Geometric Sampling of Images, Vector Quantization and Zador's Theorem

Emil Saucan (1), Eli Appleboim (2) and Yehoshua Y. Zeevi (2)

(1) Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel.
(3) Electrical Engineering Department, Technion - Israel Institute of Technology, Haifa 32000, Israel.
semil@tx.technion.ac.il, eliap@ee.technion.ac.il, zeevi@ee.technion.ac.il

Abstract:

We present several consequences of the geometric approach to image sampling and reconstruction we have previously introduced. We single out the relevance of the geometric method to the vector quantization of images and, more important, we give a concrete and candidate for the optimal embedding dimension in Zador's Thorem. An additional advantage of our approach is that that this provides a constructive proof of the aforementioned theorem, at least in the case of images. Further applications are also briefly discussed.

1. Introduction

In recent years it became common amongst the signal processing community, to consider images and other signals as well, as Riemannian manifolds embedded in higher dimensional spaces. Usually, the embedding manifold is taken to be $\mathbb{R}^n$, but other options can, and had been considered. Along with that, sampling is an essential preliminary step in processing of any continuous signal by a digital computer. This step lies at heart of any digital processing of any (presumably continuous) data/signal. It is therefore natural to strive to achieve a sampling method for images, viewed as such, that is as higher dimensional dimensional objects (i.e. manifolds), rather than their representation as 1-dimensional signals. In consequence, our sampling and reconstruction techniques stem from the the fields of differential geometry and topology, rather than being motivated by the traditional framework of harmonic analysis. More precisely, our approach to Shannon's Sampling Theorem is based on sampling the graph of the signal, considered as a manifold, rather than a sampling of the domain of the signal, as is customary in both theoretical and applied signal and image processing. In this context it is important to note that Shannon's original intuition was deeply rooted in the geometric approach, as exposed in his seminal work [14].

Our approach is based upon the following sampling theorem for differentiable manifolds that was recently presented and applied in the context image processing [12]:

Theorem 1 *Let $\Sigma^n \subset \mathbb{R}^N, n \geq 2$ be a connected, not necessarily compact, smooth manifold, with finitely many compact boundary components. Then, there exists a sampling scheme of Σ^n, with a metric density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\Big(\frac{1}{k(p)}\Big)$, where $k(p) = \max\{|k_1|, ..., |k_n|\}$, and where $k_1, ..., k_n$ are the principal curvatures of Σ^n, at the point $p \in \Sigma^n$.*

In particular, if Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density. Note, however, that this is not necessarily the optimal scheme (see [12]).

The constructive proof of this theorem is based on the existence of the so-called *fat* (or *thick*) triangulations (see [11]). The density of the vertices of the triangulation (i.e. of the sampling) is given by the inverse of the maximal principal curvature. An essential step in the construction of the said triangulations consists of isometrically embedding of Σ^n in some $\mathbb{R}^N$, for large enough N (see [10]), where the existence of such an embedding is guaranteed by Nash's Theorem ([9]). Resorting to such a powerful tool as Nash's Embedding Theorem appears to be an impediment of our method, since the provided embedding dimension N is excessively high (even after further refinements due to Gromov [4] and Günther [5]). Furthermore, even finding the precise embedding dimension (lower than the canonical N) is very difficult even for simple manifolds. However, as we shall indicate in the next section, this high embedding dimension actually becomes an advantage, at least from the viewpoint of information theory.

The resultant sampling scheme is in accord with the classical Shannon theorem, at least for the large class of (bandlimited) signals that also satisfy the condition of being $\mathcal{C}^2$ curves. In our proposed geometric approach, the radius of curvature substitutes for the condition of the Nyquist rate. To be more precise, our approach parallels, in a geometric setting, the *local bandwidth* of [7] and [16]. In other words, manifolds with bounded curvature represent a generalization of the *locally band limited signals* considered in those papers.

We concentrate here only on some of the consequences of Theorem 1. More precisely, we present, in Sections 2 and 3, two applications of our geometric sampling method and of the embedding technique employed in the proof, namely to the vector quantization of images and to determining the embedding dimension in Zador's Theorem, respectively. Further directions of study are briefly discussed in the concluding section.

2. Vector Quantization for Images

A complementary byproduct of the constructive proof of Theorem 1 is a precise method of *vector quantization* (or *block coding*). Indeed, the proof of Theorem 1 consists in the construction of a Voronoi (Dirichlet) cell complex $\{\bar{\gamma}_k^n\}$ (whose vertices will provide the sampling points). The centers a_k of the cells (satisfying a certain geometric density condition) represent, as usual, the *decision vectors*. An advantage of this approach, besides its simplicity, is entailed by the possibility to estimate the error in terms of length and angle distortion when passing from the cell complex $\{\bar{\gamma}_k^n\}$ to the Euclidean cell complex $\{\bar{c}_k^n\}$ having the same set of vertices as $\{\bar{\gamma}_k^n\}$ (see [10]). Indeed, in contrast to other related studies, our method not only produces a piecewise-flat simplicial approximation of the given manifold, it also actually renders a simplicial complex on the manifold. Moreover, one can actually compute the local distortion resulting by passing from the Euclidean geometry of the piecewise-flat approximation to the intrinsic geometry of its projection on the manifold. If $M = M^n$ is a manifold without boundary, then locally, for any triangulation patch the following inequality holds [10]:

$$\frac{3}{4}d_M(x,y) \leq d_{eucl}(\bar{x},\bar{y}) \leq \frac{5}{3}d_M(x,y)\,;$$

where d_{eucl}, d_M denote the Euclidean and intrinsic metric (on M) respectively, and where $x, y \in M$ and $\bar{x}, \bar{y}$ are their preimages on the piecewise-flat complex. For manifolds with boundary, the same estimate holds (for the $intM$ and ∂M), except for a (small) zone of "mashing" triangulations (see [11]), where the following weaker distortion formula is easily obtained:

$$\frac{3}{4}d_M(x,y) - f(\theta)\eta_\partial \leq d_{eucl}(\bar{x},\bar{y}) \leq \frac{5}{3}d_M(x,y) + f(\theta)\eta_\partial\,;$$

where $f(\theta)$ is a constant depending on the $\theta = \min\{\theta_\partial, \theta_{\mathrm{int}\,M}\}$ – the fatness of the triangulation of ∂M and int M, respectively, and η_∂ denotes the *mesh* of the triangulation of a certain neighbourhood of ∂M (see [11]). In other words, the (local) projection mapping π between the triangulated manifold M and its piecewise-flat approximation Σ is (locally) *bi-lipschitz* if M is open, but only a *quasi-isometry* (or *coarsely bi-lipschitz*) if the boundary of M is not empty.

But the main advantage of a geometric sampling of images resides in the fact that the sampling is done according to the geometric, hence intrinsic, features of the image, rather in the arbitrary (as far as features are concerned) manner of classical approach that transforms the image into a 1-dimensional array (signal). Therefore, the resulting sampling is adaptive, hence sparse in regions of low curvature, and, as shown in [1], it is even compressive in some special cases.

3. Zador's Theorem

A more important application stems, however, from Zador's Theorem [15], implying that we can turn into an advantage the inherent "curse of dimensionality". Indeed, by of Zador's Theorem, the *average mean squared error per dimension*:

$$\mathcal{E} = \frac{1}{N}\int_{\mathbb{R}^N} d_{eucl}(x,p_i)p(x)dx\,,$$

p_i being the *code point* closest to x and $p(x)$ denoting the *probability density function* of x, can be reduced by making avail of higher dimensional quantizers (see [2]). Since for embedded manifolds it obviously holds that $p(x) = p_1(x)\chi_M$, we obtain:

$$\mathcal{E} = \frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)p_1(x)dx\,,$$

It follows that, if the main issue is accuracy, not simplicity, then 1-dimensional coding algorithms (such as the classical Ziv-Lempel algorithm) perform far worse than higher dimensional ones. Of course, there exists an upper limit for the coding dimension, since otherwise one could just code the whole data as one N-dimensional vector (albeit of unpractically high dimension). The geometric coding method proposed here provides a *natural* high dimension for the quantization of M^n – the embedding dimension N. Moreover, it closes (at least for images and any other data that can be represented as Riemannian manifolds) an open problem related to Zador's Theorem: finding a constructive method to determine the dimension of the quantizers (Zador's proof is nonconstructive). In fact, for a uniformly distributed input (as manifolds, hence noiseless images, can assumed to be, at least in first approximation) a better estimate of the average mean squared error per dimension can be obtained, namely:

$$\mathcal{E} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\int_{M^n} dx} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\mathcal{V}_n(M^n)dx}\,,$$

where $\mathcal{V}_n$ denotes the n-dimensional volume (area) of M. Whence, for compact manifolds one obtains the following expression for $\mathcal{E}$:

$$\mathcal{E} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\sum_i^m \int_{V_i} dx} = \frac{\frac{1}{N}\int_{M^n} d_{eucl}(x,p_i)dx}{\sum_i^m \mathcal{V}_n(V_i)dx}\,,$$

where V_i represent the Voronoi cells of the partition.

Moreover, we have the following estimate for the *quantizer problem*, that is: Chose centers of cells such that the quantity

$$\mathcal{Q} = \frac{1}{N}\frac{\frac{1}{m}\int_{M^n} d_{eucl}(x,p_i)dx}{\left(\frac{1}{m}\sum_i^m \mathcal{V}_n\right)^{1+\frac{2}{N}}}\,.$$

is minimized. Here, again, the high embedding dimension N furnishes us with yet an additional advantage. Indeed, manifolds N increases dramatically, even for compact manifolds and even taking into consideration Gromov's and Günther's improvement of Nash's original method (see [4], resp. [5]). For instance, $n = 2$ requires embedding dimension $N = 10$ and $n = 3$ the necessitates $N = 14$. Hence, for large enough n one can write the following rough estimate:

$$\mathcal{Q} \approx \frac{1}{N} \frac{\int_{M^n} d_{eucl}(x, p_i) dx}{\sum_i^m \mathcal{V}_n}.$$

4. Conclusions and Future work

As we have stressed above, our geometrical approach to sampling lends itself to consideration of a much broader range of topics in communications, for such problems as Coding, Channel Capacity, amongst others (see [13]). In particular, and almost as an afterthought of the ideas presented in Section 2, it offers a new method for PCM (*pulse code modulation* – see [2] for a brief yet lucid presentation) of images, considered as such and not as 1-dimensional signals. This approach is endowed with an inherent advantage in that the sampling points are associated with relevant geometric features (via curvature) of the image, viewed as a manifold of dimension ≥ 2, and are not chosen via the Nyquist rate of some rather arbitrarily computed 1-dimensional signal. Moreover, the sampling is in this case adaptive and, indeed, compressive, lending itself to interesting technological benefits.

The implementation of the PCM method described above, as well as experimenting with the geometric quantization method, represent the applicative directions of study that are natural and interesting to pursue further. A better understanding of the geometry of images, included color, texture and other relevant features, in terms of curvature, represent the theoretical directions to be pursued in future. In particular, determining the lowest embedding dimension and finding global curvature constraints are, as we have seen, important for a highly compressive sampling.

5. The role of curvature

We briefly discuss here the crucial role of curvature in determining the embedding dimension (and hence the Zador dimension) by illustrating it on a "toy" example, namely that of the torus.

For a "round" torus of revolution T_r^2 in $\mathbb{R}^3$, the embedding dimension is $N = 3$, since the metric of T_r^2 is the intrinsic one induced by the Euclidian one of the ambient space $\mathbb{R}^3$, thus in this case our method does not depart too much from standard ones. However, if one considers the *flat* torus T_f^2, i.e. of Gaussian curvature $K \equiv 0$, then the minimal dimension needed for isometric embedding is $N = 4$ (see, e.g. [3]). (Before we proceed further, let us note that such tori arise naturally when considering planar rectangles with opposite sides identified – that is, "glued" – via translations. In a practical context, these would model 2-dimensional repetitive patterns on a computer screen, e.g. screen savers. Flat tori also appear in another context relevant to Computer Graphics and Image Processing, namely as solutions for discrete curvature flows (on triangular meshes), see e.g. [8].) In general, given a 2-dimensional torus, equipped with generic Riemannian metric, the whole range of dimensions, up to, and including, the one prescribed by the Nash-Gromov-Günther Theorem, is possible. There are huge differences arising not only from the sign of the curvature, but from its "speed of change" as well – for a exhaustive treatment of this subject see [6].

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