

The Ordered Timeline Game: Strategic Posting Times over a Temporally-Ordered Shared Medium

Eitan Altman · Nahum Shimkin

Revised, March 5, 2015

Abstract We consider a game of timing between a random number of content creators, who compete for position and exposure time over an ordered shared medium such as an on-line classified list. Contents (such as ads, messages, multimedia items or comments) are ordered according to their submission times, with more recent submissions displayed at the top (and better) positions. The instantaneous effectiveness of each item depends on its current display position, as well as on a time-dependent site exposure function which is common to all. Each content creator may choose the submission time of her item within a finite time interval, with the goal of maximizing the total exposure of this item. We formulate the problem as a non-cooperative game, and analyze its symmetric Nash equilibrium. We show existence of the equilibrium profile, characterize it in terms of a differential boundary value problem, provide sufficient conditions for its uniqueness, and devise a numerical scheme for its computation. We further compute the equilibrium profile explicitly for certain special cases, which include a two-player small match and a Poisson-distributed number of players, and evaluate the social efficiency of these equilibria.

Keywords Timelines game · Timing games · User equilibrium · Social networks

Eitan Altman
INRIA, 2004 Route des Lucioles, 06902 Sophia-Antipolis Cedex, France
E-mail: altman@sophia.inrial.fr

Nahum Shimkin
Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel
E-mail: shimkin@ee.technion.ac.il

1 Introduction

Consider an online media site, arranged as an ordered list, to which each of several users can post one item such as advertisements, news items, on-line comments, discussion messages, pictures, video clips, and so on. We shall henceforth refer to these users as *players*. Each individual player can choose the time to post his or her item. Posted items are positioned on the list in descending order according to their arrival time, with the latest item displayed at the top (best) position, and is gradually moved to lower positions (and eventually removed) as new items arrive and take its place.

Each player is interested in maximizing the exposure, or effectiveness, of his own posted item. A couple of time-related factors determine this effectiveness (in addition, of course, to the appeal of the item's content). One is the overall exposure of the site, or viewer potential, which in general depends on time (e.g., on the time of day or day in the week, according to the application). The second is presumed rate of postings by other players, which may lead to closely spaced posts that compete for the viewers' attention. In our model, taking the view point of a specific player, each new post that is submitted after our own lowers our position on the ordered list, leading to lower exposure potential. The latter interaction between the players leads to a noncooperative game of timing, which we analyze in this paper.

The above-described list structure bears some similarity to a standard LIFO (last in first out) queue, in that later arrivals are placed at more favorable positions. However, there is no service here, and the queue advances solely due to new arrivals. In terms of applications, our model fits various chronologically-arranged lists on the Web, which include online classified ad sites (such as Craigslist and many others), comment and response posting on news websites, Internet discussion forums, online book and product customer reviews and equity-specific commentary items on financial sites.

There are of course many other situations that present a similar timing tradeoff between peak demand periods and the level of peer competition. These may range from finding the best time to share a link on a social network, to marketing oriented decisions of choosing the time to release a new product or launch a publicity campaign. See, for example, the discussions in [17,16,3] regarding the seasonality of demand and new product introduction in the food market and in the motion picture industry, and the game formulation of these timing problems in [12,11]. In the online advertising context, practical guides address the reduction of ad bid prices by avoiding periods of high advertiser competition, see [1].

In the present model, The instantaneous effectiveness (or exposure) of a displayed item is determined by its position in the list (with higher positions considered better), modulated by a time-dependent site exposure function $u(t)$ that represents the expected traffic or interest in the site at time t . The number of players is finite and possibly random, which accounts for stochastic user demand. Each participating player chooses the submission time of his own item, and aims to maximize the cumulative exposure of this item. Our interest

will be in the Nash equilibrium of the resulting game, and in particular in the *symmetric* equilibrium point.

Related literature: The closest work to ours is [15], which in fact introduced a simplified version of the present model that features a single list position. In that paper the symmetric equilibrium was explicitly computed for the two-player game, and a discretization scheme was proposed for computing an ϵ -optimal equilibrium for the n -player case. Our model extends the one studied in [15] in several respects:

- A time-varying exposure function $u(t)$ is considered, rather than a constant one.
- We allow for multiple list positions, rather than a single one.
- We allow the number of participating players to be stochastic, representing uncertainly with regard to the advertiser demand in a particular instance of the game.

As we will see, considering the game with a random number of players will have particularly interesting consequences when that number has a Poisson distribution.

Several classes of timing games have been considered in the literature, where the sole action of each player is a stopping time. Some classical two-player games of timing, which include the the War of Attrition and preemption game, are reviewed in [4], Chapter 4.5; more recent pointers may be found in [13]. In the transportation literature, Vickrey’s bottleneck model [18] addresses strategic timing decisions of commuters who balance road congestion delay with late or early arrival to their destination, where congestion is modeled as a fluid queue. This model has been extensively studied and extended in various directions – see, e.g., [14,9] and their references. A similar model which incorporates a stochastic queue with exponentially distributed service times was introduced in [5], and further extended recently in [7,10,8]. It is interesting to note that in these papers, since the players are lined up in a First Come First Served queue, only players that arrive before us affect our payoff. In the present model the opposite is true – due to the pushback queue mechanism, only players that arrive *after* us affect our payoff.

Main results: For our general model, we characterize the symmetric equilibrium point and provide a numeric procedure for its computation. It is first shown that the equilibrium strategy of each player is mixed, namely a probability distribution over the game interval $[0, T]$, which is continuous and supported on a sub-interval $[0, L]$. We refer to that distribution as the *equilibrium profile*. The equilibrium profile is characterized in terms of an integral equation, or equivalently in terms of a functional differential equation (FDE) that evolves backward in time. Using the latter characterization we establish existence of an equilibrium profile, and provide a numeric procedure that involves a single-parameter search for computing all symmetric equilibria. We also provide a sufficient condition for *uniqueness* of the symmetric equilibrium, in terms of a certain convexity requirement on the *relative utility parameters* of the different list positions. That condition holds trivially in the case of a single

list position ($K = 1$), and more generally when the relative utility parameters decreases to zero linearly or geometrically in the list position. We also provide an expression for the *social efficiency* of the equilibrium solution, namely the under-usage of the board in equilibrium relative to its full potential occupancy.

In some specific cases of interest, the equilibrium profile (and associated social efficiency) are obtained explicitly. The first is the case of *Poisson demand*, where the number of participating players follows a Poisson distribution. With a single list position, the arrival density in equilibrium turns out to be directly proportional to the *instantaneous value* of the exposure function $u(t)$. That property, while intuitively appealing, is not at all self-evident in the general model, and indeed we show that it holds only for Poisson demand. We further extend this explicit solution to the case of an infinite (unlimited) number of list positions, whose relative effectiveness is geometrically decreasing. The second case that is solved explicitly is that of a two-player game: The solution here is similar in form to the one in [15], but is derived under the general framework of our model, namely time-varying exposure function, multiple list positions, and random player participation.

The remainder of the paper is organized as follows. Section 2 presents the system and game model. Section 3 presents our main results for the general model, including existence, uniqueness, structure, and computation of the equilibrium profile. The proof of these results are collected in Section 3. Sections 4 and 5 discuss the special cases of Poisson demand and two-player game, respectively, while Section 6 presents some numerical results. We conclude the paper in Section 7.

2 Model Description

We consider an on-line display area, or advertising board, on which users (players) can post their items during a given time interval $[0, T]$. The board is arranged as an ordered list of $K \geq 1$ positions, where position 1 is the most effective, and last position K the least effective. The position of the posted items on the list is dynamically determined according to their order of arrival: a newly arrived item is placed at the top position, while existing items are pushed one position lower (from 1 to 2, 2 to 3 etc.). The item in position K (if any) is dropped from the list.

For $t \in [0, T]$ and $k = 1, \dots, K$, let $u_k(t)$ denote the expected utility rate (per unit time) for a displayed item at time t in position k . The total expected utility over the entire life cycle of an item is therefore

$$U(\mathcal{T}_1, \dots, \mathcal{T}_K) = \sum_{k=1}^K \int_{t \in \mathcal{T}_k} u_k(t) dt,$$

where \mathcal{T}_k is the time interval on which the item was displayed at position k . We assume that the functions $u_k(t)$ are decomposed as

$$u_k(t) = r_k u(t), \quad t \in [0, T], \quad k = 1, \dots, K,$$

where

- The *exposure function* $u(t)$, which is common to all positions, captures the the temporal dependence of the utility, due to variation in the exposure of the entire list.
- The constants $(r_k, k = 1, \dots, K)$ are the *relative utility parameters*, which capture the relative effectiveness of the different positions in the list. That is, the relative utilities are positive, and are decreasing in the list position k . It will be convenient for our analysis to define $r_k = 0$ for $k \geq K + 1$.

The following assumptions are imposed throughout the paper.

Assumption 1

- (i) The exposure function $u : [0, T] \rightarrow \mathbb{R}$ is continuous and strictly positive, namely $u(t) > 0$ for $t \in [0, T]$. Let $u_{\min} > 0$ and u_{\max} denote the extremal values of u .
- (ii) The relative utility parameters (r_k) are decreasing in the list position. Specifically,

$$r_1 > r_2 \geq r_3 \cdots \geq r_K > 0.$$

Remarks

1. Note that we require a strict inequality in $r_1 > r_2$, namely, the first list position is strictly better than the second (if the latter exists). This is convenient for technical reasons that will be pointed out in Section 6. The case of $r_1 = r_2$ may still be handled directly, but is more conveniently approximated by taking r_1 slightly larger than r_2 .
2. The requirement for *strict* positivity of $u(t)$ is not essential, and is imposed here as it simplifies the statement of the results. In case $u(t) = 0$ on some interval, all our main results can be seen to hold except that $F'(t) = 0$ on that interval.

The game formulation involves several players, who compete for a place in the list and wish to to maximize their individual utilities. Each player i chooses the submission time t_i of his own item. As mentioned, upon submission, the item is placed in the top position, but goes down in rank as further items are posted. Clearly, the utility of each player depends on his or her own choice of submission time (t_i), as well as the submission times of the other player. We therefore consider the problem as a non-cooperative game, and analyze the Nash equilibrium of this game.

To complete the game description, we specify some additional properties.

1. The number of players who participate in a given instance of the game is a random variable, denoted D_0 . We refer to D_0 as the *objective demand*.
2. The belief of each participating player regarding the number of *other* players in the game is another random variable, denoted D . We refer to D as the *subjective demand*. Clearly, if D_0 is deterministic then $D = D_0 - 1$. The general relation between D and D_0 is discussed in Subsection 2.3. Let $p_D = (p_D(n), n \geq 0)$ denote the distribution of D . We assume that

$E(D) < \infty$, and further, to avoid triviality, that $r_{D+1} < r_1$ with positive probability¹.

3. A player cannot observe the submission times of others before choosing his own submission time; in particular, the players do not observe the list status before their arrival. (As we shall see, the latter assumption can be relaxed when D follows a Poisson distribution.)
4. The submission time t_i of player i can be chosen randomly, according to a probability distribution on $[0, T]$ with cumulative distribution function $F_i(t)$, $t \in [0, T]$. The corresponding density function, when it exists, will be denoted by $f_i(t)$. We refer to F_i as the (mixed) *strategy* of player i .

We shall be interested in the Nash equilibrium point (NEP) of this game. Specifically, we consider the *symmetric* NEP in which the strategies of all players are identical, namely $F_i \equiv F$ all players. The restriction to symmetric strategies, besides its analytical tractability, seems natural in the present scenario where players are essentially anonymous. We proceed to calculate the players' utilities for the symmetric case.

2.1 Expected Utility

Consider a certain player i who posts his item at time t . Suppose that each of the other D players uses an identical strategy F . We proceed to calculate the expected utility $U(t; F)$ of the player in that case.

Suppose first that F has no point mass at t , so that with probability 1 there are no simultaneous arrivals at t . Let $N_{(t,s]}$ denote the number of arrivals (by other players) during the time interval $(t, s]$, for $t < s \leq T$. Since i arrives at t , his position in the list at time s will be $k+1$ if $N_{(t,s]} = k$, for $0 \leq k \leq K-1$, and he would have left the list if $N_{(t,s]} \geq K$. It follows that

$$\begin{aligned} U(t; F) &= \mathbb{E}_F \left(\int_t^T \sum_{k=0}^{K-1} r_{k+1} \mathbf{1}_{\{N_{(t,s]}=k\}} u(s) ds \right) \\ &= \int_t^T \sum_{k=0}^{K-1} r_{k+1} \mathbb{P}_F(N_{(t,s]} = k) u(s) ds. \end{aligned} \quad (1)$$

To compute the probability $\mathbb{P}_F(N_{(t,s]} = k)$, recall that the number of participating players other than i is a random variable D . The probability that each of these players submits his item on $(s, t]$ is $F(s) - F(t)$. Therefore, conditioned on $D = n$, $N_{(t,s]}$ follows a Binomial distribution $\text{Bin}(n, p)$ with success probability $p = F(s) - F(t)$. Denoting

$$B_{k,n}(p) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n \quad (2)$$

¹ Otherwise, all $D+1$ players can arrive at $t=0$ and remain in positions with maximal relative utility r_1 all the way up to T .

(and setting $B_{k,n} \equiv 0$ for $k > n$), we obtain

$$\begin{aligned} \mathbb{P}_F(N_{(t,s]} = k) &= \sum_{n \geq k} p_D(n) \mathbb{P}_F(N_{(t,s]} = k | D = n) \\ &= \sum_{n \geq k} p_D(n) B_{k,n}(F(s) - F(t)). \end{aligned} \quad (3)$$

Substituting (3) in (1) gives

$$U(t; F) = \int_t^T h(F(s) - F(t)) u(s) ds. \quad (4)$$

where

$$h(p) \triangleq \sum_{k=0}^{K-1} r_{k+1} \sum_{n \geq k} p_D(n) B_{k,n}(p), \quad p \in [0, 1].$$

We mention some special cases of this expression.

Single List Position: When $K = 1$, noting that $B_{0,n}(p) = (1 - p)^n$ we obtain

$$h(p) = r_1 \sum_{n \geq 0} p_D(n) (1 - p)^n.$$

Deterministic Demand: Suppose D is a deterministic positive integer (which corresponds to a game with $D_0 = D + 1$ players). In that case

$$\mathbb{P}_F(N_{(t,s]} = k) = B_{k,D}(F(s) - F(t)),$$

and

$$h(p) = \sum_{k=0}^{K-1} r_{k+1} B_{k,D}(p).$$

Poisson Demand: Suppose $D \sim \text{Pois}(\Lambda)$, a Poisson random variable (RV) with parameter $\Lambda > 0$, namely $p_D(n) = \Lambda^n e^{-\Lambda} / n!$ for $n \geq 0$. Since a Bernoulli dilution of a Poisson RV remains Poisson, it follows that $N_{(t,s]}$ is a Poisson RV with parameter $\Lambda(F(s) - F(t))$, and

$$\mathbb{P}_F(N_{(t,s]} = k) = \frac{1}{k!} \Lambda^k (F(s) - F(t))^k e^{-\Lambda(F(s) - F(t))}.$$

This expression can be directly substituted in Equation (1).

Simultaneous arrivals: If F has a point mass at t , then there is a positive probability of simultaneous arrivals of several players at that time. In that case we assume that their order of arrival (and subsequent positioning on the list) is determined uniformly at random. The utility $U(t; F)$ needs to be modified accordingly. We need not bother here with writing the straightforward but cumbersome expression, as we argue in Lemma 1 below that in equilibrium F does not possess point masses.

2.2 Nash Equilibrium

A symmetric Nash equilibrium point (NEP) is represented by a strategy F , which is a probability distribution on $[0, T]$, such that F is a best response for each player when all others use the same strategy F . More formally, for any pair of strategies G and F , let

$$\bar{U}(G; F) = \mathbb{E}_{t \sim G}(U(t; F)) = \int_t U(t; F) dG(t)$$

denote the expected utility of a player for using strategy G when all others follow F . Then F represents a symmetric NEP if

$$F \in \operatorname{argmax}_G \bar{U}(G; F),$$

where the maximum is taken over all probability distributions on $[0, T]$. We shall refer to a symmetric equilibrium strategy F as an *equilibrium profile*.

An equivalent definition of the symmetric NEP, that is more useful for the analysis, requires $U(t; F)$ to be maximized on a set of times t of F -probability 1. That is:

There exists a constant u^* and a set $A \subset [0, T]$, such that $\int_A dF(t) = 1$, and

$$U(t; F) = u^* \text{ for } t \in A, \quad (5)$$

$$U(t; F) \leq u^* \text{ for } t \notin A. \quad (6)$$

The equivalence of the two definitions is readily verified. We refer to the constant $u^* = u_F^*$ as the *equilibrium utility* corresponding to an equilibrium profile F .

2.3 Demand Distributions

Recall that the random variable D stands for the subjective demand, which represents the belief of a participating player regarding the number of other participating players, while the random variable D_0 stands for the objective demand, namely the number of players that participate in a given instance of the game. The objective load D_0 does not play a direct role in determining the equilibrium profile; it is required however for determining social utility, and of course influences D .

A well known formula from [2] connects the distributions of D and D_0 under symmetry conditions. Let $p_0(n) = P(D_0 = n)$ denote the distribution of D_0 . We are interested in the conditional probabilities $P(D = n - 1) = P(D_0 = n | A_i)$, where A_i is the event that a certain player i participates in the game. Then

$$P(D = n - 1) = \frac{1}{c} n p_0(n), \quad n \geq 1 \quad (7)$$

where $c = \sum_{n \geq 1} n p_0(n)$ is the normalization constant. The argument that leads to (7) is roughly as follows. Suppose that the D_0 participants are selected

symmetrically from a finite population of size N . By symmetry, the probability of a particular subset B of the population to be chosen depends only on its size $|B|$. By comparing the number of subsets of size n in which a given player i is selected to the total number of such subsets, we obtain:

$$p(A_i|D_0 = n) = \binom{N-1}{n-1} / \binom{N}{n} = \frac{n}{N}.$$

Invoking Bayes formula now leads to (7).

The relation (7) should be used with some caution. It relies on the assumption that participation decisions are fully symmetric, and that the potential participants know exactly the distribution of D_0 . Furthermore, the significant skewness (by a factor n) in the distribution of D in (7) may be attributed to correlation in the joining decisions of the players. However, if this correlation is induced by a public signal, then the players can actually condition on that signal before choosing their strategy.

Rather than dealing further with the general case, when considering specific demand distributions we will focus on two special cases where the above-mentioned correlation is absent. The first case is the deterministic case, $D_0 = n$, for which indeed $D = n - 1$ according to (7). The second case is the Poisson distribution, $D_0 \sim \text{Poiss}(\lambda)$, for which we obtain from (7) that $D \sim \text{Poiss}(\lambda)$, i.e., the same distribution as D_0 .

We note that both these cases can be viewed as extremes of the Binomial distribution, where n potential players decide independently whether to join or not. For $D_0 \sim \text{Bin}(n, p)$, we obtain $D \sim \text{Bin}(n - 1, p)$. The deterministic case is obtained for $p = 1$, while the Poisson case can be viewed as the limit of taking $n \rightarrow \infty$ while keeping $np = \lambda$.

3 Equilibrium Analysis

We present in this section the main properties of the equilibrium profile that apply to the general model. We first establish existence, as well as some useful structural properties that include characterization of the equilibrium distribution in terms of a functional differential equation, and a computational procedure that relies on this equation. We then present sufficient conditions for uniqueness of the equilibrium, and finally discuss the social efficiency of the equilibrium. The proofs of these results are presented in the next section.

3.1 Existence and Characterization

For an arrival profile F , let $F'(t)$ denote the time derivative of F at $t \in [0, T]$. Recall that the *support* of a probability measure η is the smallest closed set of η -probability 1. For brevity, we denote by $\text{supp}(F)$ the support of the probability measure η_F induced by a distribution function F .

Recall that $U(t; F)$ is the expected utility which is specified in (4). We will show in Lemma 3 that $\frac{d}{dt}U(t; F) = -r_1 u(t) + F'(t)g(t, F)$, where

$$g(t, F) = \sum_{k=0}^{K-1} (r_{k+1} - r_{k+2}) \sum_{n \geq k} (n+1) p_D(n+1) \int_t^T B_{k,n}(F(s) - F(t)) u(s) ds. \quad (8)$$

Thus, $g(t)$ represents the sensitivity of the utility $U(t)$ to variations in $F(t)$.

Theorem 1 (Existence and Characterization) *An equilibrium profile $F = (F(t), t \in [0, T])$ exists. Any equilibrium profile satisfies the following properties.*

- (i) F is a continuous function, and there exists a number $L \in (0, T)$ such that $\text{supp}(F) = [0, L]$. Specifically, $F(0) = 0$, $F(L) = 1$, and $F(t)$ is strictly increasing in $t \in [0, L]$.
- (ii) Consequently, a continuous probability distribution function F on $[0, T]$ is an equilibrium profile if, and only if, there exists a number $L \in (0, T)$ such that $F(0) = 0$, $F(L) = 1$, and $U(t; F) = u_L$ for $t \in [0, L]$ and some constant $u_L > 0$.
- (iii) Equivalently, a continuous probability distribution function F on $[0, T]$ is an equilibrium profile if, and only if, there exists a number $L \in (0, T)$ such that: $F(0) = 0$, $F(L) = 1$, and the derivative $F'(t)$ exists for $t \in (0, L)$ and satisfies the equality

$$F'(t) = \frac{r_1 u(t)}{g(t, F)}, \quad t \in (0, L), \quad (9)$$

where $g(t, F)$ is defined in (8).

- (iv) For an equilibrium profile F with support $[0, L]$, the equilibrium utility u_F^* is given by $u_F^* = r_1 \int_L^T u(s) ds$.

Proof outline: The full proof is presented Appendix A. Here we provide a brief outline, for the reader's benefit.

(i) To argue that F is continuous, suppose (see Lemma 1(i)) that F has an upward jump (i.e., a point mass) at t . This means that there is a positive chance that several ads are submitted simultaneously at t . But then it would be better to submit the ad just after t , leading to $U(t; F) < U(t^+; F)$, which contradicts the assumption that t is in $\text{supp}(F)$.

To show that $\text{supp}(F)$ is an interval $[0, L]$, suppose by contradiction that there exists a *gap* in that support, namely numbers $0 \leq a < b$ such that $F(a) = F(b) < 1$. We may extend b to the right till it hits $\text{supp}(F)$. But since there are no arrivals in (a, b) , it follows that $U(a; F) < U(b; F)$, which means that b cannot belong to the support $\text{supp}(F)$; see Lemma 1(iii).

(ii) If F is an equilibrium profile then $\text{supp}(F) = [0, L]$ by (i), and $U(t; F) = c$ on $t \in [0, L]$ follows by definition of the equilibrium and the continuity of $U(t; F)$ in t (Lemma 1(ii)). The converse follows by noting that if $F(L) = 1$, then there are no arrivals on $(L, T]$ which implies that $U(t; F) > U(L; F) = u_L$ for $t > L$.

(iii) Differentiating the expression (4) for $U(t; F)$ gives $\frac{d}{dt}U(t; F) = -r_1 u(t) + F'(t)g(t, F)$ (Lemma 3). Now, if F is an equilibrium profile then $U(t; F)$ is constant on $[0, L]$, then $\frac{d}{dt}U(t; F) = 0$ on $(0, L)$, and we obtain the differential relation (9). The converse statement follows by arguing that (9) implies that $U(t; F)$ is constant on $[0, L]$; see Lemma 4.

Part (iv) of the Theorem is straightforward, upon noting that there are no new arrivals on $(L, T]$ so that an ad submitted at $t = L$ remains at the top position till T .

The proof of *existence* of an equilibrium relies on the differential characterization in part (iii). For that purpose, we consider Equation (9) as a (functional) differential equation with terminal conditions $F(t) = 1$ for $t \in [L, T]$. It is argued that the solutions $F_L(t)$ to this equation are well defined and continuous in L (Proposition 1). From this we deduce (Lemma 5) that $F_L(0) = 0$ for some L , which implies that the corresponding solution $F_L(t)$ is an equilibrium profile. \square

Remarks

1. Note that property (iii) of the Theorem implies that F admits a density f , which is strictly positive on $[0, L)$ and zero on $(L, T]$.
2. The differential relation in (9) provides a *functional* differential equation for F , as the right-hand side depends on entire function F and not only on its value at t . This relation provides the basis for the existence and uniqueness claims in this and the next subsection, as well as for the computational procedure that follows.
3. The results of the last Theorem do not imply uniqueness of the equilibrium profile. However, if multiple equilibria do exist, we obtain a strict ordering among them. First, we note that to each number L and support $[0, L]$, there correspond at most one equilibrium profile (this follows by uniqueness of solutions to Equation (9), see Proposition 1 below). Consider now two equilibrium profiles F_1 and F_2 , indexed by L_1 and L_2 respectively, with $L_1 < L_2$. By part (iv) of the Theorem it follows that the equilibrium utility u^* of F_1 is strictly higher than that of F_2 . That is: the equilibrium with the smaller support $[0, L]$ is better (in terms of individual utilities).

3.2 Uniqueness

To establish uniqueness of the equilibrium profile, we require an additional condition on the relative utilities (r_k). We observe that this is only a sufficient condition, and uniqueness may well hold in greater generality.

Assumption 2 *The relative utility parameters satisfy the following convexity condition:*

$$r_k \leq \frac{1}{2}(r_{k-1} + r_{k+1}), \quad k = 2, \dots, K \quad (10)$$

(recall that $r_{K+1} = 0$ by definition).

Theorem 2 (Uniqueness) *Suppose Assumption 2 holds, then the equilibrium profile F is unique.*

We note that Assumption 2 holds trivially for a single list position, namely $K = 1$. It also holds for linearly-decreasing utilities of the form $r_1 = (K + 1)d, \dots, r_K = d$, for some $d > 0$, as well as for geometrically decreasing utilities: $r_k = \gamma^k$, $1 \leq k \leq k_0$ for $\gamma \in (0, 1)$ and $k_0 \leq K$ (if $\gamma \leq 0.5$ we can take $k_0 = K$; otherwise we need to continue r_k beyond k_0 , say linearly, so that the convexity condition holds up to $r_{K+1} = 0$). The assumption does *not* hold in the interesting case where $K \geq 2$ and some list positions have equal weights, say $r_1 = r_2$.

3.3 Computation

We outline next a numeric procedure for computing all equilibrium profiles. This procedure relies on computing the solution F of the functional differential equation (FDE) (9) for different values of the parameter $L \in (0, T)$, and searching for values of L for which the boundary conditions $F(L) = 1$ and $F(0) = 0$ are satisfied. The latter essentially involves an exhaustive search over the scalar parameter L . When the uniqueness condition in Assumption 2 is satisfied, that search can be expedited by observing monotonicity properties of the solution F in L .

For each $L \in (0, T)$, we consider the differential equation (9), with terminal conditions $F(t) = 1$, $t \in [L, T]$. As mentioned, this is a *functional* differential equation since the derivative $F'(t)$ at time t depends on values of F at other times as well. However, a key property of that equation, which follows from the definition of $g(t, F)$ in (8), is that this dependence is one-sided: $F'(t)$ depends only on 'future' values of F , namely on $(F(s), s \geq t)$. In the terminology of [6], this equation is a *retarded* FDE (up to time reversal). This property allows to back-integrate this equation, starting with the above-mentioned terminal conditions, and proceeding backward in time.

We collect some properties of the FDE (9) in the following Proposition. These support the numerical computation procedure that follows, and are also used in the proofs of the previous results.

Proposition 1 *For $L \in (0, T)$, define $F_L(t) = 1$ for $t \in [L, T]$. Consider the FDE (9) for F_L , namely*

$$F'_L(t) = \frac{r_1 u(t)}{g(t, F_L)}, \quad t \leq L \quad (11)$$

where g is defined in (8).

(i) *The FDE (11) admits a unique solution $F_L(t)$ over $t \in [t_L, L]$, where*

$$t_L = \inf\{s \in [0, L] : F_L(s) > 0\}.$$

- (ii) There exists a positive constant $\epsilon_L > 0$ such that $g(t, F_L) \geq \epsilon_L$ for all $t \in [t_L, L]$. Consequently, $F_L'(t) \leq (r_1 u_{\max}/\epsilon_L) < \infty$ for $t \in [t_L, L]$.
- (iii) t_L is a continuous function of L , and $F_L(t)$ is a continuous function of L for each $t \in (t_L, L]$.

Note that t_L is the first time $t \geq 0$ at which $F_L(t)$ becomes zero. As $g(t, F_L)$ is meaningless for $F_L < 0$, the solution of equation (11) cannot be extended beyond this point. For the sake of exposition, it will be convenient to linearly extend $F_L(t)$ below t_L when $t_L > 0$, using

$$F_L(t) = -(t_L - t), \quad t \in [0, t_L].$$

This gives $F_L(0) < 0$ when $t_L > 0$.

The equilibrium profiles now correspond to those values of L for which $F_L(0) = 0$. To search for these values, we may use the following crude exhaustive search approach:

- For values of L in a grid over $(0, T)$, integrate equation (11) numerically (using Euler approximation or a more accurate method) and obtain $F_L(t)$, $t \in [0, T]$.
- The equilibrium points correspond to values of L where $F_L(0) \simeq 0$.

The search grid may of course be refined around points of interest. We note that under the uniqueness condition (10), we know that $F_L(0)$ crosses 0 at a unique value of L , which clearly simplifies the search.

It should be possible to compute explicit bounds on the variation of $F_L(t)$ in L , which may lead to formal guarantees on ϵ -optimality of computed equilibrium profile. This is however beyond the scope of the present paper. We merely point out here that the level of sub-optimality of any candidate equilibrium profile F can be easily tested numerically by computing the utility function $U(t; F)$ in (4) and evaluating its deviation from a constant over $[0, L]$.

Remarks

1. We observe that if $F(t)$ is an equilibrium profile for the model with the given exposure function $u(t)$, then $\tilde{F}(t) = F(g(t))$ is an equilibrium profile for the model with unit exposure function, $u(t) \equiv 1$, where $g(t)$ denotes the time change

$$g(t) = \frac{T}{\int_0^T u(s) ds} \int_0^t u(s) ds.$$

This follows by verifying, through a change of integration variable, that $U(t; F)$ in equation (4) is equal to $U(g(t), \tilde{F})$ in the model with $u = 1$. Therefore, once the equilibrium profile is computed for one exposure function $u(t)$, it can be readily obtained for any other.

2. Our standing Assumption 1 requires a strict inequality in $r_1 < r_2$. If this is violated, namely $K \geq 2$ and $r_1 = r_2$, it may be seen that $g(L, F_L) = 0$, which leads to difficulties in initializing the backward solution of equation (11). A direct solution therefore requires more elaborate numeric schemes than the ones considered here. Rather than go into these, this case may be handled by solving an approximate problem with $r_1 = r_2 + \epsilon$, for ϵ small enough.

3.4 Social Efficiency

Under an equilibrium profile F with support $\text{supp}(F) = [0, L]$, the expected utility u_F^* per player is given by Theorem 1(iv). Recalling that D_0 is the subjective demand, the expected social utility may be seen to be

$$S(F) = E(D_0)u_F^* = E(D_0)r_1 \int_L^T u(s)ds.$$

Let us compare that to the optimal social utility. Under full utilization, namely all K positions occupied over $[0, T]$, the social utility is clearly

$$\sum_{k=1}^K r_k \int_{s=0}^T u(s)ds.$$

However, taking into account that if $n < k$ player arrive then some lower positions remain empty, the optimal social utility becomes

$$S^* = \sum_{k=1}^K r_k \left(\sum_{n \geq k} p_0(n) \right) \int_{s=0}^T u(s)ds. \quad (12)$$

The *relative efficiency* of the equilibrium is now defined and evaluated by the ratio:

$$\rho(F) \triangleq \frac{S(F)}{S^*} = \frac{r_1 E(D_0)}{\sum_{k=1}^K r_k \sum_{n \geq k} p_0(n)} \frac{\int_L^T u(s)ds}{\int_0^T u(s)ds}. \quad (13)$$

This expression still depends on the equilibrium parameter L . Explicit expressions for some specific cases will be derived in the following sections.

4 Special Case I: Poisson Demand

We consider in this section the special case of Poisson demand. After discussing some general relations, we further specialize to the case of a single list position, namely $K = 1$. We will also consider formally the more general model of an infinite list with geometrically decreasing utility parameters. In both cases we provide explicit formulae for the equilibrium profile, as well as for the associated efficiency loss.

Suppose, then, that system demand D_0 is a Poisson random variable with parameter $\Lambda > 0$, namely $D_0 \sim \text{Pois}(\Lambda)$, and $p_0(n) = \Lambda^n e^{-\Lambda} / n!$ for $n \geq 0$.

As mentioned in the Introduction, the Poisson assumption models the case where the participating players are chosen randomly and independently from a large population of potential participants. Also, as mentioned above, a unique property of the Poisson demand is that the subjective and objective demand distributions, related by (7), are equal. Hence, $D \sim \text{Pois}(\Lambda)$, and $p_D(n) = \Lambda^n e^{-\Lambda} / n!$ for $n \geq 0$.

The utility function $U(t; F)$ can now be evaluated by (4), using some algebra. More directly, since Bernoulli dilutions of a Poisson random variable remain Poisson, it follows that $N_{(t,s]}$ in (1) is a Poisson random variable with parameter $\Lambda(F(s) - F(t))$, and

$$\mathbb{P}_F(N_{(t,s]} = k) = \frac{1}{k!} \Lambda^k(F(s) - F(t)) e^{-\Lambda(F(s) - F(t))}.$$

Therefore,

$$U(t; F) = \int_t^T \sum_{k=0}^{K-1} r_{k+1} \frac{1}{k!} \Lambda^k(F(s) - F(t)) e^{-\Lambda(F(s) - F(t))} u(s) ds \quad (14)$$

4.1 Unit List

Consider now the case of a single list position, $K = 1$, with utility function $u_1(t) = r_1 u(t)$. From (14) we obtain

$$U(t; F) = r_1 \int_t^T e^{-\Lambda(F(s) - F(t))} u(s) ds \quad (15)$$

We proceed to derive (9) directly. By differentiating the last expression,

$$\frac{d}{dt} U(t; F) = -r_1 u(t) + r_1 \Lambda F'(t) \int_t^T u(s) e^{-\Lambda[F(s) - F(t)]} ds.$$

Since $\frac{d}{dt} U(t; F) = 0$ on $(0, L)$, we obtain

$$F'(t) = \frac{r_1 u(t)}{\Lambda r_1 \int_t^T e^{-\Lambda[F(s) - F(t)]} u(s) ds}, \quad t \in (0, L), \quad (16)$$

which is the explicit form of (9) in this case.

Observe now that the denominator of the last expression is just Λ times $U(t; F)$, while the latter identically equals u_F^* on $[0, L]$ by the basic equilibrium property. Therefore,

$$F'(t) = \frac{r_1}{\Lambda u_F^*} u(t), \quad t \in (0, L). \quad (17)$$

We see that $F'(t)$ is proportional to $u(t)$ on its support!

It remains to determine L and u_F^* . Recall from Theorem 1 that

$$u_F^* = U(L; F) = r_1 \int_L^T u(s) ds. \quad (18)$$

On the other hand, by (17) and the definition of L ,

$$1 = F(L) = \int_0^L F'(s) ds = \frac{r_1}{\Lambda u_F^*} \int_0^L u(s) ds,$$

which implies

$$u_F^* = \frac{r_1}{\Lambda} \int_0^L u(s) ds. \quad (19)$$

Comparing the last two expressions for u_F^* , we arrive at the following equation, from which L (and hence u_F^*) can be computed:

$$\int_L^T u(s) ds = \frac{1}{\Lambda} \int_0^L u(s) ds \quad (20)$$

Observe that this equation has a unique solution, since the left-hand side is continuously and strictly decreasing to zero, while the right-hand side is continuously increasing from zero.

We summarize these findings in the following Proposition:

Proposition 2 (Equilibrium Profile) *For the case of Poisson demand with mean Λ and a single list position ($K = 1$), the equilibrium profile F is given in terms of its density F' by*

$$F'(t) = \frac{1}{\int_0^L u(s) ds} u(t), \quad t \in (0, L), \quad (21)$$

where L is the unique solution in $(0, T)$ of equation (20).

For a simple example, consider the case of time-invariant exposure rate, $u(t) \equiv 1$. Here we obtain that the equilibrium profile corresponds to a uniform distribution on $[0, L]$, and from (20) we obtain $(T - L) = L/\Lambda$, so that $L = \frac{\Lambda}{\Lambda+1}T$. Note that for $\Lambda \rightarrow \infty$ (heavy load), $L \rightarrow T$, i.e., the support of the equilibrium profile extends to the entire interval $[0, T]$, as may be expected. This in fact holds for any exposure function u , as may be seen from (20).

It may be seen from (21) that the equilibrium density $F'(t)$ is proportional to the exposure function $u(t)$. This appears intuitively appealing – higher arrival probabilities are assigned to times with larger utility. However, it is actually remarkable that the arrival rate at time t depends on u only through $u(t)$ at t , and not at later times. Indeed, this *myopic* appearance of the equilibrium is unique to the case of Poisson-distributed demand, as the following converse shows.

Proposition 3 *For the model with $K = 1$, suppose that the equilibrium profile is of the form*

$$F'(t) = C_L^{-1} u(t), \quad t \in [0, L]$$

for some L , with $F(L) = 1$ (here $C_L = \int_0^L u(s) ds$ is the normalization constant). Then the demand D has a Poisson distribution.

Proof For $K = 1$, the utility function (4) reduces to

$$U(t; F) = r_1 \sum_{n \geq 0} p_D(n) \int_t^T B_{0,n}(F(s) - F(t)) u(s) ds.$$

By Theorem 1, $U(t; F) = u^*$ on $t \in [0, L]$. In the following let us abbreviate for convenience $F(t)$ as F_t , and similarly $F'(t)$ as F'_t and $u(t)$ as u_t . Note that $B_{0,n}(q) = (1 - q)^n$. Substituting $u_s = C_L F'_s$ for $s \leq L$ and $F_s = 1$ for $s \geq L$, we obtain for each of the last integrals

$$\begin{aligned} \int_t^T B_{0,n}(F_s - F_t)u_s ds &= \int_t^L (1 - F_s + F_t)^n C_L F'_s ds + F_t^n \int_L^T u_s ds \\ &= C_L \left[-\frac{1}{n+1} (1 - F_s + F_t)^{n+1} \right]_t^L + U_L F_t^n \\ &= \frac{C_L}{n+1} (1 - F_t^{n+1}) + U_L F_t^n, \end{aligned}$$

where $U_L \triangleq \int_L^T u_s ds$. Substituting the above expressions in the equilibrium equation $U(t; F) = u^*$ gives, for $t \in [0, L]$,

$$C_L \sum_{n \geq 0} \frac{p_D(n)}{n+1} (1 - F_t^{n+1}) + U_L \sum_{n \geq 0} p_D(n) F_t^n = \frac{u^*}{r_1}.$$

Observe that F_t is continuous and therefore assumes all values in $[0, 1]$ as t ranges from 0 to L . Therefore, for the last equation to hold for all $t \in [0, L]$, the respective coefficients of each power of F_t must sum up to zero. This gives:

$$-C_L \frac{p_D(n)}{n+1} + U_L p_D(n+1) = 0, \quad n \geq 0,$$

or $p_D(n+1) = \frac{\Lambda}{n+1} p_D(n)$, with $\Lambda = \frac{C_L}{U_L}$. This implies of course that p_D is a Poisson distribution with parameter Λ . \square

We turn to consider the efficiency of the Nash equilibrium in this case, which can be computed in closed form.

Proposition 4 (Efficiency) *The equilibrium utility is given by*

$$u_F^* = \frac{1}{1 + \Lambda} r_1 \int_0^T u(s) ds.$$

Consequently, the relative efficiency at equilibrium, as defined in (13), equals

$$\rho(F) = \frac{1}{(1 + \Lambda^{-1})(1 - e^{-\Lambda})}.$$

Proof The expression for the equilibrium utility follows from (18)-(19), which imply that

$$\int_0^T u(s) ds = \frac{\Lambda u_F^*}{r_1} + \frac{u_F^*}{r_1} = \frac{1 + \Lambda}{r_1} u_F^*.$$

Noting that $E(D_0) = E(D) = \Lambda$ under the Poisson distribution, we obtain $S(F) = \Lambda u_F^*$. Next, evaluating (12) with $K = 1$ and $\sum_{n \geq 1} = 1 - p_0(n) = 1 - e^{-\Lambda}$, we obtain

$$S^* = (1 - e^{-\Lambda}) r_1 \int_0^T u(s) ds.$$

Substituting $S(F)$ and S^* in (13) gives the required ratio $\rho(F)$. \square

Some observations are in order regarding last result.

- It is remarkable that the relative efficiency $\rho(F)$ does not depend on the time-dependent exposure function $u(t)$, but only on the mean demand Λ .
- The relative efficiency $\rho(F)$, which is smaller than 1 by definition, converges to 1 both when $\Lambda \rightarrow 0$ (where essentially at most one player shares the system) and, more importantly, when $\Lambda \rightarrow \infty$ (heavy demand).

4.2 Infinite Board with Geometric Utilities

The results of the previous Subsection can be readily be extended to the asymptotic model of an infinite list, $K = \infty$, with geometrically decreasing relative utility parameters: $r_k = r_1 a^{k-1}$, $k \geq 1$, with $0 < a < 1$.

Proceeding formally, with these definitions the expected utility function in (14) becomes,

$$\begin{aligned} U(t; F) &= r_1 \int_t^T \sum_{k=0}^{\infty} a^k \frac{1}{k!} \Lambda^k (F(s) - F(t))^k e^{-\Lambda(F(s)-F(t))} u(s) ds \\ &= r_1 \int_t^T e^{-\Lambda(1-a)(F(s)-F(t))} u(s) ds. \end{aligned} \quad (22)$$

It may be seen that (22) is identical to (15), once $\Lambda(1-a)$ is substituted for Λ . Thus, all the results of the previous subsection regarding the equilibrium profile F and equilibrium utility u_F^* are valid in the present model as well, after this substitution.

The relative efficiency can also be evaluated using a similar substitution. To see this, observe that

$$S(F) = \Lambda u_F^* = \frac{\Lambda}{1 + \Lambda(1-a)} r_1 \int_0^T u(s) ds.$$

Also, evaluating (12) can be seen to give in this case

$$S^* = \frac{1 - e^{-\Lambda(1-a)}}{1-a} r_1 \int_0^T u(s) ds.$$

Therefore,

$$\rho(F) = \frac{S(F)}{S^*} = \frac{1}{(1+x^{-1})(1-e^{-x})}, \quad x \triangleq \Lambda(1-a).$$

5 Special Case II: Two Players

The Poisson demand model of the previous section is well suited for a system with a large number of potential users. In this section we consider the opposite case – where exactly two players compete for position on the advertising board. As in the Poisson case, it will be possible here to attain explicit expressions for the equilibrium profile F .

We thus consider two players, where each participates in the game (by submitting an item) with probability $q \in (0, 1]$. The exogenous demand D_0 has the binomial distribution $\text{Bin}(n, q)$ with $n = 2$. Accordingly, the endogenous demand D as seen by a participating player is a Bernoulli random variable, with $p_D(1) = q$ and $p_D(0) = 1 - q$.

Since there are at most two participating players, it is clear that the maximal list size that need be considered is $K = 2$. Recall that r_1 and r_2 are the relative utility parameters of these two list positions, with $r_2 \leq r_1$. Note that when $r_1 = r_2$ the game becomes trivial, since then the two players are then indifferent between the two list positions. We therefore assume in the following that $r_2 < r_1$. Note further that $r_2 = 0$ corresponds to the single position case, $K = 1$. We shall return to this special case as an example towards the end of the section.

With $D \sim \text{Bern}(q)$, the function $g(t, F)$ in (8) becomes

$$g(t; F) = q(r_1 - r_2) \int_t^T u(s) ds. \quad (23)$$

Therefore, by Theorem 1,

$$F'(t) = \frac{r_1 u(t)}{q(r_1 - r_2) \int_t^T u(s) ds} = C^{-1} \frac{u(t)}{\int_t^T u(s) ds}, \quad t \in (0, L), \quad (24)$$

where $C \triangleq q(1 - \frac{r_2}{r_1}) < 1$. Integrating $F'(t)$, we further obtain that for $t \leq L$,

$$\begin{aligned} F(t) &= \int_0^t F'(\tau) d\tau = -C^{-1} \ln \left(\int_\tau^T u(s) ds \right) \Big|_{\tau=0}^t \\ &= C^{-1} \ln \left(\frac{\int_0^T u(s) ds}{\int_t^T u(s) ds} \right). \end{aligned}$$

It only remains to specify L . By the normalization condition $F(L) = 1$, we obtain

$$\int_L^T u(s) ds = e^{-C} \int_0^T u(s) ds, \quad (25)$$

and L is the unique solution of this equation. Uniqueness follows by monotonicity in L of the left-hand side. Hence, the equilibrium is unique in this case.

We can now compute the individual and social utility at equilibrium. First, by Theorem 1 and the last equation,

$$u_F^* = r_1 \int_L^T u(s) ds = r_1 e^{-C} \int_0^T u(s) ds. \quad (26)$$

The social utility at equilibrium is therefore

$$S(F) = E(D_0)u_F^* = 2qr_1 e^{-C} \int_0^T u(s) ds.$$

The *optimal* social utility, as defined in (12), evaluates here as

$$S^* = [2q(1-q)r_1 + q^2(r_1 + r_2)] \int_0^T u(s) ds.$$

Therefore,

$$\begin{aligned} \rho(F) &= \frac{S(F)}{S^*} = \frac{2qr_1 e^{-C}}{(2q - q^2) + q^2 r_2} \\ &= \frac{2e^{-C}}{2 - C}. \end{aligned}$$

It is easy to verify that $\rho(F)$ is a strictly decreasing function of the game parameter C .

We summarize our findings next.

Proposition 5 *For the two-player game with participation probability q :*

- (i) *The equilibrium profile is given by (24), where $C = q(1 - \frac{r_2}{r_1})$, and L is the unique solution of equation (25). The equilibrium utility u_F^* is specified in (26).*
- (ii) *The relative efficiency is given by $\rho(F) = 2e^{-C}/(2 - C)$. It approaches 1 as $C \rightarrow 0$, and attains its minimal value of $\rho(F) = \frac{2}{e}$ for $C = 1$.*

Remarks:

- Comparing (24) with (21), we see that here $F'(t)$ at t does depend on the entire exposure function beyond t . In fact, the denominator can be seen to be decreasing in t , thereby skewing the arrival density $F'(t)$ towards larger arrival times.
- The relative efficiency $\rho(F)$ depends solely on the single parameter $C = q(1 - \frac{r_2}{r_1})$. Notably, it does not depend on the exposure function $u(t)$.
- The extreme case of $C \rightarrow 0$ corresponds either a small joining probability $q \rightarrow 0$ for each player, or to $r_2 \approx r_1$. These values minimize the conflict between the players, which explains the convergence of the relative efficiency to 1. Indeed, it may be seen from (25) that $L \rightarrow 0$ correspondingly, and the equilibrium profile F approaches a unit atom at $t = 0$.

- The case of $C = 1$ corresponds to a deterministic number of participants ($D_0 = 2$), and a single list position (as $r_2=1$). As observed, the relative efficiency in this case equals $2/e$, independently of the exposure function $u(t)$.

Example. We close the section with the simple example of *constant exposure rate*: Suppose $u(t) = 1$, $t \in [0, T]$. From (25) we obtain that $(T - L) = e^{-CT}$, or

$$L = (1 - e^{-C})T,$$

and by (24),

$$F'(t) = \frac{C^{-1}}{T-t}, \quad t \in (0, L).$$

We observe that the density F' is strictly increasing in t , from $F'(0) = (TC)^{-1}$ to $F'(L) = \frac{C^{-1}}{T-L} = e^C(TC)^{-1}$.

6 Numeric Examples

We present next some numerical computations, that illustrate the equilibrium profile obtained for different choices of model parameters. We also compare the numerical results to our theoretical findings. The numerical results are obtained using the computational procedures outlined in Section 6, implemented in MATLAB. The basic Euler method is used to integrate the relevant differential equations.

We present two case studies – one with deterministic demand, and the other with Poisson demand. In both cases we assume a uniform exposure function, $u(t) = 1$ for $t \in [0, T]$. In view of Remark 1 in Section 6, the obtained equilibria for other exposure functions will be similar up to a time change. We further fix the time horizon at $T = 10$.

Case 1: Deterministic demand, with single list position ($K = 1$).

We first consider a deterministic number of players, and a single list position. The obtained equilibria profiles are shown in Figure 1 for $N = 2, 4, 11, 51$ players (note that the total number of players is $D+1$). We note that for $K = 1$ the equilibrium is unique by Theorem 2. For $N = 2$, the numeric results coincide with the theoretical results of Subsection 5. Indeed, following the example at the end of that subsection with $C = 1$, we obtain $F'(t) = 1/(T - t)$, and $L = (1 - e^{-1})T = 6.3$.

As the demand increases, L is seen to approach T , and the equilibrium distribution is close to uniform. This can be explained with the help of a high-demand fluid model, which is outside the scope of the present paper.

Case 2: Poisson demand.

Here we consider the system with a fixed Poisson demand, $D, D_0 \sim \text{Pois}(\Lambda)$, with $\Lambda = 3$.² Several values for the list size K were considered, with relative

² Recall from Subsection 2.3 that D and D_0 , the subjective and objective demands, has the same distribution in the Poisson case.

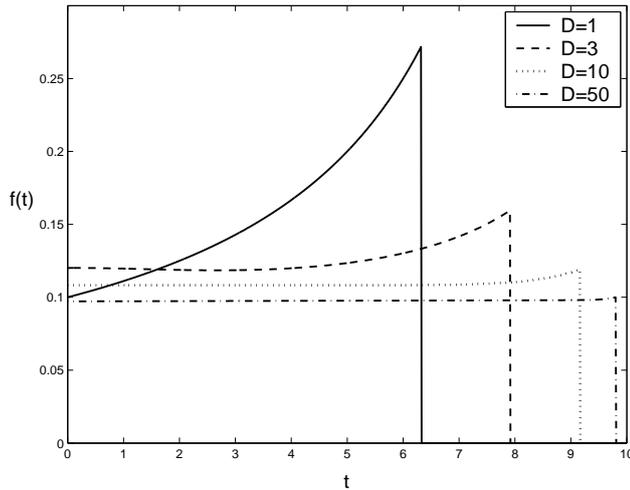


Fig. 1 The equilibrium density functions for deterministic demand (Case 1). The number of players is $N = D + 1$.

utility parameters than decrease geometrically according to $r_k = (0.5)^{k-1}$, $k = 1, \dots, K$. Thus for $K = 1$ we have $r_1 = 1$, and for $K = 3$ we have $(r_1, r_2, r_3) = (1, \frac{1}{2}, \frac{1}{4})$.

The results obtained for $K = 1, 2, 3, 6$ are depicted in Figure 2. For $K = 1$, the results confirm the theoretical prediction in Proposition 2, namely a uniformly-distributed equilibrium profile with L solving equation (20). Since $u = 1$ this equation gives $(10 - L) = L/3$, or $L = 7.5$.

For $K = 6$, since the r_k 's decay exponentially, the model can be approximated by the asymptotic model with $K = \infty$ that was studied in Subsection 4.2. As observed there, the obtained equilibrium is identical to the one obtained for a single-position case, with an effective demand of $\Lambda' = \Lambda(1 - a) = 1.5$. Thus, we expect a uniform distribution on $[0, L]$, with L determined from $(10 - L) = L/1.5$, which gives $L = 6$. This is indeed what was obtained numerically.

In contrast to the above extreme case, for intermediate values of $K = 2$ and $K = 3$, the equilibrium distribution may be seen to be non-uniform.

7 Conclusion

We have considered in this a stylized model of advertiser timing competition, where advertisers (or other media users) compete for position over a shared publication medium that gives priority to recently submitted items. Our results address the existence, uniqueness, and computation of the symmetric equilibrium profile, as well as some explicit solutions for certain special cases of the model.

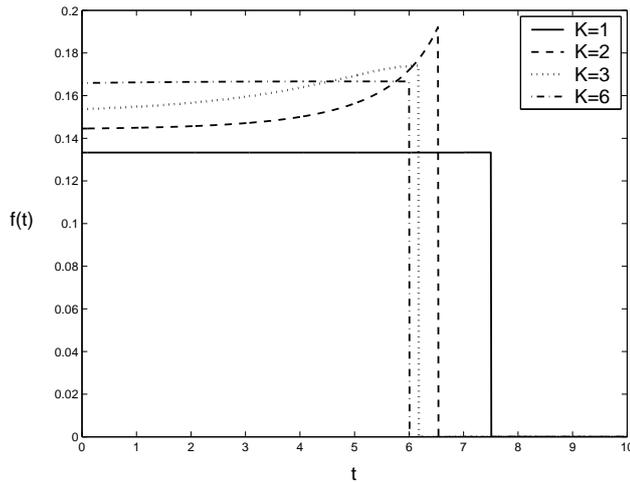


Fig. 2 The equilibrium density functions for Poisson demand (Case 2).

The basic model of this paper may be enriched in several directions, that include:

- Additional cost components, such as a one-time submission cost.
- Long-range periodic model: Our model assumes a finite and per-specified display interval $[0, T]$. It is of practical interest to consider a variant of this model that operates on a long (indefinite) duration, with a continuous supply of new ads (which may fluctuate periodically, say on a daily basis, along with the exposure function $u(t)$).
- Large-demand limits: When the number of submitted ads is large, fluid-scale models that assumes a deterministic rather than stochastic submission rates may be computationally attractive. The analysis of such fluid models and their relations to the stochastic ones are of interest here.
- Multi-class models: It may be of interest to consider several classes of advertisers with different preferences – for example, the exposure function $u(t)$ may differ among advertisers that address different segments of the population.

These topics present challenges with various degrees of difficulty, which are left for future investigation.

Acknowledgements This research was supported by the Israel Science Foundation grant No. 1319/11.

References

1. J. Constine, “Minimizing Your Facebook Ad CPC Bid Prices by Avoiding Bid Competition,” <http://www.insidefacebook.com/2011/09/08/suggested-bid-cpc/>, Sept. 2011.

2. R. B. Cooper, *Introduction to Queueing Theory*, 2nd Ed., North-Holland, New York, 1981.
3. L. Einav, "Seasonality in the U.S. Motion Picture Industry," *RAND Journal of Economics* 38(1):127-45, 2007.
4. D. Fudenberg and J. Tirole, *Game Theory*, MIT press, Cambridge, MA, 1991.
5. A. Glazer and R. Hassin, "?/M/1: On the equilibrium distribution of customer arrivals," *European J. of Oper. Research* 13:146-150, 1983.
6. J. K. Hale and S. M. Verduyn Lune, *Introduction to Functional Differential Equations*, Springer, New York, 1993.
7. R. Hassin and Y. Kleiner, "Equilibrium and optimal arrival patterns to a server with opening and closing times," *IEE Transactions* 43(3):164-175, March 2011.
8. M. Haviv, "When to arrive at a queue with tardiness costs?" *Performance evaluation* 70:387-399, 2013.
9. R. Jain, S. Juneja and N. Shimkin, "The Concert Queueing Game: To Wait or to be Late," *Discrete Event Dynamic Systems* 21(1):103-138, April 2011.
10. S. Juneja and N. Shimkin, "The Concert Queueing Game: Strategic arrivals with waiting and tardiness costs," *Queueing Systems* 74(4):369-402, August 2013.
11. T. Klastorin and W. Tsai, "New Product Introduction: Timing, Design, and Pricing," *Manufacturing and Service Operations Management* 6(4):302-320, 2004.
12. R. E. Krider and C. B. Weinberg, "Competitive Dynamics and the Introduction of New Products: The Motion Picture Timing Game," *Journal of Marketing* 35(1):1-15, 1998.
13. R. Laraki, E. Solan and N. Vieille, "Continuous-time Games of Timing," *Journal of Economic Theory* 120(2):206-238, February 2005.
14. R. Lindley, "Existence, uniqueness, and trip cost function properties of user equilibrium in the bottleneck model with multiple user classes," *Transportation Science* 38(3):293-314, 2004.
15. Z. Lotker, B. Patt-Shamir, and M. R. Tuttle, "A game of timing and visibility," *Games and Economic Behavior* 62(2):643-660, 2008.
16. S. Radas and S. M. Shugan, "Seasonal Marketing and Timing New Product Introductions," *Journal of Marketing Research* 35(3):296-315, 1998.
17. D. Rajagopal, "New Product Introduction and Seasonality Effect in Food Products Retailing," *Journal of Food Products Marketing* 14(4):54-75, 2008.
18. W. S. Vickrey, "Congestion theory and transport investment," *The American Economic Review* 59:251-260, 1969.

Appendix

A Proofs for Section 3

We next present the proofs for our main results in the previous Section, namely Theorems 1-2 and Proposition 1. We start with the latter, which is also needed for the proofs of the two other theorems.

Proof of Proposition 1: Fixing L , we consider the differential equation (11) with the stated terminal conditions over $t \in [t_L, L]$. Denote the right hand side of (11) by $H(t, F) \triangleq r_1 u(t)/g(t, F)$. Since H is non-negative it follows that $F' \geq 0$, and noting that $F_L(L) = 1$ we may restrict attention to non-decreasing functions F with values in $[0, 1]$. For simplicity, when $t_L > 0$ we may set $F_L(t) = 0$ for $t < t_L$. We further consider F_L to be a member of the Banach space of continuous functions on $[0, T]$, equipped with the sup-norm topology.

(i) As mentioned, Equation (11) is a retarded FDE in the sense of [6]. Existence and uniqueness of solutions follow by standard results for these equations, once we verify that the right hand side has appropriate continuity and boundedness properties. Observing H above, recall that $u(t)$ is a continuous and bounded function. It is easily verified by straightforward bounds that g in (8) is (Lipschitz) continuous in t and in F (with the sup-norm topology). Therefore, $H(t, F)$ is also Lipschitz continuous provided that $g(t, F)$ is bounded away from zero.

We first show that the latter property of g holds for any $\epsilon > 0$ over $t \in [t_L(\epsilon), L]$, where

$$t_L(\epsilon) = \inf\{s \in [0, L] : F(s) > \epsilon\}.$$

Indeed, for such t , we obtain from (8) that

$$g(t, F) \geq (r_1 - r_2) \sum_{n \geq 0} q(n) \int_L^T B_{0,n}(F(s) - F(t))u(s)ds,$$

where $q(n) \triangleq (n+1)p_D(n+1) > 0$. Observe that $r_1 - r_2 > 0$ by assumption, and similarly $q(n_0) > 0$ for some $n_0 \geq 1$, since $E(D) > 0$ by assumption. Furthermore, $F(s) = 1$ for $t \geq L$, $F(t) \geq \epsilon$ for $t \in [t_L(\epsilon), L]$, and $B_{0,n}(p) = (1-p)^n$ is decreasing in p . Therefore,

$$\begin{aligned} g(t, F) &\geq (r_1 - r_2)q(n_0)u_{\min} \int_L^T B_{0,n_0}(1 - \epsilon)ds \\ &= (r_1 - r_2)q(n_0)u_{\min}B_{0,n_0}(1 - \epsilon)(T - L) \triangleq \zeta(\epsilon, L) > 0. \end{aligned} \quad (27)$$

It follows that $H(t, F)$ is indeed continuous in (t, F) for $t \in [t_L(\epsilon), L]$. Existence of a unique local solution to $F'(t) = H(t, F)$ for $t \leq L$ now follows by Theorem 2.1 in [6]. Since $H(t, F) \leq r_1 u_{\max}/\zeta(\epsilon, L) < \infty$, it follows by Theorem 3.1 there that the solution $F(t)$ can be extended to $t \in [t_L(\epsilon), L]$. Finally, by taking $\epsilon \rightarrow 0$, so that $t_L(\epsilon) \rightarrow t_L$, it follows that a unique solution F exists on $[t_L, L]$.

(ii) We wish to show that $g(t, F_L)$ is bounded away from 0 over $t \in [t_L, L]$. It was already shown in (i) that $g(t, F) \geq \zeta(\epsilon, L) > 0$ for $t \in [t_L(\epsilon), L]$ and any $\epsilon > 0$. However, since $\zeta(\epsilon, L) \rightarrow 0$ as $\epsilon \rightarrow 0$, we need to strengthen this bound for the solution F_L . Setting $\epsilon = 0.5$, we have from that bound that $g(t, F_L) \geq \zeta(0.5, L) \triangleq \delta_L > 0$ for $t \in [t_L(0.5), L]$. Clearly, if $t_L(0.5) = 0$ we are done. Otherwise, we still need to bound $g(t, F)$ for $t < t_L(0.5)$. Since F_L is a continuous function, there exists an interval $I_L \subset [t_L(0.5), L]$ of length $|I_L| > 0$, over which $F_L(t) \in [0.5, 0.6]$. (In fact, since $g(t, F) \geq \delta_L$ implies that $F'(t) \leq r_1 u_{\max}/\delta_L$, it may be seen that $|I_L| > 0.1\delta_L/r_1 u_{\max}$.) Therefore, similarly to (27), we obtain that

$$\begin{aligned} g(t, F_L) &\geq (r_1 - r_2)q(n_0)u_{\min} \int_{s \in I_L} B_{0,n_0}(F_L(s) - F_L(t))ds \\ &\geq (r_1 - r_2)q(n_0)u_{\min}|I_L|B_{0,n_0}(0.6), \quad t \in [t_L, t_L(0.5)], \end{aligned}$$

where we have used that fact that $F_L(s) \leq 0.6$ for $s \in I_L$, so that $F_L(s) - F_L(t) \leq 0.6$ as well. Using the above-mentioned bound on $|I_L|$, we finally obtain

$$g(t, F_L) \geq 0.1 \frac{r_1 - r_2}{r_1} q(n_0) \frac{u_{\min}}{u_{\max}} \delta_L B_{0,n}(0.6) \triangleq \eta_L > 0, \quad t \in [t_L, t_L(0.5)].$$

This provides the required uniform bound over $g(t, F_L)$ for $t < t_L(0.5)$. Combining the above bounds on g , we obtain that $g(t, F_L) \geq \min\{\delta_L, \eta_L\} \triangleq \epsilon_L > 0$ for $t \in [t_L, L]$, as stated.

(iii) Continuity in L of the solutions F_L to our equation $F'_L(t) = H(t, F_L)$ follows by Theorem 2.2 (Continuous dependence) in [6], once we show each $L \in (0, T)$ has a neighborhood $N_L = (L - \epsilon, L + \epsilon)$ such that $H(t, F_{L'})$ is uniformly bounded over all t and solutions $F_{L'}, L' \in N_\epsilon(L)$. Since $H(t, F) \leq r_1 u_{\max}/g(t, F)$, the required uniform boundedness follows by observing the bound $g(t, F_L) \geq \epsilon_L > 0$ developed in item (ii) above, and further noting that ϵ_L is continuous (in fact linear) in L , as may be seen in the above proof. Continuity of t_L in L follows immediately from that of the solution F_L .

This completes the proof of Proposition 1. \square

We next address the proof of Theorem 1, which relies on several Lemmas. The next two establish properties (i), (ii) and (iv) of the Theorem.

For a given distribution function F , let η_F denote the corresponding probability measure on the reals. Recall that $\text{supp}(F)$ denotes the support of η_F , i.e., the smallest closed set of η_F -probability 1.

Lemma 1 *Let F be an equilibrium profile. Then*

- (i) $F(t)$ is a continuous function of $t \in [0, T]$ (i.e., η_F does not contain any point masses).
- (ii) $U(t, F)$ is a continuous function of $t \in [0, T]$.
- (iii) $\text{supp}(F)$ is an interval $[0, L]$, with $L < T$.

Proof (i) Suppose η_F contains a point mass at t , namely $F(t) - F(t-) = q > 0$. We claim that, for $\epsilon > 0$ small enough, $U(t + \epsilon; F)$ will be strictly larger than $U(t; F)$. But this contradicts the equilibrium properties in (5)-(6), since $U(t, F) < u^* \triangleq \max_s U(s, F)$ and $\eta_F(\{t\}) > 0$.

To see that $U(t; F) < U(t + \epsilon; F)$ for ϵ small enough, consider a specific player i who arrives at t , and note that having a point mass at t implies that, with positive probability, at least one other player arrives at t simultaneously. Recall that the ordering of simultaneous arrivals is random, so that in that case player i has a probability of at least 0.5 to be immediately placed in position 2 or worse. On the other hand, arriving at a slightly later time $t + \epsilon$ guarantees that the player will be placed higher than all other arrivals at t , thereby strictly improving his expected utility over $[t + \epsilon, T]$ relative to the previous case by a quantity δ which is bounded away from 0 even as $\epsilon \rightarrow 0$, while the utility loss over $[t, t + \epsilon]$ is bounded by ϵr_1^* , which is arbitrarily small. Here $r_1^* \triangleq r_1 \max_{t \in [0, T]} u(t)$.

(ii) Again, we compare $U(t; F)$ to $U(t + \epsilon; F)$. For $\epsilon > 0$, since there are no point masses in η_F (as just established), then $\eta_F\{[t, t + \epsilon]\} \rightarrow 0$ as $\epsilon \rightarrow 0$, so that the probability $\delta(\epsilon)$ of an arrival on that interval converges to 0 as $\epsilon \rightarrow 0$. Now, a player that arrives at $t + \epsilon$ rather than t will incur a utility loss of at most ϵr_1^* over $[t, t + \epsilon]$, and a utility gain of $\delta(\epsilon) T r_1^*$ at most over $[t + \epsilon, T]$. Therefore, $\lim_{\epsilon \rightarrow 0} |U(t + \epsilon; F) - U(t; F)| \leq r_1^* |\epsilon - \delta(\epsilon) T|$ converges to 0 as $\epsilon \rightarrow 0$, establishing continuity from the right. A similar argument holds for $\epsilon < 0$.

(iii) To show that the support $\text{supp}(F)$ is an interval of the form $[0, L]$, suppose to the contrary that there exists an interval (t, s) , $t < s$ such that $\eta_F\{[t, s]\} = 0$ and $F(s) < 1$. We can extend s to the right while keeping the above properties till it hits $\text{supp}(F)$. Then, by property (5) of the NEP and the above-established continuity of $U(\cdot; F)$, it follows that $U(s; F) = u^*$. But now, since there are no arrivals on (t, s) , an arrival at $s - \epsilon$ will remain in position 1 until time s , implying that $U(s - \epsilon; F) \geq U(s; F) + \epsilon r_1 [\min_{t \in [s - \epsilon, s]} u(t)] > u^*$, which contradicts the definition of u^* .

Finally, we show that $L < T$. It is clear that $U(T - \epsilon; F) \leq \epsilon r_1^*$. Therefore, $U(t; F) < u^*$ for $t > t_0 = T - u^*/r_1^*$, implying that $(t_0, T]$ does not belong to $\text{supp}(F)$. \square

Lemma 2 *Any equilibrium profile F satisfies properties (i), (ii), (iv) of Theorem 1.*

Proof Property (i) was established in Lemma 1. For property (ii), suppose F is an equilibrium profile. Since $U(\cdot; F)$ is continuous by Lemma 1(ii), it follows by property (i) and the definition of the equilibrium that $U(t; F) = u_F^*$ on $[0, L]$, which is property (ii). Conversely, suppose that some continuous strategy F satisfies property (ii). By definition of the equilibrium, to establish that F is indeed an equilibrium strategy it only remains to show that $U(t; F) \leq u_L$ for $t > L$. But since there are no other arrivals on $[L, T]$ under F , then any arrival there would remain in the first position till T , so that $U(t; F) = r_1 \int_t^T r_1 u(s) ds$ for $t \in [L, T]$. Since $u(s) > 0$ it follows that $U(t; F) < U(L; F) = u_L$ for $t > L$, as required, and property (ii) is established. Property (iv) follows by observing that $u_L^* = U(L; F)$ and substituting the last integral. \square

It remains to establish the differential characterization of Theorem 1(iii), and the existence claim in that Theorem. We begin by computing the time derivative of the utility function $U(t; F)$ in (4).

Lemma 3 *For any strategy F , the following hold:*

(i) *At any point t at which F is differentiable,*

$$\frac{d}{dt}U(t; F) = -r_1 u(t) + F'(t)g(t, F), \quad (28)$$

where g is defined in equation (8).

(ii) *Conversely, if U is differentiable at t and $g(t, F) > 0$ there, then F is differentiable at t , hence $F'(t)$ satisfies Equation (28).*

(iii) *The function $g(\cdot; F)$ is continuous over $t \in [0, T]$, and strictly positive for all $t < T$ (except for the degenerate case where $F(T^-) = 0$, in which $g \equiv 0$).*

Proof (i) By formally differentiating (4) under the integral sign,

$$\begin{aligned} \frac{d}{dt}U(t; F) &= - \sum_{k=0}^{K-1} r_{k+1} \mathbb{P}_F(N_{(t,s]} = k) u(s) \Big|_{s=t} + \int_t^T \sum_{k=0}^{K-1} r_{k+1} \frac{\partial}{\partial t} \mathbb{P}_F(N_{(t,s]} = k) u(s) ds \\ &= -r_1 u(t) + \int_t^T \sum_{k=0}^{K-1} r_{k+1} \frac{\partial}{\partial t} \mathbb{P}_F(N_{(t,s]} = k) u(s) ds. \end{aligned} \quad (29)$$

Further, observing (3),

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{P}_F(N_{(t,s]} = k) &= \sum_{n \geq k} p_D(n) \frac{\partial}{\partial t} B_{k,n}(F(s) - F(t)) \\ &= - \sum_{n \geq k} p_D(n) F'(t) \frac{\partial}{\partial p} B_{k,n}(p) \Big|_{p=F(s)-F(t)}. \end{aligned} \quad (30)$$

Recalling the definition of $B_{k,n}$ in (2),

$$\begin{aligned} \frac{\partial}{\partial p} B_{k,n}(p) &= \binom{n}{k} k p^{k-1} (1-p)^{n-k} \mathbf{1}_{\{k > 0\}} - \binom{n}{k} (n-k) p^k (1-p)^{n-k-1} \mathbf{1}_{\{k < n\}} \\ &= n B_{k-1, n-1}(p) - n B_{k, n-1}(p) \end{aligned} \quad (31)$$

(where we use the convention that $B_{-1, n+1} = B_{n, n-1} = 0$), so that

$$\begin{aligned} \frac{d}{dt}U(t; F) &= -r_1 u(t) + \dots \\ &+ F'(t) \int_t^T \sum_{k=0}^{K-1} r_{k+1} \sum_{n \geq k} p_D(n) n [B_{k, n-1}(p) - B_{k-1, n-1}(p)]_{p=F(s)-F(t)} u(s) ds. \end{aligned} \quad (32)$$

It remains to simplify the sum in the last expression. Denoting $q_n = np_D(n)$, observe that

$$\begin{aligned}
& \sum_{k=0}^{K-1} r_{k+1} \sum_{n \geq k} q_n [B_{k,n-1}(p) - B_{k-1,n-1}(p)] \\
&= \sum_{k=0}^{K-1} r_{k+1} \sum_{n \geq k-1} q_{n+1} B_{k,n}(p) - \sum_{k=-1}^{K-2} r_{k+2} \sum_{n \geq k} q_{n+1} B_{k,n}(p) \\
&= \sum_{k=0}^{K-1} r_{k+1} \sum_{n \geq k} q_{n+1} B_{k,n}(p) - \sum_{k=0}^{K-1} r_{k+2} \sum_{n \geq k} q_{n+1} B_{k,n}(p) \\
&= \sum_{k=0}^{K-1} (r_{k+1} - r_{k+2}) \sum_{n \geq k} q_{n+1} B_{k,n}(p), \tag{33}
\end{aligned}$$

where the first equality follows by an index shift, and the second by observing again that $B_{k,n} \equiv 0$ for $n = k - 1$ or $k = -1$, while $r_{K+1} = 0$ by definition. Substituting (33) in (32) gives the equality in (28), with g as defined in (8).

Existence of the derivative of U at t now follows by observing that, on the right-hand side of (28), both u and g are continuous functions of t (the first by assumption, and the second by its definition as an integral).

(ii) Recall that $u(t)$ is continuous by assumption, and so is g (as observed in (iii)). Observing (28) and the assumed positivity of $g(t, F)$, existence of $F'(t)$ may now be inferred by basic calculus from the expression (4) for U .

(iii) Observing (8), continuity follows by the definition of g as an integral over a bounded function. Note that the sum $\sum_{n \geq k} (n+1)p_D(n+1)$ is bounded by $E(D)$, which is finite by assumption. Strict positivity of $g(t, F)$ follows as in Proposition 1(ii). \square

The next Lemma establishes the differential characterization of the equilibrium in Theorem 1.

Lemma 4 *Part (iii) of Theorem 1 holds true. That is,*

- (i) *A continuous distribution function $F : [0, T] \rightarrow [0, 1]$ that satisfies the properties in Theorem 1(iii) is an equilibrium profile.*
- (ii) *Conversely, any equilibrium profile F satisfies the properties stated in Theorem 1(iii).*

Proof (i) From (9) and (28) we obtain that $\frac{d}{dt}U(t; F) = 0$ for $t \in (0, L)$. Since U is continuous it follows that $U(t; F) = u$, $t \in [0, L]$ for some constant u . Therefore, by Theorem 1(ii), F is an equilibrium profile.

(ii) Suppose F is an equilibrium profile. In view of Theorem 1(ii), it only remains to verify (9). Now, Theorem 1(ii) also implies that $U(t; F) = u_F$ for $t \in [0, L]$, hence $\frac{d}{dt}U(t; F) = 0$ for $t \in (0, L)$. Observing Lemma 3(ii)-(iii) and Equation (28), it follows that $-r_1 u(t) + F'(t)g(t, F) = 0$ (with $g(t, F) > 0$), and Equation (9) follows. \square

Lemma 5 *There exists an equilibrium profile F .*

Proof We demonstrate existence of a distribution function F that satisfies the requirements of Theorem 1(iii). For each $L \in (0, T)$, let $F_L(t)$, $t \in [t_L, T]$ denote the solution of Equation $F_L(t)$ as defined in Proposition 1. Note that $F_L(t_L) = 0$ if $t_L > 0$. To simplify the exposition, if $t_L > 0$ extend F_L linearly up to $t = 0$ using $F_L(t) = -(t_L - t)$, $t \in [0, t_L]$. By part (iii) of Proposition 1, $F_L(0)$ is continuous in L . We show below that $t_L > 0$ (hence $F_L(0) < 0$) for L close enough to T , while $F_L(0) > 0$ for L close enough to 0. By the above-mentioned continuity, this implies that $F_L(0) = 0$ for some intermediate $L \in (0, T)$. It follows that for this value of L , F_L satisfies all the requirements in Theorem 1(iii), and is therefore an equilibrium profile.

To verify that $F_L(0) > 0$ for L close enough to 0, observe that $g(t, F)$ in (8) is bounded away from zero while $F(t)$ is positive: In particular, using the bound in Equation (27), if $L \leq T/2$ and $F(t) \geq 0.5$ we have

$$g(t, F) \geq \zeta(0.5, \frac{1}{2}T) \triangleq \zeta_0 > 0,$$

Therefore $F'_L(t) = r_1 u(t)/g(t, F_L) \leq r_1 u_{\max}/\zeta_0$ under these conditions, and since $F_L(L) = 1$ this implies that $F_L(0) \geq 0.5$ if $L < \frac{1}{2}\zeta_0/r_1 u_{\max}$.

To verify that $t_L > 0$ for L close enough to T , observe from (8) that g is upper bounded by

$$g(t, F) \leq r_1 E(D) \int_t^T u(s) ds.$$

Therefore, by (9),

$$F'_L(t) = \frac{r_1 u(t)}{g(t, F_L)} \geq E(D)^{-1} \frac{u(t)}{\int_t^T u(s) ds}, \quad t \in [t_L, L].$$

By integration, this implies

$$F_L(L) - F_L(t) \geq E(D)^{-1} \ln \left(\frac{\int_t^T u(s) ds}{\int_L^T u(s) ds} \right).$$

Since the last denominator is arbitrarily small for L close enough to T , and recalling that $u(s) > 0$, it follows that $F_L(t) = 0$ for some $t > 0$ (namely $t_L > 0$) for such L .

This completes the proof of existence of an equilibrium profile. \square

Proof of Theorem 1: Items (i)-(iv) of this Theorem were proven in Lemmas 2 and 4, while existence of an equilibrium profile was established in Lemma 5. \square

We may now proceed to the proof of the uniqueness claim in Theorem 2. The proof relies on the following monotonicity properties of the function g , as defined in (8).

Lemma 6 *Let Assumption 2 hold.*

(i) *For each $n \geq 0$, the function*

$$f_n(p) = \sum_{k=0}^{K-1} (r_{k+1} - r_{k+2}) B_{k,n}(p), \quad p \in [0, 1]$$

is non-increasing in p .

(ii) *Consequently, let F_1 and F_2 be two strategy profiles (namely distribution functions over $[0, T]$) such that, for some $t \in [0, T]$,*

$$F_2(s) - F_2(t) \geq F_1(s) - F_1(t) \quad \text{for all } s \in [t, T]. \quad (34)$$

Then $g(t, F_2) \leq g(t, F_1)$.

(iii) *If, in addition, the inequality in (34) is strict over some nonempty interval $I \subset [t, T]$, then either $g(t, F_2) < g(t, F_1)$, or else $g(t, F_2) = g(\tau, F_1)$ for all $\tau \in [0, T]$.*

Proof (i) Denote $\delta_k = r_k - r_{k+1}$. A similar calculation to equations (31) and (33) yields,

$$f'_n(p) \triangleq \frac{d}{dp} f_n(p) = n \sum_{k=1}^K (\delta_{k+1} - \delta_k) B_{k-1, n-1}(p). \quad (35)$$

But Assumption 2 implies that $\delta_{k+1} - \delta_k = 2r_{k+1} - r_k - r_{k+2} \leq 0$, so that $f'_n(p) \leq 0$.

(ii) Changing the order of summation in (8), and recalling that $B_{k,n} \equiv 0$ for $k > n$, g may be rewritten as

$$g(t, F) = \sum_{n \geq 0} q_n \int_t^T f_n(F(s) - F(t)) ds, \quad (36)$$

where $q_n = (n+1)p_D(n+1) \geq 0$. The claim now follows directly from (i) and the assumed relation (34) between F_1 and F_2 .

(iii) For each $n \geq 0$, it may be seen from (35) that either $q_n f'_n(q) \equiv 0$ for $q \in [0, 1]$, or $q_n f'_n(q) < 0$ for all $q \in (0, 1)$. Suppose first that the former holds for all $n \geq 0$. Then, by (36), $g(t, F)$ does not depend on F , so that $g(\cdot, F_1) = g(\cdot, F_2)$. Otherwise, suppose that the

latter holds for some $m \geq 0$, namely $q_m f'_m(q) < 0$ for $q \in (0, 1)$. Then, wherever (34) holds with a strict inequality,

$$q_m f_m(F_2(s) - F_2(t)) < q_m f_m(F_1(s) - F_1(t)),$$

which holds over $s \in I$. Substituting in (36) obtains $g(t, F_2) < g(t, F_1)$. \square

Proof of Theorem 2 (uniqueness): Recall that an equilibrium profile F satisfies the properties in Theorem 1 with some parameter L . In particular, $F(t) = 1$ on $t \in [L, T]$, F satisfies equation (9) on $[0, L]$, and $F(0) = 0$. Let F_1 and F_2 denote two equilibrium profiles with corresponding parameters L_1 and L_2 . We will show below that $L_1 < L_2$ implies that $F_2(0) < F_1(0)$, so that only one can be an equilibrium. Therefore $L_1 = L_2$. But Proposition 1 implies that L defines F uniquely, hence the equilibrium is unique.

Consider then F_1 and F_2 as above, and suppose that $L_1 < L_2$. Since $F_1(s) = 1$ for $s \in [L_1, T]$ and F_2 is strictly increasing over $t < L_2$, it follows that

$$F_2(L_1) < F_1(L_1) = 1, \quad (37)$$

and

$$F_2(s) - F_2(L_1) > F_1(s) - F_1(L_1) = 0, \quad s \in (L_1, T]. \quad (38)$$

Therefore inequality (34) is satisfied with strict inequality for $t = L_1$. By Lemma 6(iii), exactly one of the following two conclusions holds:

(a) $g(t, F_2) = g(t, F_1)$ for all $t \in [0, T]$. In that case it follows from (9) that $F'_2(t) = F'_1(t)$ holds for $t \leq L_1$, so that

$$F_1(0) - F_2(0) = F_1(L_1) - F_2(L_1) > 0.$$

(b) $g(t, F_2) < g(t, F_1)$ at $t = L_1$. By (9), this implies that $F'_2(L_1) > F'_1(L_1^-)$. We argue that this inequality extends to all $t < L_1$. Suppose, to the contrary, that $F'_2(\tau) \leq F'_1(\tau)$ for some $\tau < L_1$. Noting that F'_1 and F'_2 are continuous on $[0, L_1]$ by (9), there must exist a time $t_0 < L_1$ so that $F'_2(t_0) = F'_1(t_0)$, while $F'_2(s) > F'_1(s)$ for $s \in (t_0, L_1]$. By integration, it follows that

$$F_2(s) - F_2(t_0) \geq F_1(s) - F_1(t_0), \quad s \in [t_0, L_1].$$

Combined with (38), we may apply Lemma 6(iii) to deduce that $g(t_0, F_2) < g(t_0, F_1)$, hence $F'_2(t_0) > F'_1(t_0)$. But this contradicts the definition of t_0 . We have thus verified that $F'_2(t) > F'_1(t)$ for all $t < L$. But this implies that $F_2(0) - F_1(0) < F_2(L_1) - F_1(L_1) < 0$, where the first inequality follows by integration, and the second from (37).

We have thus shown that $L_1 < L_2$ implies $F_2(0) < F_1(0)$, which completes the proof of Theorem 2. \square