Monte Carlo Methods for Computation and Simulation (048715)

Problem Set 2 (part 1: Conditional MC)

Submission: May 20 (both parts)

Guidelines: As in Problem Set 1

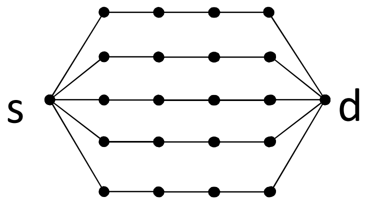
1. *Conditional MC:* Prove that $\mathrm{Var}(E(H(X \mid Y)) \le \mathrm{Var}(H(X))$

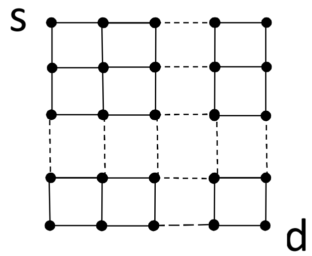
2. *Permutation MC for reliability (Example 2, page 3-8):* Prove that $t_{b(\pi)}$, conditioned on $\Pi = \pi$, has the stated distribution.
Note: If you do not manage to show this analytically, demonstrate equality in simulation on some simple example.

3. Conditional MC – Permutation MC for counting the number of solutions in monotone CNFs
A Boolean expression Ψ is in Conjunctive Normal Form (CNF) if it is a conjunction (logical 'and') of m clauses C_i, and each clause is a disjunction (logical 'or') of t_i literals. That is,
$$\Psi(\mathrm{x}_1,\ldots,\mathrm{x}_n) = \wedge_{i=1}^{m}(\vee_{k=1}^{t_i} \mathrm{y}_{i,k})$$
where $\mathrm{y}_{i,k}$ is either one of the Boolean variables (x_j) or its negation ($\overline{\mathrm{x}}_j$). The CNF Ψ is monotone if no literal appears as a negation. For example, the following Ψ_1 is monotone but Ψ_2 is not:
$$\Psi_1 = (\mathrm{x}_1 \vee \mathrm{x}_2 \vee \mathrm{x}_3) \wedge (\mathrm{x}_3 \vee \mathrm{x}_4)\,,\quad \Psi_2 = (\mathrm{x}_1 \vee \mathrm{x}_2 \vee \mathrm{x}_3) \wedge (\overline{\mathrm{x}}_3 \vee \mathrm{x}_4)$$
An assignment of True/False values to $\mathrm{x}_1,\ldots,\mathrm{x}_n$ satisfies Ψ if Ψ evaluates as True under this assignment. Clearly, it is easy to find a satisfying assignment for a monotone CNF – just set all literals to True.
Here, we are interested in counting (or estimating) the number of all satisfying assignments for a given monotone CNF. This is already a hard problem, which belongs to the $\#P$-complete complexity class.
   a. Propose a Crude Monte Carlo Algorithm to perform the counting task. Provide an expression for the variance of the proposed estimator and discuss the results.
   b. We described in class the Permutation Monte-Carlo method for Reliability models. (page 3.8). Show that this method can be adapted for the counting task at hand. Describe in detail the proposed algorithm.
   c. Show that the proposed estimator is unbiased.

Simulation problem:

4. *Permutation MC for reliability:* Consider the network reliability problem in Example 2 on page 3-8. In this problem we will implement the algorithm described there on some specific network examples.
   1. Network 1: A network of 5 parallel (and disjoint) paths, each consisting of 5 serial links. (This network is easily solved analytically and serves to verify the algorithm).
   2. Network 2: A square grid of 10 by 10, connecting the upper left corner to the lower right.





In either case the network fails if there is no connectivity between the source and destination is lost. Choose the failure probability of each link independently and uniformly between 0.1 and 0.2.

Preparation:

a. Suggest an algorithm to check connectivity for a given set of failed links.

b. Suggest a specific method for computing $g(\pi)$.

c. Compute analytically the failure probability of network 1.

Simulation:

d. Apply the Permutation MC algorithm to network 1, to compute the failure probability. Estimate the number of samples required to obtain 10% accuracy. Compare with the analytical result.

e. Apply the Permutation MC algorithm to network 2. Estimate the failure probability to within 10% accuracy.

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Problem Set 2, part 2: Importance Sampling

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5. (Cross Entropy Estimation) Let $\{g(\cdot,\theta), \theta \in \Theta \subset \mathbb{R}^m\}$ be a (multidimensional) exponential family of the general form

$$g(x,\theta) = c(\theta)\exp(v(\theta)\cdot t(x))h(x) \equiv \exp(v(\theta)\cdot t(x) - \xi(\theta))h(x).$$

Let θ^* be a maximizer of $L(\theta)$ (page 4-7). Show that, under suitable technical conditions, and denoting $h(x) = |H(x)|$ for simplicity,

$$\nabla \xi(\theta^*) = \frac{E_f(h(X)t(\mathrm{X}))}{E_f(h(X))}\frac{\partial v(\theta^*)}{\partial \theta}.$$

Furthermore, if the parameterization is chosen such that $\theta = E_\theta(t(X))$, then

$$\theta^* = \frac{E_f(h(X)t(\mathrm{X}))}{E_f(h(X))}.$$

Note: You may use any technical conditions you need, but state them explicitly.

6. (Continued) Let X be a discrete RV with probabilities $(p_1,\dots p_n) \triangleq \theta$.
   a. Show that this discrete distribution can be represented as an exponential family, with $\theta = E_\theta(t(X))$.
   b. Suggest an approximate solution for θ^*, based on samples $(X_1,\dots,X_K)$ from f. Write explicitly the expression for each component p_i.

7. The next three problems demonstrate the application of the Variance Minimization and the CE methods on a simple scalar example.
Consider the problem of estimating $\ell = E_f(H(X))$, with $f = N(0,1)$, and

$$H(x) = \exp(-10x - 20).$$

   a. Compute ℓ analytically.
   b. Consider using Crude Monte Carlo to estimate ℓ. Compute the sample variance, and the number of samples required to obtain 5% accuracy with 95% confidence.
   c. What is the optimal Importance Sampling distribution here? How many samples are needed to estimate ℓ with this distribution?

Simulation:

8. For the model in the previous problem, use Importance Sampling to estimate ℓ, with IS distribution $g = N(\mu,1)$. Here μ is a parameter to be optimized. If needed you can start with some initial guess for μ, but not too close to the optimal one.

   a. Using the VM method, write down an optimization problem for μ, and compute analytically (if possible) the optimal μ.

b. Write down the empirical version of this optimization problem, using 10 samples from f. Solve numerically or analytically for μ, and iterate several times. Compare the obtained values to the optimal one.

9. Repeat Problem 8, using the Cross Entropy method in place of VM. Compare briefly the application of the two methods in this case.