Monte Carlo Methods for Computation and Simulation (048715)

Problem Set 1

Submission: April 22 (in class)

Guidelines:
- Submission is individual.
- Limited verbal consultation is allowed, but should be mentioned in your solution. Similarly, consultation with any written source other than the course textbooks should be mentioned.
- A handwritten solution is encouraged, but type if you must.
- Simulations should be done individually. Any explanations of methods and result analysis should be presented in text and not on program printout or figures.
- Append your code listing. If too long send it by e-mail.

1. a. *Needle throws* (Lect. 1 p. 2): Show that the probability of a needle of length l intersecting a grid of parallel (say, horizontal) lines with spacing $D > \ell$ is $2l / \pi D$.

   b. *Acceptance-rejection method* (Section 2.3): Assume a *continuous* RV X. Show that the returned sample X has the required densify f.

   c. *Box-Muller formula* (p. 2-4): Show that X and Y are Normal and independent.

   d. *Uniform sampling from the unit ball:* Show that the scheme on page 2-6 obtains uniform samples.

2. RV generation: The standard Cauchy distribution has the pdf

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

   a. What are the mean and variance of the Cauchi RV? What is the median?

   b. Show that if X and Y are independent standard Normal RVs, then $Z = X / Y$ has the Cauchy distribution.

   c. Using this observation, provide a formula for generating a Cauchy RV from random numbers.

   d. Show how to generate a Cauchy RV using the inverse transform method.

3. Acceptance-rejection: The truncated exponential distribution with $\lambda = 1$ has the pdf

$$f(x) = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-a}}, \quad 0 \le x \le a$$

   a. Devise an acceptance-rejection algorithm for sampling from this distribution, using as proposal distribution the uniform distribution on $[0, a]$. Specify the efficiency parameter (C) as a function of a.

   b. Repeat the above using an exponential $\mathrm{Exp}(\lambda)$ proposal distribution with an appropriate parameter λ. Provide a (possibly implicit) formula for the efficiency-maximizing λ. Evaluate the efficiency (C^{-1}) of this algorithm for $a = 1$, $a \to \infty$ and $a \to 0$, and compare to (a).

4. Common Random Numbers: Suppose we wish to evaluate the integral

$$I = \int_0^1 h(x)dx$$

for a given function $h(x)$. This can be clearly done by evaluating $\ell = E(h(U))$ using MC methods, where $U \sim \mathrm{U}(0,1)$.

a. Compute the variance of the crude MC estimator.

b. Suggest an application of the Common Random Numbers method to this problem to reduce the variance.

c. Compute the obtained variance for your method and compare to (a). When (for what functions) will the improvement be most significant?

5. Stratified Sampling: For Example 2 on page 3-6, compute the optimal choice of (N_k) to minimize the variance σ_{str}^2 (subject to $N_1 + \ldots + N_K \le N$).

Simulation problems:

6. Random numbers: For the LCG random number generator on page 2-1,

   a. Choose a random seed and generate N=1000 samples. Compute the Kolmogorov-Smirnov statistics for this sample:
   $$D_N = \max_{x\in[0,1]} |\hat{F}_N(x) - F(x)|,$$
   where $\hat{F}_N$ is the empirical cdf and F the cdf of the uniform distribution.

   b. Assuming that the samples are independent, use the Kolmogorov-Smirnov test to compute the confidence level that the samples are drawn from the uniform distribution. [You can use the Internet to learn about the K-S test].

   c. Repeat the above for $N = 10^4$

7. Uniform sampling: Use the "General n-simplex" procedure on p. 2-6 to sample uniformly 100 points from a (2-D) equilateral triangle of your choosing. Plot the obtained samples in the plane.

8. Crude MC: Evaluate π to an accuracy of 10% (with confidence 99%) using the two methods in Section 1.2. Use the empirical variance to estimate confidence. Compare briefly the two methods and discuss obtained accuracies.

9. Control Variates: Use the method of Example 1, p.3-4 to reduce the variance. Instead of the two triangles mentioned there, choose a polygon that gives a better fit to the circle. Compare the results to those of the previous problem.

10. Control Variates for Shortest Path: Apply the control variates method to the Stochastic Shortest Path problem (Example 2, p.3-4).
Guidelines: Choose a small network, say two hops and several alternative paths from source to destination. Use Bernoulli link costs. Make sure to choose the parameters so that several paths have significant chances of being optimal. Compare three choices of control variates: none, a path with least expected cost, and a path with maximal expected cost. Continue the simulations to obtain a relative error on the order or 1%. Plot the resulting estimates and their variances. Discuss the results.