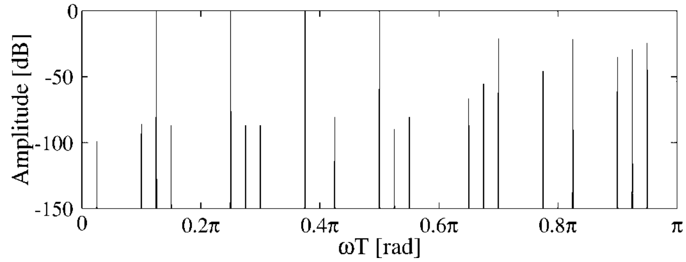

Fig. 11. Spectrum of the sequence $y_1(n)$ in Fig. 7 with $M = N = 5$.

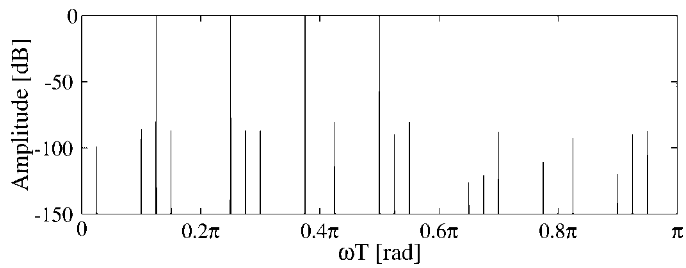

Fig. 12. Spectrum of the sequence $y_1(n)$ in Fig. 7 with $M = N = 5$.

Next, to reduce the amplitudes of the unwanted frequency components, we use the proposed system with $M = N = 5$ (i.e., $\omega_0 T = 0.6\pi$). We do the reconstruction in two steps according to Fig. 7. The filters $E_k(z)$, $k = 0, 1, \ldots, 4$ are FIR filters with eight zeros (eighth-order filters if one disregards extra delays). These filters are optimized in such a way that δ_k is minimized subject to the constraints $|\Delta a_k(\omega T)|/a_k \leq \delta_k$ and $|\Delta t_{pk}(\omega T)| \leq \delta_k$ for $|\omega T| \leq 0.6\pi$ [see (74)]. The maximum value of $\delta_k a_k$ becomes 0.000 166. Hence, by (73) and (76), we can expect the amplitudes of the undesired frequency components to be at most -69.6 dB. However, in practice, they will generally be smaller since (73) is based on worst-case assumptions. This can be seen in Fig. 11, which plots the spectrum of the output sequence from the RPR system, i.e., $y_1(n)$ in Fig. 7. The largest amplitude is below -80 dB in the low-frequency region $|\omega T| < 0.6\pi$. We also note that we still have undesired frequency components with large amplitudes in the high-frequency region $\omega_0 T \leq |\omega T| \leq \pi$, which is expected since the system as seen from $x_a(t)$ to $y_1(n)$ is an (approximately) RPR system. These high-frequency components are removed by the lowpass filter $F(z)$, producing an overall (approximately) PR system, as illustrated in Fig. 12. Note that, in practice, we must let $F(z)$ have a transition band between $\omega_0 T - \Delta$ and $\omega_0 T$. Thus, the "actual $\omega_0 T$" will be smaller than the theoretical value. The smaller the value Δ, the higher the filter order of $F(z)$.

IX. Fractional Delay Filter Structures

The reconstruction can only be carried out if the time skews t_k are known. In an implementation of the system, one must therefore be able to measure t_k or, equivalently, measure the time-skew errors Δt_k in (50) and (53). This can be performed either before or during normal operation [7]. Since the time-skew errors may vary over time due to, e.g., component aging and temperature variations, it may be necessary to do the measurements during normal operation. Using the proposed system under these conditions, it is essential that we design the fractional delay filters $G_k(z)$ in such a way that they need not be redesigned when Δt_k change values since it is not practical to design filters on line. If $G_k(z)$ are in the form of (20), i.e., $G_k(z) = E_k(z)F(z)$, then $E_k(z)$ are fractional delay filters, whereas $F(z)$ is a conventional lowpass filter. In this case, it suffices to make sure that $E_k(z)$ need not be redesigned since $F(z)$ can be kept fixed.

Online design of $E_k(z)$ can be avoided if we employ so-called Farrow structures [16]. The transfer functions $E_k(z)$ are, in this case, expressed as

$$E_k(z) = a_k \sum_{l=0}^{L-1} (\Delta t_k)^l Z^{-K_{kl}} D_{kl}(z) \tag{99}$$

where

$$D_{kl}(z) = \sum_{n=0}^{N_{kl}} d_{kl}(n) z^{-n}. \tag{100}$$

After optimizing $E_k(z)$ in the form of (99), $D_{kl}(z)$ will be kept fixed. It thus suffices to adjust the values of a_k and Δt_k. To make sure that $E_k(z)$ meets their specifications when the values of a_k and Δt_k are allowed to vary, one must, of course, take care of this in the optimization. Optimization of FIR filters with adjustable fractional delay has been considered in, e.g., [17] and [18]. Here, we also have adjustable gain constants, but this is easily handled when a_k only vary slightly. Finally, we note that when $M = N$, all $D_{kl}(z)$, $k = 0, 1, \ldots, N-1$ in (99) can be made equal. This reduces the number of distinct subfilters required in an implementation.

X. Conclusion

This paper has introduced a synthesis system composed of digital fractional delay filters for reconstructing a class of nonuniformly sampled bandlimited signals. The overall system can be viewed as a generalization of time-interleaved ADC systems to which the former reduces as a special case. By generalizing these systems, it is possible to eliminate the errors at the output that are introduced in practice due to time-skew errors. We showed how to obtain perfect reconstruction by selecting the (ideal) fractional delay filters properly. Further, we provided error analysis that is useful in the analysis of the quantization noise as well as when designing practical filters approximating the ideal ones. We also gave some expressions for the average quantization noise at the output of the overall system. Finally, we provided an example and briefly discussed fractional delay filter structures suitable for real-time applications.

The main reason for introducing the new system is that it is very attractive from an implementation point of view. In particular, it has two major advantages. One is that we can approximate PR as close as desired by properly designing the digital fractional delay filters. The second advantage is that, if properly implemented, the fractional delay filters need not be redesigned in case the time skews t_k are changed. It suffices to adjust some multiplier coefficient values that are uniquely determined by the time skews t_k. The price to pay for these facilities is that we need to use a slight oversampling. The oversampling factor is, however, always less than two as compared with the Nyquist rate; it is thus a small oversampling factor. In addition, using our system to correct errors in time-interleaved ADCs, the individual ADCs in each channel will still work at a lower sampling rate compared with a single ADC working at the Nyquist rate (except when $M = 2$). In our case, we have a reduction of the sampling rate requirements of some $M/2$ instead of M.

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