Curvature – Smooth, Piecewise-Linear and Metric

Emil Saucan

Abstract

The notion of curvature is explored, first in the classical context and then in more general spaces. Emphasis is placed on the approximation of various curvatures defined on smooth Riemnanian manifolds by piecewise-linear and metrical analogues. The correlations between the different aspects are considered. Applications of metric methods to other fields are also presented.

1 Introduction

"The fundamental notion of differential geometry is the concept of curvature." ([13]) Indeed, it can be stated (only slightly exaggerating) that the subject of differential geometry *is* the study of curvature. Therefore, any attempt to summary present even only the main aspects of this deep and multifaceted subject would be vain and condemned to utter failure. Hence, given the vastness of the expounded material and the enormous extent of the bibliographical matter, one has to restrict himself to exploring (and even this only partially) certain selected aspects of the subject. Such a choice is, of course, dependent on the tastes and inclinations of the exposer. However, it seems to us that largely this choice is predetermined by the subject matter for "while all of us have an intuitive concept of the difference between straight and curved, it is surprisingly difficult to make the intuition precise"([13]). Therefore, the main task of any exposition of the subject at hand – be it a course, a book or a short survey – is to convey at least some of the geometric intuition behind such a fundamental concept. This is even more true when the matter at hand is usually presented in an extremely technical manner, via a large number of intricate formulas and tedious, meandering computations.

Nevertheless, it is important to understand that the use (and abuse) of the technical apparatus is not just a consequence of the ossification of curricula or of the commodity of the exposer, but it is intrinsically embedded in the very fabric of the study matter. Indeed, intuition by itself does not suffice, since: "the motivating force behind differential geometry is not a set of notions and intuitions" but rather "the ability and the desire to translate these intuitions into calculations" ([13]). The reason behind the quest for optimal formulas is partly innate, a consequence of the structure and functionality of our brains, for: "Our brain has two halves: one is responsible for the multiplication of polynomials and languages, and the other half is responsible for the orientation of figures in space and all the things important in real life. Mathematics is Geometry when you have to use both halves."([1])).

It is precisely in the choice of computational material one presents that individual tastes, interests and field of research are more evident and play a more important role. We have settled to emphasize certain aspects in which intuition and computation interact in the optimal manner, i.e. such that geometry is still evident yet calculations are feasible. Two approaches are presented: one is the more traditionalistic: the study of the approximation of differentiable structures and their invariants by piecewise-linear ones. The second approach is even more minimalistic: it dispenses with the classical world of locally Euclidean geometric objects, and explores the existence and meaning of a notion of curvature in the general setting of metric spaces. The question of the interrelation between the two approaches is naturally raised.

At this point a few explanatory words are required: as it is clear from the remarks above, this study is of an expository nature. It is not a research paper, that inevitably would be restricted in scope and technical in character. However this type of connections between PL approximations of curvature and metrical ones has not been, as far as we are informed, raised till now. (However, see [43] for a different approach.)

The nature of this work also dictates the choice of the included proofs: rather then covering the more standard subjects, we have opted to present some of the proofs pertaining to the least known material, i.e. the metrical curvatures. Even here the more lengthy proofs and many of the technical intermediate steps were omitted. Moreover, even such important notions as that of mean curvature are omitted, if they represent a digression from the main goal.

The remainder of the paper is structured as follows: in Section 2 we introduce, in logical and historical order, the classical notions, first for curves and then for surfaces. This is followed in Section 3 by a brief excursus in the generalizations of curvature to higher dimensions. In Section 4 we further widen the generalizations to include metric spaces. We conclude in Section 5 with a few remarks regarding the possibility of applying metric concepts and methods to the problem of the approximation of curvatures of smooth manifolds by those of much simpler geometrical objects, such as triangulations and graphs. We regard this problem both from a theoretical viewpoint and from that of possible practical applications.

Finally, before we start a few cautionary words regarding the apparatus involved: while we tried to remain self contained, a certain degree of mathematical literacy is needed, especially regarding fundamental notions of geometry and topology.

2 Classical Curvature

We first define the notion of curvature for plane curves. This not only follows the historical development, it is also relevant for the generalizations considered in Section 2.2 and even more so, in Section 4.1. The definition of curvature for surfaces in $\mathbb{R}^3$ is introduced in Section 2.2 and various generalizations for higher dimensional manifolds are considered in Section 3.

2.1 Curves

The definition of curvature for plane curves is based upon the classical notion of curvature of a circle: $K \equiv 1/R$, where R denotes the radius of the circle. More precisely, the curvature of the curve at the point p is the curvature of the "best fitting" circle to γ at p. It can be defined as the largest circle that has one common point with a curve and it is entirely contained on one side of the curve. This however is a rather elusive property, in the sense that it is not readily computable. Therefore, we (following Newton, 1665) define the *osculatory circle* as a limit, more precisely the limit of circles that have 3 common points with the curve. If the curve $\gamma \subset \mathbb{R}^2$ is the image of the function $c : I = [0,1] \to \mathbb{R}^2$, then the osculatory circle at $\gamma_0 = c(t_0)$ is defined as:

$$C(\gamma_0) = C_\gamma(\gamma_0) = \lim_{\gamma_1,\gamma_2 \to \gamma_0} C(\gamma_0, \gamma_1, \gamma_2) = \lim_{t_1,t_2 \to t_0} C(t_0, t_1, t_2)\,; \gamma_i = \gamma(t_i)\,, i = 1, 2\,.$$

Of course, the *curvature* $\kappa_\gamma(\gamma_0)$ of γ at γ_0 is defined to be as $1/R\big(C(\gamma_0)\big)$, where $R\big(C(\gamma_0)\big)$ is the radius of $C(\gamma_0)$.

It can be shown that the osculatory circle satisfies the maximality property discussed above (see, e.g. [52]).

Moreover, its elements, i.e. radius and center, are easily computable if the curve is given by a $\mathcal{C}^k$-parameterization, $k \geq 2$. More precisely, the *radius of curvature* $R_C = R\big(C(\gamma_0)\big)$ and the *center of curvature* $O_C = R(C(\gamma_0))$ of γ at γ_0 are respectively given by: $R_C = \frac{||c'(t_0)||^3}{||c'(t_0) \times c''(t_0)||}$, $O_C = \gamma_0 + R\big(C(\gamma_0)\big)\mathbf{n}_0$, where $\mathbf{n}_0$ denotes the normal to γ at the point γ_0. (Here "$||\cdot||$" denotes the Euclidean norm on $\mathbb{R}^2$.)

We shall encounter both the second derivative (in Sections 2.2 and 3.1 below) and, more important for our approach, the idea of the considering the radii of circles passing through three points (in Section 4.1).

2.2 Surfaces

We have introduced above the notion of osculatory circle and we have remarked that it admits further generalizations. One would expect that it would work particularly well for one of the most immediate generalization of plane curves, i.e. for surfaces in $\mathbb{R}^3$. But this idea, however nice, does not work for surfaces –

the *osculatory sphere* (Fuss, 1829) can not be used to define a notion of curvature for surfaces.

2.2.1 Sectional Curvatures and Gauss Curvature

The first, most natural idea, which was the one in use before the seminal work of Gauss is to "dissect" the surface in all possible directions (by cutting the surfaces with normal planes) and to define curvature for the surfaces S via the curvatures of the sections. However, the idea above is far from satisfactory, since there exist infinitely many directions and it is hard to ascertain in what way do the sectional curvatures represent the curvature of the surface.

Gauss' inspired idea was to define curvature as a measure of a surface from "being straight" or equivalently, a measure of how much a surface has to be bent in order to obtain a certain standard surface, i.e. the *unit sphere* $\mathbf{S}^2$. Gauss achieved this by considering the *normal mapping* $\nu : S \to \mathbf{S}^2$ (see Figure 1 below). Then the *Gauss curvature* of S at p is defined as:

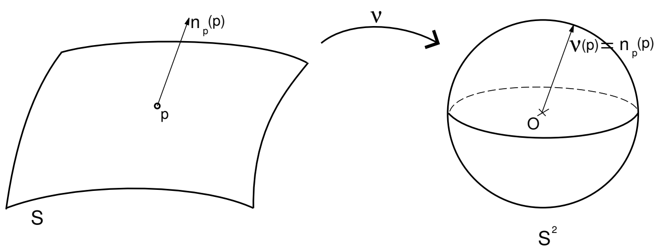

Figure 1: The Normal Mapping

$$K(p) = K_S(p) = \lim_{diam(R)\to 0} \frac{Area(\nu(R))}{Area(R)} \tag{1}$$

where R is a simple region, $p \in R \subset S$.

A sign is attached to $K(p)$ in a natural way – see Figure 2 below:

Remark 2.1 *Note that the same basic idea is also applicable for plane curves (see, e.g. [52]).*

Gauss proved also that:

$$K(p) = k_{min}k_{Max}$$

where k_{min} and k_{Max} are the *minimal*, respective *maximal normal curvatures* of the surface S at the point p. Recall that the normal curvature of $\gamma \subset S$ in the direction $\mathbf{v}$ at a point p is defined as: $\kappa_v(p) = \kappa_\gamma(p)$, where $\gamma = S \cap P$, and where

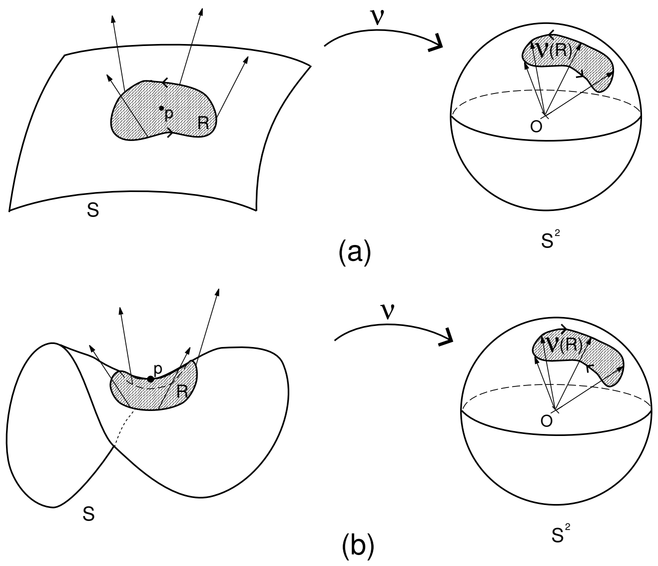

Figure 2: Gauss Curvature Definition: (a) $K(p) > 0$ and (b) $K(p) < 0$

P is a plane such that $P \perp T_p(S)$ and such that $\gamma' \parallel \mathbf{v}$. Here $T_p(S)$ denotes the *tangent plane* to S at p. This formula is unfortunately – since it is neither immediate nor natural – employed as the definition of the Gauss curvature.

Both definitions of curvature considered above render themselves to generalization, but the original, geometric idea of Gauss is the one that will provide us in Section 4.2 with an extension of the notion of curvature to rather abstract spaces.

The classical case of smooth surfaces, i.e. of class $\mathcal{C}^k, k \geq 2$, deserves, of course, special attention: Let $U = int(U)$ be an open set and let $f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a smooth function (i.e. $f \in \mathcal{C}^k, k \geq 2$). Then the expression of K in local coordinates is:

$$K = \frac{eg - f^2}{EG - F^2}\,; \tag{2}$$

where

$$E = f_u \cdot f_u\,,\ F = f_u \cdot f_v\,,\ G = f_v \cdot f_v\,; \tag{3}$$

and

$$e = \frac{det(f_u, f_v, F_{uu})}{\sqrt{EG - F^2}}\,,\ f = \frac{det(f_u, f_v, F_{uv})}{\sqrt{EG - F^2}}\,,\ g = \frac{det(f_u, f_v, F_{vv})}{\sqrt{EG - F^2}}\,; \tag{4}$$

where $f_u = \partial f / \partial u$, etc. and "$\cdot$" denotes the scalar product.

$I_f = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and $II_f = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ are called *the first*, respectively *the second fundamental form* of S.

Note that II_f depends on the position of S in space (see e.g. [18], p.154), i.e. upon the specific embedding considered. Therefore, the problem with the definition of K given by formula (2) is that it is dependent upon II_f, hence upon the specific embedding of S in $\mathbb{R}^3$, thus its relevance as an intrinsic geometric invariant of the surface S appears to be limited, to say the least. (A property is called *intrinsic* iff it depends solely upon the the first fundamental form of the surface, hence it is invariant under *local isometries*.) However, it can be proved (see [54], p. 112) that the following formula (of Frobenius) holds:

$$K = -\frac{1}{4(EG-F^2)^2}\begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial}{\partial v}\frac{E_v - F_u}{\sqrt{EG-F^2}} - \frac{\partial}{\partial u}\frac{F_v - G_u}{\sqrt{EG-F^2}}\right);$$

where $F_u = \partial F / \partial u$, etc.

Since Frobenius' formula shows K is independent of II_f it proves:

Theorem 2.2 (Gauss' Theorema Egregium) Gaussian curvature is intrinsic.

Remark 2.3 Both Theorema Egregium and the Gauss-Bonnet theorem admit proofs for PL-manifolds – see [2], [3], [20], [32], [45].

2.2.2 The Bertrand-Diguet-Puiseaux Formulas

We include only a small part of the vast formulary pertaining to the Gauss curvature (and, most notably, no references are brought to the Gauss-Bonnet formula). The formulas we selected give simple geometric interpretation to Gaussian curvature, that will be employed herein and on which we focus. They relate to the circumference (respective area) of a small circle on a surface (or *geodesic circle* – for the precise definition see, e.g. [18], p. 287).

Theorem 2.4 (Bertrand-Diguet-Puiseaux – 1848) Let S be a surface in $\mathbb{R}^3$, $p \in S$ and let $\varepsilon > 0$. Denote by $C(p,\varepsilon)$, $B(p,\varepsilon)$ the geodesic circle, respective the geodesic ball of center p and radius $\varepsilon > 0$. Then:

$$length\, C(p,\varepsilon) = 2\pi\varepsilon - \frac{\pi}{3}K(p)\varepsilon^3 + o(\varepsilon^3)\,, \tag{5}$$

and

$$area\, B(p,\varepsilon) = \pi\varepsilon - \frac{\pi}{12}K(p)\varepsilon^4 + o(\varepsilon^4)\,. \tag{6}$$

Hence:

$$K(p) = \lim_{\varepsilon \to 0} \frac{3}{\pi} \frac{2\pi\varepsilon - length\, C(p,\varepsilon)}{\varepsilon^3} = \lim_{\varepsilon \to 0} \frac{12}{\pi} \frac{\pi\varepsilon^2 - area\, B(p,\varepsilon)}{\varepsilon^4} \tag{7}$$

For a proof see e.g. [37], pp. 104-105, or [18], pp. 292 and 294.

3 Higher Dimensions

From here on we denote by M^n an n-dimensional $\mathcal{C}^k$-Riemannian manifold, $k \geq 2$, which we may presume, by Nash's theorem (see, e.g. [53]), to be isometrically embedded in R^N, for some N sufficiently large. Analogously to the notation for surfaces, let $T_p(M^n)$ denote the *tangent space* at the point $p \in M^n$, and let $T_p^{\perp}(M^n)$ stand be the orthogonal complement of $T_p(M^n)$ in $T_p(\mathbb{R}^N)$, i.e. $T_p(M^n) \oplus T_p^{\perp}(M^n) = T_p(\mathbb{R}^N)$. Then M^n can be locally written as the graph of a function $f : T_p(M^n) \to T_p^{\perp}(M^n)$.

3.1 Curvatures and their Geometric Interpretation

We restrict ourselves to the essentials necessary in the sequel. Therefore, except definitions and notations, we present only those results that satisfy both of the following conditions: they carry a meaningful geometric interpretation and are further required in this study. (Once again this selection is dictated by the large amount of existing material.) In consequence no proofs are given and even among the most meaningful geometric results (such as the generalized Gauss-Bonnet theorem), those who are not strictly needed below are (sadly) omitted.

3.1.1 The Curvature Tensor

Definition 3.1 Let M^n and f be as above, and let $p \in M^n \subset \mathbb{R}^N$. The bilinear form $II_p : T_p(M^n) \to T_p^{\perp}(M^n)$

$$II_p(M^n) = (\beta_{ij})_{1 \leq i,j \leq n} \tag{8}$$

where $\beta_{ij} = \partial^2 f / \partial x_i \partial x_j \, , 1 \leq i,j \leq n\,$; is called the *second fundamental tensor* of M^n at the point p.

The *Riemannian curvature tensor* (at a point p) is defined as the tensor of 2×2-minors of $II_p(M^n)$, i.e.:

$$R_{ijkl} = \beta_{ik}\beta_{jl} - \beta_{jk}\beta_{il}\,. \tag{9}$$

Remark 3.2 Riemannian curvature is not intrinsic.

3.1.2 Sectional Curvature

We begin by introducing the notion of *sectional curvature* – its definition being a very direct, geometric generalization of its classical 2-dimensional counterpart:

Definition 3.3 Let $p \in M^n$, and let $\Pi \subset T_p(M^n)$ be a 2-dimensional plane, and let $S = M^n \cap \big(\Pi \oplus T_p^{\perp}(M^n)\big)$. Then $dim\, S = 2$ and we define $K(\Pi) = K_p(S)$, where $K_p(S)$ represents the Gauss curvature of S at the point p.

Of course, if $n = 2$, K reduces to the classical Gauss curvature, thus justifying the name (and the notation).

There is a close connection between sectional curvature and the Riemannian curvature tensor, as can be seen from the following formula:

$$K(\Pi) = \sum_{1 \leq i,j,k,l \leq n} R_{ijkl}\, \mathbf{x_i x_j x_k x_l}\,; \tag{10}$$

where $\{\mathbf{x}_h\}_{1 \leq h \leq n}$ is an orthonormal base of $T_p(M^n)$.

Moreover, by direct computations it is easy to show that knowledge of $K(M)$ on all tangent planes is equivalent to knowing the curvature tensor (see, e.g. [19], pp. 94-95).

An analogue (and slight generalization) of the Bertrand-Diguet-Puiseaux formula (5) is given by:

Proposition 3.4

$$length\, C(p,\varepsilon,\alpha) = \alpha\varepsilon - \frac{\alpha}{3\sin^2\alpha} K(p)\varepsilon^3 + o(\varepsilon^3)\,, \tag{11}$$

where $C(p,\varepsilon,\alpha)$ denotes the arc of length α of $C(p,\varepsilon)$. In particular, for $\alpha = 2\pi$ (and $n = 2$) one gets the Bertrand-Diguet-Puiseaux formula in its classical form.

Thus sectional curvature (and the curvature tensor) measure the defect of M^n from being locally Euclidean. This is done at the 2-dimensional level. More precisely, M^n is *flat* (i.e. locally Euclidean) iff $K \equiv 0$. In addition, if $K \equiv k_0$, where k_0 is a constant, then M^n is locally isometric to the simply connected space of constant sectional curvature. (For the special case of simply connected surfaces of constant Gauss curvature, see Section 4.2.)

Remark 3.5 K behaves like a second derivative (or as a Hessian) of the metric g (see [6], p. 267).

3.1.3 Ricci Curvature

The *Ricci curvature* is obtained by *contracting* the Riemannian curvature tensor:

Definition 3.6 Let $\mathbf{v} \in T_pM^n$ be a unit vector. The *Ricci curvature* in the direction $\mathbf{v}$ is defined (in local coordinates) as:

$$Ricci_{ij} = \sum_i R_{ijil} \tag{12}$$

It also follows that:

$$Ricci(\mathbf{v}) = \sum_{i=2}^{n} K(\mathbf{v}, \mathbf{x}_i) = \sum_{i=2}^{n} R(\mathbf{v}, \mathbf{x}_i, \mathbf{v}, \mathbf{x}_i)\,, \tag{13}$$

where $\{\mathbf{v}, \mathbf{x}_1\,, \ldots, \mathbf{x}_{n-1}\}$ represents an orthonormal base of T_pM^n.

The geometric meaning of the Ricci curvature is underlined by the following version of the first Bertrand-Diguet-Puiseaux formula:

Theorem 3.7 ([34]) Let M^n be as above. Denote by $d\alpha$ the n-dimensional solid angle in the direction of the vector $\mathbf{v} \in T_p(M^n)$ and by $\omega(\alpha)$ the $(n-1)$-volume generated by geodesics of length ε in $d\alpha$. Then:

$$vol\big(\omega(\alpha)\big) = d\alpha\, \varepsilon^{n-1} \left(1 - \frac{Ricci(\mathbf{v})}{3}\varepsilon^2 + o(\varepsilon^2)\right)\,. \tag{14}$$

Remark 3.8 See also [30], pp. 316-319, for a generalization of the result above.

Moreover, let $< \mathbf{v}, \mathbf{w} >$ denote the plane spanned by $\mathbf{v}$ and $\mathbf{w}$. Then the following holds:

$$\mathbf{v} \cdot Ricci(\mathbf{v}) = \frac{n-1}{vol\big(\mathbb{S}^n - 2\big)} \int_{\mathbf{w} \in T_p(M^n),\ \mathbf{w} \perp \mathbf{v}} K(< \mathbf{v}, \mathbf{w} >)\,; \tag{15}$$

that is the Ricci curvature represents an average of sectional curvatures.

The counterpart of Remark 3.5 is:

Remark 3.9 The Ricci curvature behaves as the Laplacian of the metric g (see [6], p. 267).

3.1.4 Scalar Curvature

Formally, *scalar curvature* is defined as the trace of the Ricci curvature:

Definition 3.10

$$Scal = \sum_i Rici_{ii}\,. \tag{16}$$

It follows immediately from the definition that, for any orthonormal basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of $T_p(M^n)$, the following holds:

$$Scal = 2 \sum_{1 \leq i < j \leq n} K(< \mathbf{x}_i, \mathbf{x}_j >) = \frac{n(n-1)}{vol(\Pi)} \int_{\Pi \in \mathcal{P}} K(\Pi)\,; \tag{17}$$

where $\mathcal{P}$ represents the collection of 2-planes in $T_p(M^n)$.

This curvature also admits an analogue of the second Bertrand-Diguet-Puiseaux formula:

Theorem 3.11 ([31], p. 166)

$$vol\, B(p, \varepsilon) = \omega_n \varepsilon^n \left(1 - \frac{1}{6(n+2)} Scal(p) \varepsilon^2 + o(\varepsilon^2)\right)\,; \tag{18}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

That is, scalar curvature measures the defect of the manifold from being locally Euclidean at the level of volumes of small geodesic balls.

Remark 3.12 For an application of these formulas is the theory of subharmonic functions see [22].

3.2 PL-Manifolds and Curvature Approximation

One of the most natural questions that arises when confronted with the technical definitions and tedious derivations of cumbersome, intricate, complicated formulas, as compared to the simple geometric meaning of these notions, is the following: do these notions translate to simpler geometrical objects, such as polyhedra or PL-manifolds? That is: do there exist natural extensions of the definitions developed for the differentiable context and do the results obtained there still hold in the PL setting? Or, more concretely: are there "good" PL-approximations to the various notions of curvature?

The question was raised – and answered – several times, e.g. in the context of the Gauss-Bonnet theorem and that of the Tube Formula, see, for example, [2], [20], [25], [54] for the first problem, and [21], [26], [28], [33] for the second, (amongst others). Yet, however tempting to cover these subjects, we restrict ourselves to one aspect the problem of good approximations.

We concentrate on *Lipschitz-Killing curvatures* without, however, presenting all the technical details regarding his generalized notion of curvature (see [20]), since they are beyond the scope of this paper. However, we emphasize that, as a particular case of Lipschitz-Killing curvature one gets, for instance, the scalar curvature. Therefore an approximation theorem for these curvatures includes many separate results regarding other curvatures (such as the Chern-Gauss-Bonnet theorem). Such a convergence result was proved in [20], namely that *Lipschitz-Killing curvatures* of smooth Riemannian manifolds can be approximated in polyhedral approximation. More precisely, we have the following theorem:

Theorem 3.13 ([20]) Let M^n be a smooth Riemannian manifold and let $\{M_j^n\}_{j\geq 1}$ be a sequence of PL-manifolds that approximate M^n well (see Definition 3.17 below). Denote by the R, R_j Lipschitz-Killing curvatures of M^n, M_j^n, respectively. Then $R_j \to R$ in the sense of measures.

Definition 3.14 Let $\tau \subset \mathbb{R}^n$; $0 \leq k \leq n$ be a k-dimensional simplex. The *fatness* φ of τ is defined as being:

$$\varphi = \varphi(\tau) = \inf_{\substack{\sigma < \tau \\ dim\,\sigma\,=\,l}} \frac{vol(\sigma)}{diam^l\,\sigma} \tag{19}$$

The infimum is taken over all the faces of τ, $\sigma < \tau$, and $vol_{eucl}(\sigma)$ and $diam\,\sigma$ stand for the Euclidian l-volume and the diameter of σ respectively. (If $dim\,\sigma = 0$, then $vol_{eucl}(\sigma) = 1$, by convention.)

The simplex τ is φ_0*-fat*, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$.

Remark 3.15 There exists a constant $c(k)$ that depends solely upon the dimension k of τ s.t.

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\substack{\sigma < \tau \\ dim\,\sigma\,=\,l}} \measuredangle(\tau,\sigma) \leq c(k)\cdot\varphi(\tau)\,, \tag{20}$$

and

$$\varphi(\tau) \leq \frac{vol(\sigma)}{diam^l\,\sigma} \leq c(k)\cdot\varphi(\tau)\,; \tag{21}$$

where $\measuredangle(\tau,\sigma)$ denotes the (*internal*) *dihedral angle* of $\sigma < \tau$. (For a formal definition, see [20], pp. 411-412.)

Remark 3.16 The definition above is the one introduced in [20]. Other equivalent definitions were given in [16], [17], [40], [42], [56], amongst others. A somewhat more general definition was given in [25].

Definition 3.17 Let M^n be a smooth Riemannian manifold and let $\{M_j^n\}_j$ be a sequence of PL-manifolds. We say that $\{M_j^n\}_j$ approximate M^n *well* iff $M_j^n \to M^n$ as secant approximations and if all the simplices of $\{M_j^n\}_j$, $j \geq 1$ are φ_0-fat, for some given φ_0.

The meaning of "*convergence in the sense of measures*" is that, when approximating the Lipschitz-Killing curvatures by their PL counterparts, at a specific vertex of the approximating space, any of these individually computed PL versions may fail to represent a good approximation of R_j's, however they do so in average. (For the convergence of PL-spaces and secant approximations, see [40].)

4 Generalization – Metric Spaces

A number of metric versions of curvature in metric spaces are given. First are introduced metric analogues for the curvature of plane curves, then a number of metric definitions for Gauss curvature are considered.

4.1 Menger, Alt and Haantjes Curvatures

The *Menger*, *Alt* and *Haantjes* curvatures are metric definitions of curvature for curves.

We begin by introducing the most elementary of them: the *Menger* curvature: this is a metric expression for the circumradius of a triangle – thus giving in the limit a metric definition of the osculatory circle – and it is based upon some elementary high-school formulas:

Definition 4.1 Let (M, d) be a metric space, and let $p, q, r \in M$ be three distinct points. Then:

$$K_M(p,q,r) = \frac{\sqrt{(pq+qr+rp)(pq+qr-rp)(pq-qr+rp)(-pq+qr+rp)}}{pq \cdot qr \cdot rp}\,;$$

is called the *Menger Curvature* of the points p, q, r.

We can now define the Menger curvature at a given point by passing to the limit:

Definition 4.2 Let (M,d) be a metric space and let $p \in M$ be an accumulation point. Then M has at p *Menger curvature* $\kappa_M(p)$ iff for any $\varepsilon > 0$, there exists $\delta > 0$, such that if $d(p, p_i) < \delta$, $i = 1, 2, 3$; then $|K(Q) - \kappa_M(p)| < \varepsilon$.

Remark 4.3 The apparent equivalent notion of *Alt* curvature, in which one uses only two points converging to the third, is in fact more general, where we define the Alt curvature by:

Definition 4.4 Let (M,d) be a metric space and let $P \in M$ be an accumulation point. Then M has at p *Alt curvature* $\kappa_A(p)$ iff the following limit exists

$$\kappa_A(p) = \lim_{q,r \to p} K(p,q,r)\,;$$

where $K(p,q,r) = 1/R$ and R is the circumradius of the triangle of vertices p, q, r.

However, both $\kappa_M(p)$ and $\kappa_A(p)$ suffer from the same imperfection: since they are both modelled closely after the Euclidean Plane, they convey this Euclidean type of curvature upon the space they are defined on. However, the next definition does not closely mimic $\mathbb{R}^2$, therefore is better fitted for generalizations (see e.g. [51]):

Definition 4.5 Let (M,d) be a metric space and let $c : I = [0,1] \xrightarrow{\sim} M$ be a homeomorphism, and let $p, q, r \in c(I)$, $q, r \neq p$. Denote by $\widehat{qr}$ the arc of $c(I)$ between q and r, and by qr segment from q to r.

Then c has *Haantjes curvature* $\kappa_H(p)$ at the point p iff:

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\big(l(\widehat{qr})\big)^3} \ ;$$

where "$l(\widehat{qr})$" denotes the length – in intrinsic metric induced by d – of $\widehat{qr}$.

Remark 4.6 κ_H exists only for rectifiable curves, but if κ_M exists at any point p of c, then c is rectifiable.

Remark 4.7 Evidently, the existence of κ_M implies that of κ_A, while the existence of κ_A does not automatically imply that of κ_M (see [7], p. 76). However, we can prove the following theorem:

Theorem 4.8 Let $c : I \to M$ be a rectifiable curve, and let $p \in M$. If $\kappa_A(p)$ (or $\kappa_M(p)$) exists, then $\kappa_H(p)$ exists and $\kappa_A(p) = \kappa_H(p)$.

4.2 The Embedding or Wald Curvature

This approach stems from the Gauss' original method of comparing surface curvature to a standard, model surface (i.e. the unit sphere in $\mathbb{R}^3$). Wald's idea was to use more types of gauge surfaces and to restrict oneself to the study of the minimal geometric figure that allows this comparison.

Definition 4.9 Let (M,d) be a metric space, and let $Q = \{p_1, ..., p_4\} \subset M$, together with the mutual distances: $d_{ij} = d_{ji} = d(p_i, p_j)$; $1 \leq i, j \leq 4$. The set Q together with the set of distances $\{d_{ij}\}_{1 \leq i,j \leq 4}$ is called a *metric quadruple*.

Remark 4.10 One can define metric quadruples in slightly more abstract manner, without the aid of the ambient space: a metric quadruple being a 4 point metric space; i.e. $Q = \big(\{p_1, ..., p_4\}, \{d_{ij}\}\big)$, where the distances d_{ij} verify the axioms for a metric.

Before we proceed to the next definition, let us introduce the following notation: S_κ denotes *the complete, simply connected surface of constant Gauss curvature* κ, i.e. $S_\kappa \equiv \mathbb{R}^2$, if $\kappa = 0$; $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$, if $\kappa > 0$; and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$, if $\kappa < 0$. Here $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$ denotes the Sphere of radius $R = 1/\sqrt{\kappa}$, and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$ stands for the Hyperbolic plane of curvature $\sqrt{-\kappa}$, as represented by the Poincaré model of the plane disk of radius $R = 1/\sqrt{-\kappa}$.

Definition 4.11 The *embedding curvature* $\kappa(Q)$ of the metric quadruple Q is defined be the curvature κ of the gauge surface S_κ into which Q can be isometrically embedded. (See Figures 3 and 4 for embeddings of a metric quadruple in $\mathbb{S}^2_{\sqrt{\kappa}}$ and $\mathbb{H}^2_{\sqrt{-\kappa}}$, respectively.)

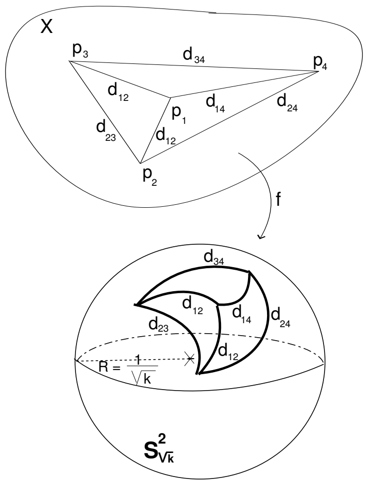

Figure 3: Embedding of a Metric Quadruple in $\mathbb{S}^2_{\sqrt{\kappa}}$

We can now define the embedding curvature at a point in a natural way by passing to the limit (but without neglecting the existence conditions), more precisely:

Definition 4.12 (M, d) be a metric space, let $p \in M$ and let $N \in \mathcal{N}(p)$. Then N is called linear iff N is contained in a geodesic. (Here $\mathcal{N}(p)$ denotes the set of neigbourhoods of N.)

Definition 4.13 Let (M, d) be a metric space, and let $p \in M$ be an accumulation point. Then p is said to have *Wald curvature* $\kappa_W(p)$ iff
(i) There exists $N \in \mathcal{N}(p)$, N *linear* (i.e. N is contained in a geodesic);
(ii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $Q = \{p_1, ..., p_4\} \subset M$ and such that if $d(p, p_i) < \delta\,, i = 1, ..., 4$; then $|\kappa(Q) - \kappa_W(p)| < \varepsilon$.

Remark 4.14 1. If one uses the second (abstract) definition of the metric curvature of quadruples, then the very existence of $\kappa(Q)$ is not assured, as it is shown by the following counterexample:

Counterxample 4.15 *The metric quadruple of lengths*

$$d_{12} = d_{13} = d_{14} = 1;\ d_{23} = d_{24} = d_{34} = 2$$

admits no embedding curvature.

2. Even if a quadruple has an embedding curvature, it still may be not unique (even if Q is not linear), indeed, one can study the following examples:

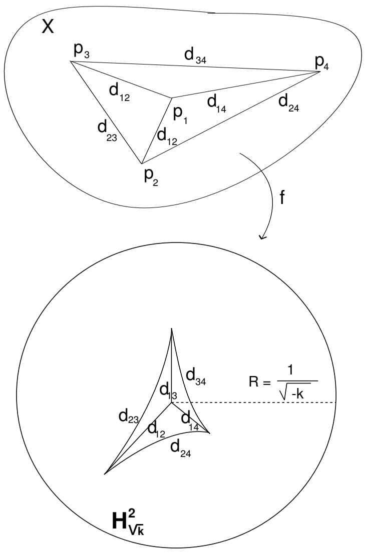

Figure 4: Embedding of a Metric Quadruple in $\mathbb{H}^2_{\sqrt{\kappa}}$

Example 4.16 *(a) The quadruple Q of distances $d_{ij} = \pi/2, 1 \leq i < j \leq 4$ is isometrically embeddable both in $S_0 = \mathbb{R}^2$ and in $S_1 = \mathbb{S}^2$.*

(b) The quadruple Q of distances $d_{13} = d_{14} = d_{23} = d_{24} = \pi$, $d_{12} = d_{34} = 3\pi/2$ admits exactly two embedding curvatures: $\kappa_1 \in (1.5, 2)$ and $\kappa_2 = 3$. (See [11].)

However, for "good" metric spaces (i.e. spaces that are locally "plane like") the embedding curvature exists and it is unique. And, what is even more relevant for us, this embedding curvature coincides with the classical Gaussian curvature. The proof of this result is rather long and tedious, therefore we shall present here only a brief sketch of it. (This will prove to be somewhat redundant anyhow, in view of the more general results presented in the previous section, a fact but we shall emphasize later in our presentation.)

The main ingredient for this proof and for the analysis of yet another approach to curvature (the CAT one – see Section 5) is provided by the following string of propositions (which are just generalizations of the classical triangle inequalities):

Proposition 4.17 Let $\triangle(p_1, q_1, r_1) \subset \mathcal{S}_{\kappa_1}$ and $\triangle(p_2, q_2, r_2) \subset \mathcal{S}_{\kappa_2}$, such that $p_1q_1 = p_2q_2,\ p_1r_1 = p_2r_2$ and $\measuredangle(q_1, p_1, r_1) = \measuredangle(q_2, p_2, r_2)$.
Then: if $\kappa_1 < \kappa_2$, then $q_1r_1 > q_2r_2$. (Here $\triangle(p_1, q_1, r_1)$ denotes the triangle of vertices p_1, p_2, p_3 .)

Proposition 4.18 Let $p_1, q_1, r_1 \in \mathcal{S}_{\kappa_1},\ p_2, q_2, r_2 \in \mathcal{S}_{\kappa_2}\ ,\ \kappa_1 < \kappa_2$ be two isometric triples of points, such that the triple p_1, q_1, r_1 is not linear. Then:
$\measuredangle(q_1, p_1, r_1) < \measuredangle(q_2, p_2, r_2), \measuredangle(p_1, q_1, r_1) < \measuredangle(p_2, q_2, r_2), \measuredangle(q_1, r_1, p_1) < \measuredangle(q_2, r_2, p_2)$.

Proposition 4.19 Let $Q_1 = \{p_1, q_1, r_1, s_1\}$, $Q_2 = \{p_2, q_2, r_2, s_2\}$ be non-linear and non-degenerate quadruples in $\mathcal{S}_{\kappa_1}$, $\mathcal{S}_{\kappa_2}$, respectively. If $\triangle(p_1, q_1, r_1) \cong \triangle(p_2, q_2, r_2)$ and $\kappa_1 < \kappa_2$, then:

1. $p_1s_1 = p_2s_2,\ q_1s_1 = q_2s_2 \Longrightarrow r_1s_1 > r_2s_2\,$;
2. $r_1s_1 = r_2s_2,\ q_1s_1 = q_2s_2 \Longrightarrow p_1s_1 > p_2s_2\,$;
3. $p_1s_1 = p_2s_2,\ r_1s_1 = r_2s_2 \Longrightarrow q_1s_1 < q_2s_2\,$.

To fully exploit the results above we need the following definition:

Definition 4.20 A metric quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j),\ i = 1, ..., 4$, is called *semi-dependent* (or a *sd-quad*, for brevity), iff 3 of its points are on a common geodesic, i.e. there exist 3 indices – e.g. 1,2,3 – such that: $d_{12} + d_{23} = d_{13}$.

Now we can easily formulate the following immediate consequence of Proposition 4.19 :

Corollary 4.21 A sd-quad admits at most one embedding curvature.

Unfortunately – as we have already noticed – in the general case the uniqueness of the embedding curvature is not guaranteed. However we can be a bit more explicit regarding the existence of the embedding curvature by using the following definition:

Definition 4.22 Let $Q = \{p, q, r, s\}$ be a non-linear and non-degenerate quadruple. Q is called *planar* iff $\measuredangle(q, p, r) + \measuredangle(q, p, s) + \measuredangle(s, p, r) = 2\pi$.

Proposition 4.23 Let $Q = \{p, q, r, s\}$ be a non-linear and non-degenerate quadruple in $\mathcal{S}_\kappa$. Then

1. If Q is planar, then it admits no isometric embedding in $\mathcal{S}_{\kappa_1},\ \kappa_1 > \kappa$.
2. If Q is not planar, then it admits no isometric embedding in $\mathcal{S}_{\kappa_2},\ \kappa_2 < \kappa$.

Corollary 4.24 Let $Q = \{p, q, r, s\}$ be a non-linear and non-degenerate quadruple. Then Q has at most two different embedding curvatures.

In fact we can state a much stronger assertion, of which Example 4.16(a) is just a very particular case:

Proposition 4.25 For any $p \in \mathcal{S}_\kappa$, and for any $\kappa > 0$, there exists $U \in \mathcal{N}(p)$ such that there exists a non-linear, non-degenerate quadruple $Q \subset U$ of embedding curvature 0.

Proof: Let $\gamma_1, \gamma_2 \in U$, two great-circle arcs such that $\gamma_1 \cap \gamma_2 = p$. Let $q_1, q_2 \in \gamma_1$ such that $pq_1 = pq_2 \neq 0$ and let $q \in \gamma_2$ such that $pq < \pi/2\sqrt{\kappa}$. Consider $\triangle(q_1'q_2', q') \subset \mathbb{R}^2$, $\triangle(q_1'q_2', q') \cong \triangle(q_1q_2, q)$, let $p' = \frac{1}{2}q_1'q_2'$, and let $h = q'p'$. (Here $\triangle(q_1'q_2', q')$ denotes the triangle of base $q_1'q_2'$ and vertex q', etc.)

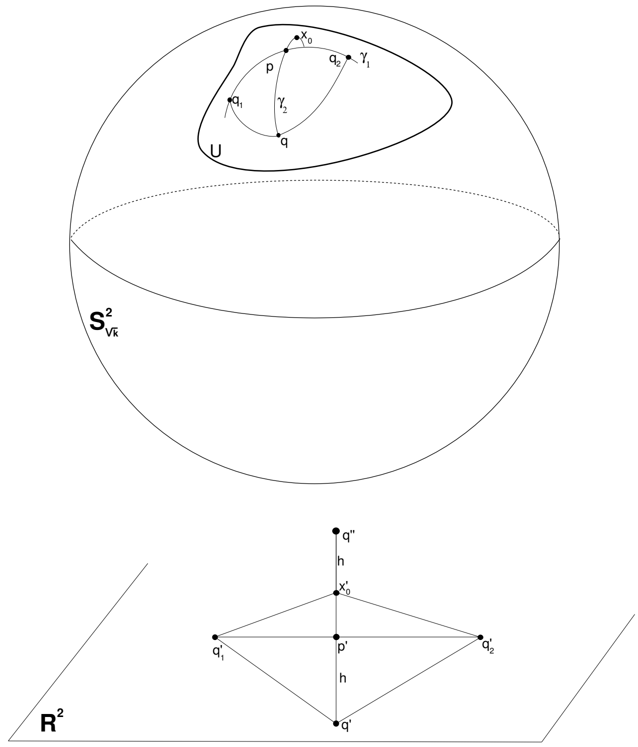

Figure 5: A Non-degenerate Quadruple of Zero Embedding Curvature

Then since $0 < \kappa$, Proposition 4.19(3) applied to the quadruples $\{q, q_1, q_2, p\}$ and $\{q', q_1', q_2', p'\}$ implies that $h < pq$.

Now let $x \in \gamma_2$, x between p and q, and let $x' \in \mathbb{R}^2$ such that $\triangle(q_1'q_2', x') \cong \triangle(q_1q_2, x)$ and such that x and q' are on different sides of the line $\overleftrightarrow{q_1q_2}$. Then, if $x = p$, then $xq > x'q'$, and if $x = q$, then $xq = 0 < x'q' = 2h$, where, in this case $x' = q''$. (See Figure 5.) Then it follows from a continuity argument that there exists $x_0 \in \gamma_2$, x_0 between p and q, such that $x_0q = x_0'q'$, thus implying that $\{q_1, q_2, q, x\} \cong \{q_1', q_2', q', x'\}$. QED

4.3 Wald Curvature and Gauss Curvature Comparison

The discussion above would be nothing more than a nice intellectual exercise where it not for the fact that the metric (Wald) and the classical (Gauss) curvatures coincide whenever the second notion makes sense, that is for smooth (i.e. of class $\geq \mathcal{C}^2$) surfaces in $\mathbb{R}^3$. More precisely the following theorem holds:

Theorem 4.26 (Wald) Let $S \subset \mathbb{R}^3$, $S \in \mathcal{C}^m$, $m \geq 2$ be a smooth surface. Then, given $p \in S$, $\kappa_W(p)$ exists and $\kappa_W(p) = \kappa_G(p)$.

Moreover, Wald also proved that a partial reciprocal theorem holds, more precisely he proved the following:

Theorem 4.27 Let M be a compact and convex metric space. If $\kappa_W(p)$ exists, for all $p \in M$, then M is a smooth surface and $\kappa_W(p) = \kappa_G(p)$, for all $p \in M$.

Remark 4.28 If one tries to restrict oneself, in the building of Definition 4.13 only to sd-quads, then Theorem 4.27 holds only if the following presumption is added:

Condition 4.29 *M is locally homeomorphic to $\mathbb{R}^2$.*

However the proof of this facts is involved and, as such, beyond the scope of this presentation. Therefore we shall restrict ourselves to a succinct description of the principal steps towards the proofs. The basic idea is to show that if a metric M space admits a Wald curvature at any point, than M is locally homeomorphic to $\mathbb{R}^2$, thus any metric proprieties of $\mathbb{R}^2$ can be translated to M, (in particular the first fundamental form). The first of these partial results is:

Theorem 4.30 Let M be a convex metric space. Then M admits at most one Wald curvature $\kappa_W(p)$, for any $p \in M$.

Proof: By Corollary 4.21 it suffices to prove that any disk neighborhood $B(p\,;\rho) \in \mathcal{N}(p)$ contains a non degenerate sd-quad. Without loss of generality one can assume that $B(p\,;\rho)$ contains three points p_1, p_2, p_3 such that $d(p, p_i) < \rho/2$, $i = 1, 2, 3$ (see [7]). Then, by the convexity of M it follows that there exists $q \in M$ such that $p \neq p_2, p_3$ and $p_2q + p_3q = p_2p_3$. But $p_2p_3 \leq pp_2 + pp_3 < \rho$ implies that $pq < \rho/2$ or $pp_2 < \rho/2$. If the first inequality holds, then $pq \leq pp_2 + p_2q < \rho$, i.e $q \in B(p\,;\rho)$; and if the second one holds, then $pd \leq pp_3 + p_3q < \rho$, i.e. $q \in B(p\,;\rho)$. But $p \neq q$, therefore p, p_2, p_3, q are not linear. QED

Our next step will be to analyze those neighborhoods that display "a normal behavior", both metrically and curvature-wise: that is precisely those disk neighborhoods in which the Wald curvature is defined and ranges over a small, bounded set of values prescribed by the very radius of the disk:

Definition 4.31 A disk neighborhood $B(p\,;\rho); \rho > 0$ is called *regular* iff for any non-degenerate quadruple $Q \subset B(p\,;\rho)$, $\kappa_W(Q)$ exists and $|\kappa_W(Q)| < \pi^2/16\rho^2$.

Remark 4.32 If $\kappa_W(p)$ exists, then for any sufficiently small ρ, $B(p\,;\rho)$ will be regular.

It turns out that regular neighborhoods, in *compact, convex* spaces have the following "nice" (i.e. real plane like) proprieties:

Proposition 4.33 Let M be a compact, convex metric space and let $B(p\,;\rho) \subset M$ be a regular neighborhood. If a non-degenerate quadruple $Q \subset B(p\,;\rho)$ contains two linear triples of points, then Q is linear.

Proposition 4.34 Let M be a compact, convex metric space. Then: for any $p \in M$ and for any regular neighbourhood $B(p\,;\rho)$ of p, there exist $q, r \in B(p\,;\rho)$ such that p, q, r are not linear.

Proposition 4.35 Any regular neighborhood $B(p\,;\rho)$ of a compact, convex metric space is *strictly convex*, i.e. $q, r \in B(p\,;\rho) \Longrightarrow int(qr) \subset B(p\,;\rho)$.

While the proof of this last proposition is lengthy and therefore we omit it, that of the following important corollary is not:

Corollary 4.36 Let $B(p\,;\rho)$ be a regular neighborhood. Then, for any $q, r \in B(p\,;\rho)$, the geodesic segment qr exists and $int(qr) \subset B(p\,;\rho)$.

Proof: By the convexity of $B(p\,;\rho)$ it follows the existence of at least one geodesic qr, for all $q, r \in B(p\,;\rho)$. If $s \in int(q)r$, then by the proposition above we have that $s \in B(p\,;\rho)$. It follows that $B(p\,;\rho)$ contains all the geodesics with end points q, r. Hence, by Proposition 4.33, the geodesic segment qr is unique. QED

We can now begin to prove that a compact, convex metric space locally mimics $\mathbb{R}^2$. We start by showing that the *sinus* function is defined on M, thus allowing for angle measure, hence for the definition of polar coordinates on regular neighbourhoods (in the same way geodesic polar coordinates are used on classical surfaces).

First, let M be as before, and let $p \in M$ s.t. $\kappa_W(p)$ exists. Let $q, r \in B(p\,;\rho), q \neq p \neq r$, where $B(p\,;\rho)$ is a regular neighborhood of p. Then, for any $x \in [0, min\{pq, pr\})$, define $q(x) \in pq,\ r(x) \in pr$ by: $d(p, q(x)) = x = d(p, r(x))$, and let $d(x) = d(, q(x), r(x))$ (see Figure 6 bellow).

Proposition 4.37 The following limit exists:

$$\lim_{x \to 0} \frac{d(x)}{x}\,.$$

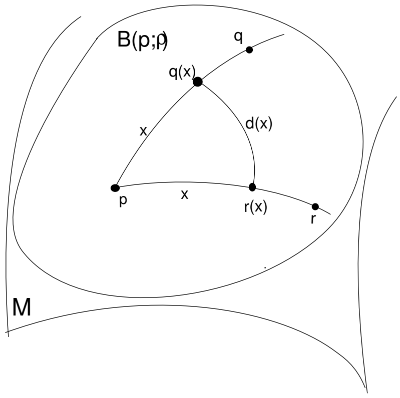

Figure 6: Metric Definition of the Sinus Function

We omit the proof since it is rather involved (but canonical for any axiomatic approach to Euclidean Geometry – see, for instance, [7], [48]).

Now we can define the measure of angles at a point p:

Definition 4.38 The measure of the angle $\measuredangle(q,p,r))$ is given by:

$$m(\measuredangle(q,p,r)) = 2arcsin\Big(\frac{1}{2}\lim_{x\to 0}\frac{d(x)}{x}\Big)\,.$$

Remark 4.39 The definition above enables us to define polar coordinates on regular neighborhoods in the following manner:

Let $p_1,p_2 \in B(p\,;\rho)$ such that p,p_1,p_2 are not collinear. (Such points exist by Proposition 4.34). To every point $q \in B(p\,;\rho)$ we associate the following pair of real numbers (defining the polar coordinates at q relative to the frame determined by p,p_1,p_2): $(r(q),\theta(q))$, where

$$r(q) = d(p,q)$$

and

$$\theta(q)) = \begin{cases} m(\measuredangle(q,p,p_1)) & \text{if } |m(\measuredangle(p_2,p,p_1)) - m(\measuredangle(q,p,p_1))| = m(\measuredangle(q,p,p_1))\,; \\ 2\pi - m(\measuredangle(q,p,p_1)) & \text{if } |m(\measuredangle(p_2,p,p_1)) - m(\measuredangle(q,p,p_1))| \neq m(\measuredangle(q,p,p_1))\,. \end{cases}$$

Once coordinates (be they polar or cartesian) are introduced, the local homomorphism with $\mathbb{R}^2$ is immediate (see e.g. [12]) and we can thus now safely state the foretold homomorphism result:

Proposition 4.40 Any convex, compact metric space is locally homeomorphic to the real plane.

4.3.1 Computation of Wald Curvature

In this section we develop formulas for the computation of embedding curvature of quadruples. First we follow the classical approach of Wald-Blumenthal that employs the so-called *Cayley-Menger determinants* (see below). Unfortunately, the formulas obtained, albeit precise are transcendental. Therefore we present, in the next subsection, the approximate formulas developed by C.V. Robinson ([47]).

The Cayley-Menger Determinant Given a general metric quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j),\ i = 1, ..., 4$; we denote by $D(Q) = D(p_1, p_2, p_3, p_4)$ the following determinant:

$$D(p_1,p_2,p_3,p_4) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 \\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix} \tag{22}$$

Then the *embedding curvature* $\kappa(Q)$ of Q is given – depending upon the embedding space (i.e. upon the sign of the curvature) – by the following formulae:

$$\kappa(Q) = \begin{cases} 0 & \text{if } D(Q) = 0\,; \\ \kappa,\ \kappa < 0 & \text{if } det(\cosh\sqrt{-\kappa}\cdot d_{ij}) = 0\,; \\ \kappa,\ \kappa > 0 & \text{if } det(\cos\sqrt{\kappa}\cdot d_{ij}) \text{ and } \sqrt{\kappa}\cdot d_{ij} \leq \pi \\ & \quad \text{and all the principal minors of order 3 are} \geq 0. \end{cases} \tag{23}$$

The determinant $D(Q) = D(p_1, p_2, p_3, p_4)$ is called the *Cayley-Menger determinant* (of the points $p_1, ...p_4$). Of course, this definition readily generalizes to any dimension, as do the results bellow. To get some geometric intuition regarding Formula (23) we look into the Euclidean case.

In order to prove the case $\kappa(Q) = 0$ of (23) we need first to investigate some of its properties (see [7], [4] for details). We start with the following proposition:

Proposition 4.41 The points $p_1, ..., p_4$ are the vertices of a simplex in $\mathbb{R}^3$ iff $D(p_1, p_2, p_3, p_4) \neq 0$.

However, we can prove the much strong result below:

Theorem 4.42 Let $d_{ij} > 0\,, 1 \leq 4\,, i \neq j$. Then there exists a simplex $T = T(p_1, ..., p_4) \subseteq \mathbb{R}^3$ such that $dist(x_i, x_j) = d_{ij}\,, i \neq j$; iff $D(p_i, p_j) < 0$, for any $\{i, j\} \subset \{1, ..., 4\}$ and $D(p_i, p_j, p_k) > 0$, for any $\{i, j, k\} \subset \{1, ..., 4\}$; where, for instance,

$$D(p_1,p_2) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & d_{12}^2 \\ 1 & d_{12}^2 & 0 \end{vmatrix}$$

and

$$D(p_1, p_2, p_3) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 \end{vmatrix};$$

etc...

In fact, the necessary and sufficient condition above can be relaxed. Indeed one can also show that the following result (which we formulate – for convenience – for the case $n = 3$ only, even if it is readily generalized to any dimension) holds:

Proposition 4.43 Let $d_{ij} > 0\,, 1 \leq 4\,, i \neq j$. Then there exists a simplex $T = T(p_1, ..., p_4) \subseteq \mathbb{R}^3$ such that $dist(x_i, x_j) = d_{ij}\,, i \neq j$; iff $D(p_1, p_2, p_3, p_4) \neq 0$ and $sign\, D(p_1, p_2, p_3, p_4) = +1$.

The proof of Formula (23) for the spherical and hyperbolical cases would prove to be to technical for this limited exposition; suffice to say that they essentially reproduce the proof given in the Euclidean case, taking into account the fact that performing computations in the spherical (resp. hyperbolic) metric one has to replace the distances d_{ij} by $\cos d_{ij}$ (resp. $\cosh d_{ij}$) – see [7] for the full details.

Approximate Formulas The formulas we just developed in are not only transcendental, but also the computed curvature may fail to be unique (see the preceding section). However, uniqueness is guaranteed for sd-quads. Moreover, the relatively simple geometric setting of sd-quads allows for the development of simple (i.e. rational) formulas for the approximation of the embedding curvature.

Theorem 4.44 ([47]) Given the metric quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j),\ i, j = 1, ..., 4$; the embedding curvature $\kappa(Q)$ is well approximated by:

$$K(Q) = \frac{6(\cos \measuredangle_0 2 + \cos \measuredangle_0 2')}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2) + d_{23}\sin^2(\measuredangle_0 2')\big)} \tag{24}$$

where: $\measuredangle_0 2 = \measuredangle(p_1 p_2 p_4)\,,\ \measuredangle_0 2' = \measuredangle(p_3 p_2 p_4)$ represent the angles of the Euclidian triangles of sides d_{12}, d_{14}, d_{24} and d_{23}, d_{24}, d_{34} , respectively.

The *error* R can be estimated by using the following inequality:

$$|R| = |R(Q)| = |\kappa(Q) - K(Q)| < 4\kappa^2(Q) diam^2(Q)/\lambda(Q) \tag{25}$$

where we put: $\lambda(Q) = d_{24}(d_{12}\sin \measuredangle_0 2 + d_{23}\sin \measuredangle_0 2')/S^2$, and where $S = Max\{p, p'\}$; $2p = d_{12} + d_{14} + d_{24}\,,\ 2p' = d_{32} + d_{34} + d_{24}$.

Proof: The basic idea of the proof is to recreate, in a general metric setting, the Gauss Map – in this case one measures the curvature by the amount of "bending" one has to apply to a general planar quadruple so that it may be "straightened" (i.e. isometrically embedded as a *sd-quad*) in some S_κ.

Consider two planar (i.e. embedded in $R^2 \equiv \mathcal{S}_0$) triangles $\triangle p_1p_2p_4$ and $\triangle p_2p_3p_4$, and denote by $\triangle p_{1,k}p_{2,k}p_{4,k}$ and $\triangle p_{2,k}p_{3,k}p_{4,k}$ their respective isometric embeddings into $\mathcal{S}_k$. Then $p_{i,k}p_{j,k}$ will denote the geodesic (of $\mathcal{S}_k$) through $p_{i,k}$ and $p_{j,k}$. Also, let $\measuredangle_k 2$ and $\measuredangle_k 2'$ denote, respectively, the following angles of $\triangle p_{1,k}p_{2,k}p_{4,k}$ and $\triangle p_{2,k}p_{3,k}p_{4,k}$: $\measuredangle_k 2 = \measuredangle p_{1,k}p_{2,k}p_{4,k}$ and $\measuredangle_k 2' = \measuredangle p_{2,k}p_{3,k}p_{4,k}$ (see Fig. 7).

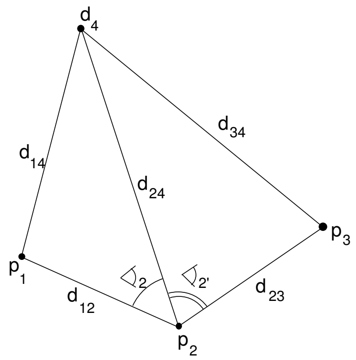

Figure 7: An sd-quad

But $\measuredangle_k 2$ and $\measuredangle_k 2'$ are strictly increasing as functions of k. Therefore the equation

$$\measuredangle_k 2 + \measuredangle_k 2' = \pi \tag{26}$$

has at most one solution k^*, i.e. k^* represents the unique value for which the points p_1, p_2, p_4 are on a geodesic in $\mathcal{S}_k$ (for instance on p_1p_4).

But that means that k^* is precisely the embedding curvature, i.e. $k^* = \kappa(Q)$, where $Q = Q(p_1, p_2, p_3, p_4)$.

Equation (26) is equivalent to

$$\cos^2 \frac{\measuredangle_{k^*} 2}{2} + \cos^2 \frac{\measuredangle_{k^*} 2'}{2} = 1$$

The basic idea being the comparison between metric triangles with equal sides, embedded in $\mathcal{S}_0$ and $\mathcal{S}_k$, respectively, it is natural to consider instead of the previous equation, the following equality:

$$\theta(k,2)\cdot\cos^2\frac{\measuredangle_0 2}{2} \;+\; \theta(k,2')\cdot\cos^2\frac{\measuredangle_0 2'}{2} \;=\; 1 \tag{27}$$

where we denote:

$$\theta(k,2)=\frac{\cos^2\frac{\measuredangle_{k^*}2}{2}}{\cos^2\frac{\measuredangle_0 2}{2}}\,;\;\;\theta(k,2')=\frac{\cos^2\frac{\measuredangle_{k^*}2'}{2}}{\cos^2\frac{\measuredangle_0 2}{2}}\,.$$

Since we want to approximate $\kappa(Q)$ by $K(Q)$ we shall resort – naturally – to expansion into MacLaurin series. We are able to do this because of the existence of the following classical formulas:

$$\cos^2\frac{\measuredangle_k 2}{2}=\frac{\sin(p\sqrt{k})\cdot\sin(d\sqrt{k})}{\sin(d_{12}\sqrt{k})\cdot\sin(d_{24}\sqrt{k})}\,;\;k>0\,;$$

$$\cos^2\frac{\measuredangle_k 2}{2}=\frac{\sinh(p\sqrt{k})\cdot\sinh(d\sqrt{k})}{\sinh(d_{12}\sqrt{k})\cdot\sinh(d_{24}\sqrt{k})}\,;\;k<0\,;$$

and, of course

$$\cos^2\frac{\measuredangle_0 2}{2}=\frac{pd}{d_{12}d_{24}}\,;$$

were $d=p-d_{14}=(d_{12}+d_{24}-d_{14})/2$ (and the analogous formulas for $\cos^2\frac{\measuredangle_{k'}2}{2}$).

By using the development into series of $f_1(x)=\frac{\sin\sqrt{x}}{\sqrt{x}}$ and $f_2(x)=\frac{\sinh\sqrt{x}}{\sqrt{x}}$; one (easily) gets the desired expansion for $\theta(k,2)$:

$$\theta(k,2)=1+\frac{1}{6}kd_{12}d_{24}\big(\cos(\measuredangle_0 2)-1\big)+r\,; \tag{28}$$

where: $|r|<\frac{3}{8}k^2p^4$, for $|kp^2|<1/16$.

By applying (28) to (27), we receive:

$$\Big[1+\frac{1}{6}k^*d_{12}d_{24}\big(\cos(\measuredangle_0 2)-1\big)+r\Big]\cos^2\frac{\measuredangle_0 2}{2}+ \tag{29}$$

$$\Big[1+\frac{1}{6}k^*d_{23}d_{24}\big(\cos(\measuredangle_0 2')-1\big)+r'\Big]\cos^2\frac{\measuredangle_0 2'}{2}=1\,;$$

for: $|r|+|r'|<\frac{3}{4}(k^*)^2(Max\{p,p'\})^4=\frac{3}{4}(k^*)^2S^4$.

By solving the linear equation (in variable k^*) (29) and using some elementary trigonometric transformation one has:

$$k^*=\frac{6(\cos\measuredangle_0 2+\cos\measuredangle_0 2')}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2)+d_{23}\sin^2(\measuredangle_0 2')\big)}+R\,;$$

where:

$$|R| < \frac{12(|r|+|r'|)}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2)+d_{23}\sin^2(\measuredangle_0 2')\big)} < \frac{9(k^*)^2\max\{p,p'\}}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2)+d_{23}\sin^2(\measuredangle_0 2')\big)}\,.$$

But $k^* \equiv \kappa(Q)$, so we get the desired formula (24) .

To prove the correctness of the bound (25) one has only to observe that:

$$S = Max\{p,p'\} < 2diam(Q),\ \big(diam(Q) = \max_{1\leq i<j\leq 4}\{d_{ij}\}\big),$$

and perform the necessary arithmetic manipulations. QED

Remark 4.45 (a) The function $\lambda = \lambda(Q)$ is continuous and 0-homogenous as a function of the d_{ij}-s. Moreover: $\lambda(Q) \geq 0$ and $\lambda(Q) = 0 \Leftrightarrow \sin\measuredangle_0 2 = \sin\measuredangle_0 2' = 0$, i.e. iff Q is linear. (Therefore for sd-quads $\lambda(Q) > 0$. Moreover, when $\lambda(Q)$ tends to 0, Q approaches linearity.)

(b) Since $\lambda(Q) \neq 0$ it follows that: $K(Q) \in \mathbb{R}$ for any quadrangle Q. Moreover: $sign(\kappa(Q)) = sign(K(Q))$.

(c) If Q is any sd-quad, then $\kappa^2(Q)diam^2(Q)/\lambda(Q) < \infty$. Moreover, $|R|$ is small if Q is not close to linearity. In this case $|R(Q)| \sim diam^2(Q)$ (for any given Q).

Since the Gaussian curvature $K_G(p)$ at a point p is given by:

$$K_G(p) = \lim_{n\to 0}\kappa(Q_n)\,;$$

where $Q_n \to Q = \Box p_1pp_3p_4\,;\ diam(Q_n) \to 0$,, from Remark 4.45(c) we immediately infer that the following holds:

Theorem 4.46 Let S be a differentiable surface. Then, for any point $p \in S$:

$$K_G(p) = \lim_{n\to 0} K(Q_n)\,;$$

for any sequence $\{Q_n\}$ of sd-quads that satisfy the following condition:

$$Q_n \to Q = \Box p_1pp_3p_4\,;\ diam(Q_n) \to 0\,.$$

Remark 4.47 In the following special cases even "nicer" formulas are obtained:

1. If $d_{12} = d_{32}$, then

$$K(Q) = \frac{12}{d_{13}\cdot d_{24}}\cdot\frac{\cos\measuredangle_0 2+\cos\measuredangle_0 2'}{\sin^2\measuredangle_0 2+\sin^2\measuredangle_0 2'}\,; \tag{30}$$

(here we have of course: $d_{13} = 2d_{12} = 2d_{32}$); or, expressed as a function of distances alone:

$$K(Q) = 12\frac{2d_{12}^2+2d_{24}^2-d_{14}^2-d_{13}^2}{8d_{12}^2d_{24}^2-(d_{12}^2+d_{24}^2-d_{14}^2)^2-(d_{12}^2+d_{24}^2-d_{34}^2)^2} \tag{31}$$

2. If $d_{12} = d_{32} = d_{24}$ and if the following condition also holds:

3. $\measuredangle_0 2' = \pi/2$; i.e. if $d_{34}^2 = d_{12}^2 + d_{24}^2$ or, considering 2., also: $d_{34}^2 = 2d_{12}^2$, then

$$K(Q) = \frac{6\cos\measuredangle_0 2}{d_{12}(1+\sin^2\measuredangle_0 2)} = \frac{2d_{12}^2 - d_{14}^2}{4d_{12}^4 + 4d_{14}^2 d_{12}^2 - d_{14}^4} \,. \tag{32}$$

4.4 Rinow Curvature

The curvatures introduced before may seem a bit archaic in comparison to the more fashionable approach of *comparison triangles* (see Section 5), with their far reaching applications. We present here one of these comparison criteria and show its equivalence with the Wald curvature. We start with the following definition:

Definition 4.48 Let (M, d) be a metric space, together with the intrinsic metric induced by d. Let $R = int(R) \subseteq M$ be a region of M. We say that R is a *region of curvature* $\leq \kappa$ ($\kappa \in$ R) iff

1. For any $p, q \in R$ there exists a geodesic segment $pq \subset R$;

2. Any triangle $T(p, q, r) \subset R$ is isometrically embeddable in $\mathcal{S}_\kappa$;

3. If $T(p, q, r) \subset R$ and $x \in pq, y \in pr$ and if the points $p_\kappa, q_\kappa, r_\kappa, x_\kappa, y_\kappa \in \mathcal{S}_\kappa$ satisfy the following conditions:

   (a) $T(p, q, r) \cong T(p_\kappa, q_\kappa, r_\kappa)$;

   (b) $T(p, q, x) \cong T(p_\kappa, q_\kappa, x_\kappa)$;

   (c) $T(p, r, y) \cong T(p_\kappa, r_\kappa, y_\kappa)$;

   then $xy \leq x_\kappa y_\kappa$.

By replacing the condition: "$xy \leq x_\kappa y_\kappa$" with: "$xy \geq x_\kappa y_\kappa$", we obtain the definition of a *region of curvature* $\geq \kappa$ (see Figure 8).

We now pass to the localization of the definition above:

Definition 4.49 Let (M, d) be a metric space, together with the intrinsic metric induced by d, and let $p \in M$ be an accumulation point. Then M has at p *Rinow curvature* $\kappa_R(p)$ iff

1. There exists a linear neighbourhood $N \in \mathcal{N}(p)$;

2. For any $\varepsilon > 0$, there exists $\delta > 0$, such that $B(p; \delta)$ is

   (a) a region of Rinow curvature $\leq \kappa_R(p) + \varepsilon$
   and

   (b) a region of Rinow curvature $\geq \kappa_R(p) - \varepsilon$.

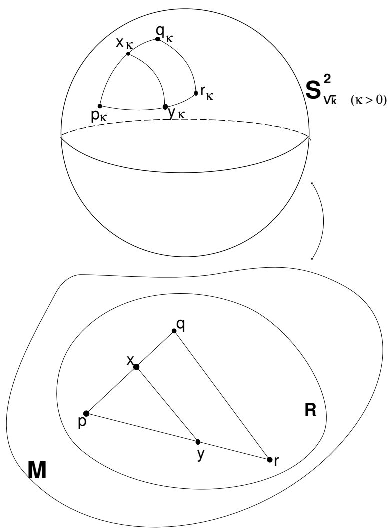

Figure 8: Positive Rinow Curvature

While its greater generality endows the Rinow curvature with more flexibility in applications and makes it easier in generalization, it is even more difficult to compute than Wald curvature. However this quandary has an almost ideal solution, due to Kirk (see [36]), solution which we briefly expose here:

Theorem 4.50 ([36]) Let M be a compact, convex metric space, and let $p \in M$. If $\kappa_W(p)$ exists, then $\kappa_R(p)$ exists, and $\kappa_R(p) = \kappa_W(p)$.

Unfortunately, since $\kappa_R(p)$ makes no presumption of dimensionality, the existence of $\kappa_R(p)$ does not imply the existence of $\kappa_W(p)$.

Counterxample 4.51 Let $M \equiv \mathbb{R}^3$. Then $\kappa_R(p) \equiv 0$, but $\kappa_W(p)$ does not exist at any point, since every neighborhood contains linear quadruples.

Kirk's solution of this problem is to consider the *modified Wald curvature* κ_{WK}, defined as follows:

Definition 4.52 Let (M, d) be a metric space, together with the intrinsic metric induced by d, and let $p \in M$. Then M has *modified Wald curvature* $\kappa_{WK}(p)$ at p iff

1. There exists a linear neighbourhood $N \in \mathcal{N}(p)$;

2. For any $\varepsilon > 0$, there exists $\delta > 0$, such that if $Q \subset B(p;\delta)$ is a non-degenerate sd-quad, then $\kappa_W(Q)$ exists and $|\kappa_{WK}(p) - \kappa_W(Q)| < \varepsilon$.

Remark 4.53 If $\kappa_W(p)$ exists, then $\kappa_{WK}(p)$ exists but the existence of $\kappa_{WK}(p)$ does not imply that of $\kappa_W(p)$. Indeed, if $p \in \mathbb{R}^3$, then $\kappa_{WK}(p) = 0$ but $\kappa_W(p)$ does not exist.

This modified curvature indeed represents the wished for solution, as proved by the following two theorems:

Theorem 4.54 ([36]) Let (M,d) be a metric space. Then: if $\kappa_R(p)$ exists, then $\kappa_{WK}(p)$ also exists and $\kappa_R(p) = \kappa_{WK}(p)$.

Theorem 4.55 ([36]) Let (M,d) be a metric space together with the associated intrinsic metric, and let $p \in M$. Suppose that

1. $\kappa_{WK}(p)$ exists

   and

2. There exists $B(p\,;\rho) \in \mathcal{N}(p)$, such that $qr \subset B(p\,;\rho)$, for all $q,r \in B(p\,;\rho)$.

   Then $\kappa_R(p)$ exists and $\kappa_R(p) = \kappa_{WK}(p)$.

5 Applications of Metric Curvatures

We first discuss a few direct applications of the various metric curvatures. The most elementary amongst them – the Menger curvature – was employed the most, both in a pure theoretical context, for estimating (via the Cauchy integral) the regularity of fractals and flatness of sets in the plane (see [41]); and for practical implementations, for curve reconstruction ([27]).

Haantjes curvature does not impose an Euclidean-type geometry upon the modelled space, therefore it is better suited for the geometrization of graphs (see [51]).

Of course, both curvatures above (as well as Alt curvature) can be employed – as approximations to sectional curvatures – in estimating curvatures of smooth curves on triangulated surfaces (see [49]).

Also, one is inclined to study the relationship between the generalizations presented in Section 3, and the more abstract ones introduced in Section 4. The relationship is rather straightforward when considering, for instance, sectional curvature on manifolds: these curvatures admit good metric approximations via "good" triangulations, whose existence is guaranteed by Theorem 3.13. It is interesting – both from the geometer's point of view and from the standpoint of Applied Mathematics, to ask whether one can use the existence of such triangulations to show directly, via the Bertrand-Diguet-Puiseaux theorem and its generalization, the convergence of curvatures in secant approximation (by PL manifolds). Such study is currently undertaken ([50]).

We could not conclude this brief excursus into the notion of curvature and its metric aspects, in particular Wald and Rinow curvatures, without at least alluding to the new developments in Comparison Geometry and their plethora of implications in a variety of fields, such as Geometric Group Theory, Partial Differential Equations, Dynamical Systems, Relativity and even Computer Science (see [14], [15], [23], [29], [30], [46]).

References

[1] Arnold, V. I. [1997] An Interview with V. I. Arnold. Notices of the A.M.S., 44, 4: 432-438.

[2] Banchoff, T.A. [1967] Critical points and curvature for embedded polyhedra. J. Differential Geometry, 1, 257-268.

[3] Banchoff, T.A. [1970] Critical Points and Curvature for Embedded Polyhedral Surfaces. Amer. Math. Monthly, 77, 475-485.

[4] Berger, M. [1987] *Geometry I,II, Universitext.* Springer-Verlag.

[5] Berger, M. [2000] Encounter with a Geometer, Part II. Notices of th AMS, 47, n3, 326-340.

[6] Berger, M. [2003] *A Panoramic View of Riemannian Geometry.* Springer-Verlag.

[7] Blumenthal L. M. [1953] *Distance Geometry – Theory and Applications.* Claredon Press.

[8] Berger, M and Gostiaux, B. [1987] *Differential Geometry: Manifols, Curves and Surfaces.* Springer-Verlag.

[9] Bernig, A. [2004] On some aspects of curvature. preprint.

[10] Bernig, A. [2005] Curvature tensors of singular spaces. Diff. Geom. Appl., to appear.

[11] L. M. Blumenthal and K. Menger, [1970] *Studies in Geometry.* Freeman and Co.

[12] Borsuk, K. and Szmielew, W. [1960] *Foundations of geometry.* North-Holland.

[13] Brooks, R. [2001] Differential Geometry – Course Notes. Technion.

[14] Bridson, M. R. and Haefliger, A. [1999] *Metric spaces of non-positive curvature, Grundlehren der mathematischen Wissenschaften.* Springer-Verlag.

[15] Burago, D., Burago, Y. and S. Ivanov [2000] A *Course in Metric Geometry, GSM.* AMS.

[16] Cairns, S.S. [1934] On the triangulation of regular loci. Ann. of Math., 35, 579-587.

[17] Cairns, S.S. [1936] Polyhedral approximation to regular loci. Ann. of Math., 37, 409-419.

[18] do Carmo, M. P. [1976] *Differential Geometry of Curves and Surfaces.* Prentice-Hall.

[19] do Carmo, M. P. [1992] *Riemannian Geometry, Mathematics: Theory & Applications.* Birkhäuser

[20] Cheeger, J., Müller, W. and Schrader, R. [1984] On the Curvature of Piecewise Flat Spaces. Comm. Math. Phys., 92, 405-454.

[21] Cheeger, J., Müller, W. and Schrader, R. [1986] Kinematic and Tube Formulas for Piecewise Linear Spaces. Indiana Univ. Math. J., 35 (4), 737-754.

[22] Ekonen, M. [2002] Generalizations of the Beckenback-Radó theorem. Ann. Acad. Sci. Fenn., Series I Math., Dissertationes.

[23] Epstein, D. B. A. et al. (editor) [1992] *Word Processing in groups.* Jones and Bartlett.

[24] Flaherty, F. J. [1973] Curvature measures for piecewise linear manifolds. Bull. Amer. Math. Soc., 79(1), 100-102..

[25] Fu, J. H. G. [1985] Tubular Neighbourhoods in Euclidian Spaces. Duke Math. J., Vol. 52, No.4, 1025-1046.

[26] Fu, J. H. G. [1993] Convergence of Curvatures in Secant Approximation. J. Differential Geometry, 37, 177-190.

[27] Giesen, J. [1999] Curve Reconstruction, the Traveling Salesman Problem and Menger's Theorem on Length. Proceedings of the 15th ACM Symposium on Computational Geometry (SoCG), 207-216.

[28] Gray, A. [1990] *Tubes.* Addison-Wesley.

[29] Gromov, M. [1987] Infinite groups as geometric objects, in *Geometric Group Theory,* (Niblo, G.A. and Roller, ed.). Spriger Verlag, MSRI Publ. 8, 75-263.

[30] Gromov, M. [1999] *Metric structures for Riemannian and non-Riemannian spaces, Progress in Mathematics 152.* Birkhauser.

[31] Gray, A. and Vanhecke, L. [1979] Riemannian geometry as determined by the volume of small geodesic balls. Acta Mathematica, 142, 157-198.

[32] Hilbert, D. and Cohn-Vossen, S. [1952] *Geometry and the Imagination.* Chelsea.

[33] Howland, J. and Fu, J. H. G. [1985] Tubular neighbourhoods in Euclidean spaces. Duke Math. J., 52(4), 1025-1045.

[34] Heintze, E. and Karcher, H. [1978] A general comparison theorem with applications to volume estimates for submanifolds. Ann. Scient. Ec. Norm Sup., 11, 451-470.

[35] Hsiung, C-C. [1981] *A first course in differential geometry.* Wiley-Interscience.

[36] Kirk, W. A. [1964] On Curvature of a Metric Space at a Point. Pacific J. Math., 14, 195-198.

[37] Klingenberg, W. [1978] *A course in differential geometry, GTM 51.* Springer.

[38] Lohkamp, J. [1998] Curvature Contents of Geometric Spaces. Doc. Math. J. Extra Volume ICM 1998, II, 381-388.

[39] Morgan, E. [1993] *Riemannian Geometry: a Beginner's Guide* , volume 68 of *CSLI Lecture Notes.* Jones and Bartlett.

[40] Munkres, J. R. [1966] *Elementary Differential Topology (rev. ed.).* Princeton University Press.

[41] Pajot, H. [2002] Analytic Capacity, Rectificabilility, Menger Curvature and the Cauchy Integral, LNM 1799. Springer-Verlag..

[42] Peltonen, K. [1992] On the existence of quasiregular mappings. Ann. Acad. Sci. Fenn., Series I Math. Dissertationes.

[43] Petrunin, A. [2003] Polyhedral approximations of Riemann manifolds Turk. J. Math., 27, 173-187.

[44] Plaut, C. [1996] Spaces of Wald-Berestowskii Curvature Bounded Below The Journal of Geometric Analysis, Vol., No. 1, 113-134.

[45] Polya, G. [1954] An elementary analogue of the Gauss-Bonnet theorem. Amer. Math. Monthly, 61, 601-603.

[46] Regge, T. [1961] General relativity without coordinates Nuovo Cimento, 19, 558-571.

[47] Robinson, C.V. [1944] A Simple Way of Computing the Gauss Curvature of a Surface. Reports of a Mathematical Colloquium, Second Series, Issue 5-6, 16-24.

[48] Ramsay, A. and Richtmayer, R. D. [1991] *Introduction to Hyperbolic Geometry, Universitext.* Spinger-Verlag, New York.

[49] Saucan, E. [2003] Surface Triangulations – the Metric Approach. Preprint.

[50] Saucan, E. [2004] A Metric Approach to Curvatures Convergence. In preparation.

[51] Saucan, E. and Appleboim, E. [2005] Curvature Based Clustering for DNA Microarray Data Analysis. Lecture Notes in Computer Science, 3523, 405-412, Springer-Verlag.

[52] Spivak, M. [1970] *A Comprehensive Introduction to Differential Geometry, Vol. 2.* Publish or Perish.

[53] Spivak, M. [1975] *A Comprehensive Introduction to Differential Geometry, Vol. 5.* Publish or Perish.

[54] Stone, D. A. [1973] Sectional curvature in piecewise linear manifolds. Bull. Am. Math. Soc., 79(5), 1060-1063.

[55] Struik, D. [1961] *Lectures in Classical Differential Geometry.* Addiso-Wesley.

[56] Tukia, P. [1985] Automorphic Quasimeromorphic Mappings for Torsionless Hyperbolic Groups. Ann. Acad. Sci. Fenn., 10, 545-560.

Emil Saucan
Department of Mathematics and Electrical Engineering Department
Technion - Israel Institute of Technology
Technion City 32000
Haifa – Israel
e-mail: semil@tx.technion.ac.il, semil@ee.technion.ac.il