Transient Behavior of Two-Machine Geometric Production Lines

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Abstract: Production systems transients describe the process of reaching the steady state throughput. Reducing transients' duration is important in a number of applications. This paper is intended to analyze transients in systems with machines obeying the geometric reliability model. The Markov chain approach is used, and the second largest eigenvalue of the transition matrices is utilized to characterize the transients. Due to large dimensionality of the transition matrices, only two-machine systems are addressed, and the second largest eigenvalue is investigated as a function of the breakdown and repair rates. Conditions under which shorter, rather than longer, up- and downtimes lead to faster transients are provided.

Keywords: Production lines; Geometric reliability model; Production rate; Transient behavior; Effects of up- and downtime

1. INTRODUCTION

Production systems often operate in transient regimes. Examples include paint shops of automotive assembly plants, where some buffers are emptied at the end of each shift due to technological constraints; this leads to production losses in the subsequent shift (until the buffer occupancy reaches its steady state). Another example are machining departments operating with so-called floats, where additional work-in-process is built up by slow machines after the end of a shift in order to prevent starvations of fast machines in the subsequent shift, leading to increased production during the transients. Clearly, to quantify the performance of these systems, a method for analysis of their transients is necessary.

Unfortunately, the literature offers very few publications in this regard. Specifically, Narahari and Viswanadham (1994) study transients in one-machine production systems, using the idea of Markov process absorption time. Mocanu (2005) develops an algorithm for a numerical solution of the partial differential equation, which describes the evolution of the probability density function of a buffer with Markov-modulated input and output flows. The closest to the current study is the paper by Meerkov and Zhang (2008), which studies transients of serial production lines with machines obeying the Bernoulli reliability model. According to this model, each machine, being neither starved nor blocked, produces a part during a cycle time with probability p and fails to do so with probability $1-p$, irrespective of what had happened in the previous cycle time. Thus, Bernoulli machines are memoryless, which simplifies the analysis of the resulting systems. While the Bernoulli model is applicable to some assembly operations, it does not describe well many others, including machining, heat treatments, washing, etc. Thus, an extension of the results reported by Meerkov and Zhang (2008) is necessary. This is carried out in the current paper for machines obeying the geometric reliability model, which is applicable to the manufacturing operations mentioned above. Due to the complexity of the resulting mathematical description, only the case of two-machine systems is addressed; longer lines will be analyzed in the future work.

The outline of this paper is follows: Section 2 presents the model and the problem formulation. In Section 3, transients of individual machines are analyzed. Sections 4 and 5 are devoted to two-machine lines with short and long buffers, respectively. The conclusions and future work are given in Section 6. All proofs and numerical justifications are included in the Appendix.

2. MODEL AND PROBLEM FORMULATION

2.1 Model

We consider a two-machine production line (see Figure 2.1) defined by the following assumptions:

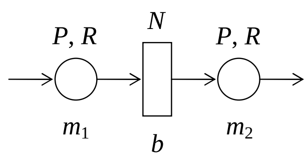

Fig. 2.1. Two-machine geometric line

(i) Both machines have an identical cycle time, τ. The time axis is slotted with the slot duration τ. The state of each machine (up or down) is determined at the beginning of each time slot.

(ii) Both machines obey the geometric reliability model, i.e., if $s(n) \in \{0 = \text{down}, 1 = \text{up}\}$ denotes the state of a machine at time slot n, the transition probabilities are given by

$$\begin{aligned} P[s(n+1) = 0|s(n) = 1] &= P, \\ P[s(n+1) = 1|s(n) = 1] &= 1 - P, \\ P[s(n+1) = 1|s(n) = 0] &= R, \\ P[s(n+1) = 0|s(n) = 0] &= 1 - R, \end{aligned}$$

where P and R are referred to as the breakdown and repair probabilities, respectively.

(iii) The buffer is characterized by its capacity $1 \le N < \infty$. The state of the buffer is determined at the end of each time slot.

(iv) Machine m_1 is never starved; it is blocked during a time slot if it is up and the buffer is full.

(v) Machine m_2 is never blocked; it is starved during a time slot if it is up and the buffer is empty.

Note that these assumptions imply, in particular, that time dependent failures are addressed and the blocked before service convention is used; that is why $N \ge 1$. Note also that the average up- and downtime of the machines are $T_{up} = 1/P$ and $T_{down} = 1/R$ and the machine efficiency is $e = T_{up}/(T_{up} + T_{down})$.

2.2 Problems

Given the above model, the production system at hand is described by an ergodic Markov chain. As it is well known (Meerkov and Zhang (2008)), the transients of such a system are characterized by the second largest eigenvalue (SLE) of its transition matrix. With this in mind, the problems addressed in this paper are as follows:

- Analyze the second largest eigenvalue of an individual geometric machine as a function of P and R. In particular, investigate the effect of T_{up} and T_{down} on SLE, under the assumption that the machine efficiency e is fixed.
- Carry out similar analyses for two-machine lines. In addition, investigate explicitly the transients of the production rate, $PR(n)$, i.e., the probability that m_2 is up and the buffer is not empty at time slot $n = 1, 2, \ldots$.

Note that the steady state production rate, $PR(\infty) =: PR_{ss}$, of a production line defined by assumptions (i)-(v) can be evaluated using the method developed in Li and Meerkov (2003). Here we are interested in how $PR(n)$ approaches the steady state value PR_{ss}.

The interest in the effect of T_{up} and T_{down} on the transients stems from the following: It is well known (see Li and Meerkov (2009)) that

- for a fixed e, shorter T_{up} and T_{down} lead to a larger PR_{ss} than longer ones;
- decreasing T_{down} by a given factor leads to a larger PR_{ss} than increasing T_{up} by the same factor.

Do similar effects exist in the case of transients as well? In other words, do shorter T_{up} and T_{down} lead to faster transients than longer ones? These and other similar questions are answered in this paper.

3. TRANSIENTS OF INDIVIDUAL MACHINES

Let $x_i(n)$, $i \in \{0, 1\}$, be the probability that the machine is in state i during time slot n. Then, the evolution of the vector $x(n) = [x_0(n)\ x_1(n)]^T$ can be described by

$$x(n+1) = Ax(n), \quad x_0(n) + x_1(n) = 1, \tag{3.1}$$

where

$$A = \begin{bmatrix} 1-R & P \\ R & 1-P \end{bmatrix}. \tag{3.2}$$

The eigenvalues of A are

$$\begin{aligned} \lambda_0 &= 1, \\ \lambda_1 &= 1 - P - R, \end{aligned}$$

and, therefore, the dynamics of the machine states can be expressed as

$$\begin{aligned} x_0(n) &= (1-e) + [x_0(0) - (1-e)](1-P-R)^n \\ &= (1-e)\left(1 - \frac{\Delta}{1-e}\lambda^n\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} x_1(n) &= e + [x_1(0) - e]\,(1-P-R)^n \\ &= e\left(1 + \frac{\Delta}{e}\lambda^n\right), \end{aligned} \tag{3.4}$$

where

$$\Delta = x_1(0) - e = (1-e) - x_0(0). \tag{3.5}$$

To investigate the effects of up- and downtime on the transients, consider λ_1 as a function of R for a fixed e, i.e.,

$$\lambda_1(R) = 1 - \left(\frac{1}{e} - 1\right)R - R = 1 - \frac{R}{e}.$$

The behavior of $|\lambda_1|$ as a function of R is illustrated in Figure 3.1. From this figure, we conclude:

- For $0 < R < e$, longer up- and downtimes lead longer transients.
- For $R = e$, the machine has no transients. Such a machine can be viewed as a Bernoulli machine.
- For $e < R < 1$, The evolution of the machine states is oscillatory (since $\lambda_1 < 0$) and, more importantly, shorter up- and downtimes lead to longer transients.

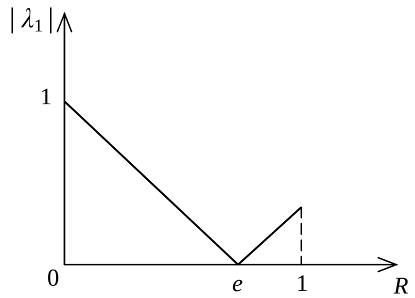

Fig. 3.1. Behavior of $|\lambda_1|$ as a function of R

Next, we address the issue of separate effects of uptime and of downtime on the transients. Recall that, as mentioned in Section 2, increasing the uptime by a factor $1+\alpha$, $\alpha > 0$, or decreasing the downtime by the same factor lead to the same steady state performance for an individual machine since

$$e' = \frac{1}{1 + \frac{T_{down}}{(1+\alpha)T_{up}}}. \quad (3.6)$$

However, the transient properties resulting from both cases are different. Indeed, consider a geometric machine with breakdown and repair probabilities P and R, respectively. Let λ_1^u denote the SLE of the machine with the uptime increased by $(1+\alpha)$, $\alpha > 0$ and λ_1^d denote the SLE for the same machine with the downtime decreased by the same factor. Then,

Theorem 3.1. For an individual geometric machine,

$$|\lambda_1^u| > |\lambda_1^d|, \quad (3.7)$$

if

$$e > 0.5, \quad \frac{T_{down}}{1+\alpha} > 2. \quad (3.8)$$

This theorem implies that if the machine efficiency is larger than 0.5 and the decreased downtime is larger than two cycle times, decreasing the downtime leads to faster transients than increasing the uptime, preserving the steady state production rate in both cases the same.

4. TRANSIENTS OF 2-MACHINE LINES WITH $N = 1$

For a serial line with two geometric machines, the state of the system can be denoted by a triple (h, s_1, s_2), where $h \in \{0, 1, \ldots, N\}$ is the state of the buffer and $s_i \in \{0, 1\}$, $i = 1, 2$, are the states of the first and the second machine, respectively. The behavior of the system is described by an ergodic Markov chain. For $N = 1$, the transition probability matrix is:

$$A = \begin{bmatrix} A_1 & \mathbf{0} & A_2 & \mathbf{0} \\ \mathbf{0} & A_3 & \mathbf{0} & A_4 \end{bmatrix}, \quad (4.1)$$

where

$$A_1 = \begin{bmatrix} (1-R)^2 & (1-R)\,P \\ (1-R)\,R & (1-R)\,(1-P) \\ R\,(1-R) & R\,P \\ R^2 & R\,(1-P) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} (1-R)\,P \\ (1-R)\,(1-P) \\ R\,P \\ R\,(1-P) \end{bmatrix}$$

$$A_3 = \begin{bmatrix} (1-R)\,P & P^2 & (1-R)^2 \\ R\,P & P\,(1-P) & (1-R)\,R \\ (1-P)\,(1-R) & (1-P)\,P & R\,(1-R) \\ (1-P)\,R & (1-P)^2 & R^2 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} (1-R)\,P & P^2 \\ R\,P & P\,(1-P) \\ (1-P)\,(1-R) & (1-P)\,P \\ (1-P)\,R & (1-P)^2 \end{bmatrix}$$

and $\mathbf{0}$'s are zero-matrices of appropriate dimensionalities. The eight eigenvalues of A are:

$$[1,\, 1-P-R,\, 1-P-R,\, (1-P-R)^2,\, (1-R)^2,\, 0,\, 0,\, 0]. \quad (4.2)$$

Clearly, the two eigenvalues $1 - P - R$ represent, as it follows from Section 3, the dynamics of the individual machines; the eigenvalue $(1 - P - R)^2$ represents the transients of a pair of individual machines (note that the states of the machines in model (i)-(v) are determined independently); therefore, the remaining non-zero eigenvalue $(1 - R)^2$ can be viewed as describing the transients of the buffer. The last statement is supported by the following two arguments:

First, using the notations

$$\lambda_m = 1 - P - R, \quad \lambda_b = (1-R)^2,$$

the transients of the states, i.e.,

$$x_{h,i,j}(n) = P[h(n) = h, s_1(n) = i, s_2(n) = j],\ n = 0, 1, \ldots,$$

can be represented as

$$x_{h,i,j}(n) = x_{h,i,j}\left(1 + B\lambda_b^n + C\lambda_m^n + D(\lambda_m^2)^n\right), \quad (4.3)$$
$$h \in \{0,1\},\ i,j \in \{0,1\},\ n = 0,1,2,\ldots,$$

where

$$x_{h,i,j} = \lim_{n\to\infty} x_{h,i,j}(n)$$

and B, C and D are constants defined by initial conditions.

Theorem 4.1. Consider a serial line with two identical geometric machines and $N = 1$. Assume that initially the machines are in the steady states, i.e.,

$$P[s_1(0) = 1] = P[s_2(0) = 1] = e. \quad (4.4)$$

Then, in expression (4.3),

$$C = D = 0, \quad \forall i, j, h \in \{0, 1\}.$$

Thus, if the machines are in the steady states, the eigenvalue $(1 - R)^2$ indeed characterizes the transients of the buffer.

The second argument is as follows: Recall that if $R = e$, the machines can be viewed as obeying the Bernoulli reliability model. In this case, the machines have no transients, and the transients of the system are defined by $\lambda_b = (1 - e)^2$, which, as it follows from Meerkov and Zhang (2008), is equivalent to the Bernoulli case with $p = e$.

From (4.2), it is not immediately clear which of the eigenvalues is the SLE. Obviously, the SLE can be either $1 - P - R$ or $(1 - R)^2$, i.e., either λ_m or λ_b. Which one is, in fact, the SLE depends on the relationship between P

and R. To investigate when λ_m or λ_b is SLE, consider the simplex $0 < P < R < 1$ in the (P, R)-plane (see Figure 4.1). Each point (P, R) implies $e > 0.5$ and each line, $P = kR$, $k < 1$, represents a set of points (P, R) with identical efficiency $e = \frac{1}{1+k}$. Let λ_1 denote the SLE, i.e.,

$$|\lambda_1| = \max\{|\lambda_m|, \lambda_b\}.$$

Then, it can be shown that

$$|\lambda_1| = \begin{cases} \lambda_m, & \text{if } 0 < P < R(1-R), \\ \lambda_b, & \text{if } R(1-R) < P < (1-R)(2-R), \\ -\lambda_m, & \text{if } (2-R)(1-R) < P < 1. \end{cases} \quad (4.5)$$

This leads to the partitioning of the simplex according to SLE as shown in Figure 4.1. Thus, in area I, the transients of the system are defined by an individual machine; in area II, the transients are defined by the buffer; in area III, the transients are again defined by the machine, however, since the eigenvalue in this area is negative, the transients in area III are oscillatory.

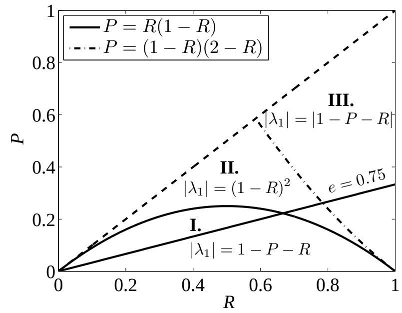

Fig. 4.1. Partitioning of the simplex $0 < P < R < 1$ according to SLE

Next, we characterize the effects of shorter and longer up- and downtimes on the duration of transients.

Theorem 4.2. Consider a geometric line with two identical machines and $N = 1$. Then, for any fixed $e > 0.5$, the SLE is a monotonically decreasing function of R for $R \in (0, 0.5)$.

Thus, for $T_{down} > 2$, shorter up- and downtimes lead to faster transients than longer ones, even if machine efficiency $e > 0.5$ remains the same. This phenomenon is illustrated in Figure 4.2.

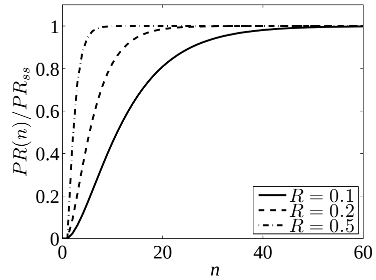

Fig. 4.2. Transients of PR for $e = 0.9$

In addition, the following can be obtained regarding the effects of increasing uptime or decreasing downtime on system transients:

Theorem 4.3. Consider a geometric line with two identical machines and $N = 1$. Let $|\lambda_1^u|$ and $|\lambda_1^d|$ denote the SLEs resulting from increasing the uptime by $(1+\alpha)$, $\alpha > 0$, and decreasing its downtime by the same factor, respectively. Then, under assumption (3.8),

$$|\lambda_1^u| > |\lambda_1^d|. \quad (4.6)$$

Thus, the qualitative effect of the uptime and the downtime on the transients in two-machine lines with $N = 1$ remains the same as that for individual machines: under (3.8), it is better to reduce the downtime than increase the uptime in order to shorten the transients. This phenomenon is illustrated in Figure 4.3.

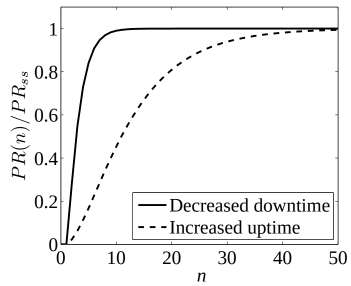

Fig. 4.3. Transients of PR with increased uptime or decreased downtime for $e = 0.7$, $R = 0.1$ and $e' = 0.9$

5. TRANSIENTS OF 2-MACHINE LINES WITH $N \geq 2$

A direct analytical investigation of transients in two-machine geometric lines with $N \geq 2$ is all but impossible due to high dimensionality of the resulting Markov transition matrices. Therefore, we resort to approximations.

Clearly, the dynamic behavior of the production rate is given by

$$PR(n) = P[\text{buffer is not empty at } n]P[m_2 \text{ is up at } n]. \quad (5.1)$$

The second term in the right hand side of this expression, as it follows from Section 3, is given by

$$1 + \frac{\Delta}{e}\lambda_m^n, \quad (5.2)$$

where Δ is defined in (3.5). We approximate the first term by reducing the geometric line to a Bernoulli one with the machines defined by

$$p^{Ber} = \frac{R}{P+R} \quad (5.3)$$

and the buffer capacity

$$N^{Ber} = [NR + 1], \quad (5.4)$$

where $[x]$ denotes the nearest integer to x. For such a line, $PR^{Ber}(n)$, $n = 0, 1, \ldots$, can be easily calculated (see Meerkov and Zhang (2008)). We use $PR^{Ber}(n)$ to approximate the first term in (5.1) taking into account that one time slot in the Bernoulli line is considered as one downtime in the original geometric line. In addition, since in the Bernoulli line, the flows in and out of the buffer are stationary, we assume that the first machine of the geometric line also reaches its steady state. This leads to the approximation

$$\widehat{PR}(n) = PR^{Ber}\left(\frac{n}{T_{down}}\right)\left(1+\frac{\Delta}{e}\lambda_m^n\right)^2, \qquad (5.5)$$

where the additional multiplier $(1+\frac{\Delta}{e}\lambda_m^n)$ accounts for the transients of the first machine.

The accuracy of (5.5) has been investigated numerically using 50,000 lines constructed by selecting the parameters randomly and equiprobably from the following sets:

$$e \in [0.6, 0.95], \qquad (5.6)$$

$$R \in [0.05, 0.5], \qquad (5.7)$$

$$N \in \{2, 3, \ldots, 40\}. \qquad (5.8)$$

A typical example is shown in Figure 5.1, where the accuracy $\epsilon(n)$ is defined by

$$\epsilon(n) = \frac{\widehat{PR}(n)}{\widehat{PR}(\infty)} - \frac{PR(n)}{PR(\infty)}. \qquad (5.9)$$

As one can see, the accuracy is sufficiently high.

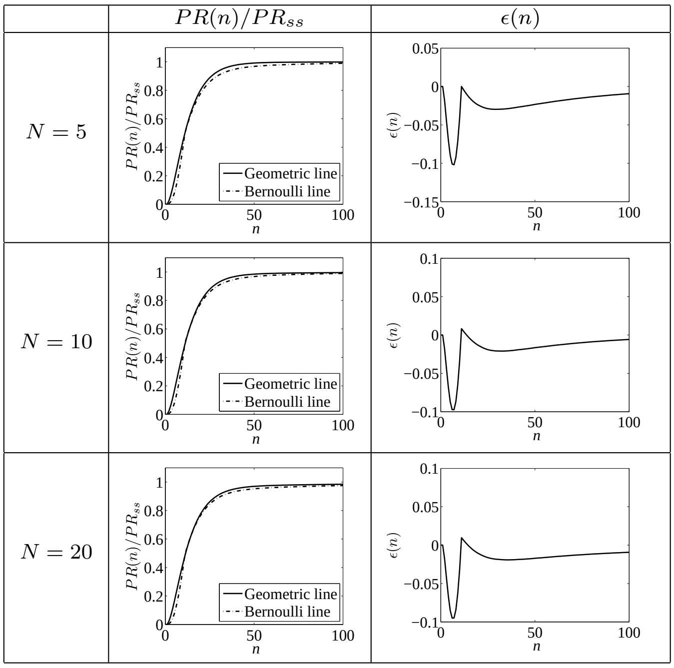

Fig. 5.1. Illustration of the accuracy of expression (5.5) for $e = 0.9$ and $R = 0.1$

Using approximation (5.5), the effects of up- and downtime on the transients can be evaluated. Since this is carried out numerically, we formulated the results as numerical facts.

Numerical Fact 5.1. Consider a geometric line with two identical machines having $e > 0.5$ and $N \geq 2$. Then, for any $T_{down} > 2$, shorter up- and downtimes lead, practically always, to faster transients than longer ones.

Numerical Fact 5.2. Under condition (3.8), reducing downtime leads, practically always, to shorter transients than increasing uptime.

As it is shown in the justification of these numerical facts, the term "practically always" is quantified as 99% for Numerical Fact 5.1 and 96% for Numerical Fact 5.2.

6. CONCLUSIONS AND FUTURE WORK

This paper provides a characterization of transients in two-machine geometric production lines. It is shown that, in some cases, the system's transients can be analyzed by separating the transients of the machines and the transients of the buffer. When the buffer is of capacity 1, this separation is exact; for longer buffers the separation is approximate. In either case, it is shown that if the machines' efficiency is greater than 0.5 and the average downtime is larger than two cycle times, shorter up- and downtimes lead to faster transients than longer ones. Under the same condition, it is shown that a reduction in downtime leads to faster transients than a similar increase of the uptime.

Future work will address transients in geometric lines with more than two machines and production lines with other machine reliability models, e.g., exponential, Weibull, log-normal, etc. For non-Markovian machines, the effect of the coefficients of variation of up- and downtime on the duration of transients will be investigated.

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Appendix A. PROOFS AND JUSTIFICATIONS

Proof of Theorem 3.1: It follows from (3.3) that

$$\lambda_1^u = 1 - \frac{P}{1+\alpha} - R, \qquad (A.1)$$

$$\lambda_1^d = 1 - P - (1+\alpha)R. \qquad (A.2)$$

Solving inequalities $|\lambda_1^u| - |\lambda_1^d| > 0$ and $|\lambda_1^u| - |\lambda_1^d| > 0$ results in

- $|\lambda_1^u| - |\lambda_1^d| > 0$, if $(1+\frac{\alpha}{2})R < e'$,
- $|\lambda_1^u| - |\lambda_1^d| < 0$, if $(1+\frac{\alpha}{2})R > e'$.

It follows immediately from (3.8) that

$$(1+\frac{\alpha}{2})R < (1+\alpha)R < 0.5 < e < e'.$$

Thus, under condition (3.8),

$$|\lambda_1^u| > |\lambda_1^d|.$$

Proof of Theorem 4.1: For matrix A given in (4.3), there exists a nonsingular matrix Q such that

$$A = Q^{-1}\tilde{A}Q, \tag{A.3}$$

where

$$\tilde{A} = diag[1 \quad \lambda_b \quad \lambda_m \quad \lambda_m \quad \lambda_m^2 \quad 0 \quad 0 \quad 0].$$

Thus,

$$x(n+1) = Ax(n) = Q^{-1}\tilde{A}Qx(n) = Q^{-1}\tilde{A}_nQx(0),$$

where

$$\tilde{A}_n = diag[1 \quad \lambda_b^n \quad \lambda_m^n \quad \lambda_m^n \quad (\lambda_m^2)^n \quad 0 \quad 0 \quad 0].$$

Hence, the evolution of the states can be expressed as

$$\begin{aligned} x_{h,i,j}(n) = x_{h,i,j}[1 + \tilde{B}\tilde{x}_2(0)\lambda_b^n + (\tilde{C}_1\tilde{x}_3(0) + \\ \tilde{C}_2\tilde{x}_4(0))\lambda_m^n + \tilde{D}\tilde{x}_5(0)(\lambda_m^2)^n], \\ h \in \{0,1\},\ i,j \in \{0,1\},\ n = 1,2,\ldots, \end{aligned} \tag{A.4}$$

where $\tilde{B}$, $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{D}$ are constants,

$$\tilde{x}_i(0) = q_i x(0) \tag{A.5}$$

and q_i is the i-th row of Q.

Then, it follows from (4.3) that

$$C = \tilde{C}_1\tilde{x}_3(0) + \tilde{C}_2\tilde{x}_4(0), \tag{A.6}$$

$$D = \tilde{D}\tilde{x}_5(0). \tag{A.7}$$

For matrix Q, it can be obtained that

$$\begin{bmatrix} q_3 \\ q_4 \\ q_5 \end{bmatrix} = \frac{P^2}{(-R+P+R^2)\,(R+P)^2} \begin{bmatrix} R^2 & -RP & R^2 & -RP & R^2 & -RP & R^2 & -RP \\ R & R & -P & -P & R & R & -P & -P \\ -\frac{R^3}{P^2} & \frac{R^2}{P} & \frac{R^2}{P} & -R & -\frac{R^3}{P^2} & \frac{R^2}{P} & \frac{R^2}{P} & -R \end{bmatrix}.$$

Moreover, initial condition (4.4) implies that

$$\sum_{h,j} x_{h,1,j}(0) = \sum_{h,i} x_{h,i,1}(0) = e,$$

$$\sum_{h,j} x_{h,0,j}(0) = \sum_{h,i} x_{h,i,0}(0) = 1-e.$$

In addition, since m_1 and m_2 are independent,

$$\sum_{h,i\neq j} x_{h,i,j}(0) = 2e(1-e),$$

$$\sum_h x_{h,0,0}(0) = (1-e)^2,$$

$$\sum_h x_{h,1,1}(0) = e^2.$$

Thus, under (4.4),

$$\tilde{x}_3(0) = \frac{P^2[R^2(1-e) - RPe]}{(-R+P+R^2)\,(R+P)^2} = 0,$$

$$\tilde{x}_4(0) = \frac{P^2[R(1-e) - Pe]}{(-R+P+R^2)\,(R+P)^2} = 0,$$

$$\tilde{x}_5(0) = \frac{R[2RPe(1-e) - R^2(1-e)^2 - P^2e^2]}{(-R+P+R^2)\,(R+P)^2} = 0.$$

Therefore, due to (A.6) and (A.7),

$$C = D = 0.$$

Proof of Theorem 4.2: Since $|1-P-R|$ and $(1-R)^2$ are both monotonically decreasing functions of R on $(0, 0.5)$ for a fixed e, the SLE of the system is a monotonically decreasing function of R on $(0, 0.5)$.

Proof of Theorem 4.3: It follows from Theorem 3.1 that

$$|\lambda_m^u| > |\lambda_m^d|. \tag{A.8}$$

In addition,

$$\lambda_b^u = (1-R)^2 > [1-(1+\alpha)R]^2 = \lambda_b^d.$$

Thus,

$$|\lambda_1^u| = \max(|\lambda_m^u|, \lambda_b^u) > \max(|\lambda_m^d|, \lambda_b^d) = |\lambda_1^d|.$$

Justification of Numerical Fact 5.1: This justification was carried out by evaluating the settling time of production rate, t_{sPR}, which is the time necessary for PR to reach and remain within $\pm 5\%$ of its steady state value, provided that the buffer is initially empty. A total of 10,000 lines were generated with e and N randomly and equiprobably selected from the sets (5.6) and (5.8), respectively. For each line, thus constructed, t_{sPR} is evaluated using approximation (5.5) as a function of R. As a result, we obtained that t_{sPR} is a monotonically decreasing function of R on $R \in (0, 0.5)$ in 99% of all cases studied. Thus, we conclude that shorter up- and downtimes lead, practically always, to faster transients, i.e., Numerical Fact 5.1 holds.

Justification of Numerical Fact 5.2: To justify this numerical fact, the 50,000 lines generated as mentioned in Section 5 were used to investigate the effects of increasing uptime or decreasing downtime on t_{sPR}. To accomplish this, we selected α randomly and equiprobably from the set

$$\alpha \in \{0.05, 0.1, \ldots, 1\}$$

and evaluated the settling times t_{sPR}^u and t_{sPR}^d, resulting from increasing uptime by $(1+\alpha)$ and decreasing downtime by $(1+\alpha)$, respectively. It turned out that t_{sPR}^u was longer than t_{sPR}^d in 96.12% of all cases studied. For the remaining 3.88% of cases, t_{sPR}^u was shorter than t_{sPR}^d by at most 1 cycle time. Therefore, we conclude that Numerical Fact 5.2 takes place.