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What is Curvature?
Our main goal is to try to answer the following basic question:

What is the curvature of a surface?
No formal proofs are given and no full formula development is provided. ... Rather the interplay continuous-discrete is emphasized, ... discretization is described, ... so no complicated differentials are considered, so no discretization is described, ... rather the natural way of describing and understanding mathematical phenomena.
For \( \text{Parametrized surfaces:} \)

\[
\mathbb{R}^2 \subseteq \mathbb{R}^3 \]

\( \gamma : f \cup (\alpha, \beta) f = S \)

We can now return to our question and begin answering by

We can now return to our question and begin answering by...
where:

\[
\frac{\alpha e}{fe} \cdot \frac{\alpha e}{fe} = \mathcal{C}, \quad \frac{\alpha e}{fe} \cdot \frac{\eta e}{fe} = \mathcal{P}, \quad \frac{\eta e}{fe} \cdot \frac{\eta e}{fe} = \mathcal{E}
\]

\[
\left( \frac{z^{2} - p_{2}^{2} \mathcal{E} \mathcal{C}}{n \mathcal{F} - n^{2} \mathcal{F} \mathcal{P} \mathcal{E}} \frac{\eta e}{e} - \frac{z^{2} - p_{2}^{2} \mathcal{E} \mathcal{C}}{n \mathcal{F} - n^{2} \mathcal{F} \mathcal{P} \mathcal{E}} \frac{\alpha e}{e} \right) \frac{\mathcal{E} \mathcal{E} - p_{2}^{2}}{1}
\]

\[
- \left| \begin{array}{ccc}
\alpha e & n & \mathcal{C} \\
\alpha & n \mathcal{F} & \mathcal{P} \\
\alpha & n \mathcal{E} & \mathcal{E}
\end{array} \right| \frac{z^{2} - p_{2}^{2} \mathcal{E} \mathcal{C}}{1} - = \chi
\]
Oooops!...Let’s start from the beginning!

First define curvature for the simpler case of planar curves. A line has no curvature so define \( K \equiv 0 \).

The next simplest curve is the circle and here too an answer was known since the Ancient Greeks: \( K \equiv \frac{1}{R} \).
But, how to find it?

The curvature of the curve at the point at $c$ is the curvature of the best "fitting" circle to $c$ at $d$.

But ... how to find it?

The answer is very simple (if you happen to be Newton!...):
Simple: use the trick employed in defining the tangent to a general curve (i.e., define the tangent as the limit of secants). In this case, define the osculating circle as the limit of circles that have 3 common points with the curve.
But this idea, however nice, won’t work for surfaces, be-

...
However, the results above are far from satisfactory, because:

- There are two many directions and even more curves.
- Indeed, do these curves curvatures represent in any way the curvature of a surface? What is the curvature of a surface?
- And, more important, what is the curvature of a surface?
- There are too many directions and even more curves.
- Even if you compute all curvatures, what should one choose?
The answer (or rather answers to all the important questions) was given by Gauss, in 1827 (in a paper entitled: "Disquisitiones generales circa superficies curvas"). We shall not try and outsmart Gauss (it would the most meaningless, hubris-laden exercise in futility, anyhow!), so we shall step in his steps and start from the more basic, interesting (and fun) question: What is the curvature of a surface (at a point)?
The idea is to define curvature as a measure of a surface from being straight, or equivalently, a measure of how much a surface has to be bent in order to obtain a certain standard surface, i.e. the unit sphere $S^2$. The idea is to define curvature as a measure of a surface...
Then the Gaussian curvature of $S$ at $p$ is defined as:

\[ K_S(p) = \lim_{\text{diam}(R) \to 0} \frac{\text{Area}(A)}{\text{Area}(A')} = (d)S_Y \]
A sign is attached to $S^d$ in a natural way for a notion defined by a integral ...
While not trivial to show that this limit exists, this formula provides us with a sound definition for surfaces’ curvature, without employing anything exterior to the surface (s.a. normal planes), i.e. this definition is intrinsic to the surface.

And indeed, it can be shown that this definition is really

Gauss’ Theorema Egregium („Excellent Theorem“)
Moreover, by considering a special coordinate system in which

\[ (\mathbf{r}, \mathbf{v}) = (d) K \]

and this formula is unfortunately neither intermediate nor natural employed as the definition of the Gaussian curvature. Gauss also proved that:

\[ ((\mathbf{r}, \mathbf{v}) f, \mathbf{v}) S = S \]

which is a graph. Moreover, by considering a special coordinate system in
But what for the Geometry relevant to such fields as Computer Graphics, Image Processing, Computer Aided Geometric Design and Bio-Geometric Modelling?
For triangulated \((PL)\) surfaces, one could (following Descartes, Hilbert, Cohn-Vossen, Pólya, Banchoff,...) define \(K\) at every vertex as the defect of the sum of angles surrounding it:...
Medical Imaging: ... is in Medical Imaging.
The following question arises naturally:

Can we define curvature without angles? Even more, can we define a notion of curvature considering only distances, i.e. in (general) metric spaces, i.e. in $\mathbb{R}^n$? Can we define curvature without angles?
First let's try and define metric curvature for curves:

The first approach is the most direct: we shall mimic the osculatory circle in the metric context.

Remember that we have to do this without referring to tangency — since we should be forced to define it, too — employing distances, in exclusivity.
This approach is based upon two most familiar high school formulations for the area of the triangle of sides \(a\), \(b\), \(c\):

\[
\sqrt{\frac{4R}{a+b+c}} = S
\]

and

\[
\frac{(c-d)(q-d)(a-d)}{4R} = S
\]

Heron's formula

\[
\text{circumradius (circumradius)}
\]

where \(R\) denotes the radius of the

This approach is based upon two most familiar high school
Now the following definition seems easy and natural.*

**Definition 1 (The Menger Curvature)** Let \((M, d)\) be a metric space, and let \(p, q, r \in M\) be three distinct points. Then:

\[
K_M(p, q, r) = \frac{(d(pq) + d(qr) + d(rp))}{(d(pq) - d(qr) + d(rp)) \cdot (d(pq) + d(qr) - d(rp))}
\]

where \(pq = d(p, q)\), etc., is called the Menger Curvature of the points \(p, q, r\).
We can now define the Menger Curvature at a given point by passing to the limit. Let \((M, d)\) be a metric space and let \(p \in M\) be an accumulation point. Then \(M\) has Menger Curvature at \(p\) iff for any \(\varepsilon > 0\) there exists \(\delta > 0\) s.t. for any \(\varepsilon > 0\) there exists \(\delta > 0\) s.t.

\[
\varepsilon > \left| (d)_{M} - (d)_{M}^{p} \right| \iff \exists \delta > 0 : \forall i, j, k \in \{1, 2, 3\}, \forall (p_{i}, p_{j}, p_{k}) \in M \]

\[
|d(p_{i}, p_{j}) - d(p_{i}, p_{k})| < \delta.
\]

**Definition 2**

Let \((M, d)\) be a metric space and let \(p \in M\) be an accumulation point then \(M\) has Menger Curvature at \(p\) iff for any \(\varepsilon > 0\) there exists \(\delta > 0\) s.t.

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\[
|d(p_{i}, p_{j}) - d(p_{i}, p_{k})| < \delta.
\]
However, the very simplicity of the definition above is its own undoing: the Menger curvature is defined in an in-trinsically Euclidean manner, so it may impose an Euclidean structure upon general spaces, with possibly paradoxical re-

structure. However, the next definition doesn’t mimic closely curves in $\mathbb{R}^2$, so it better fitted for generalizations: But not everything is lost since the Menger curvature controls (Mel-nikov, 1971) — via the Cauchy integral — of all things! — the smoothness (regularity) of fractals and flatness of sets in the plane...
Definition

Let \((M, d)\) be a metric space, let \(c : I = [0, 1] \to M\) be a homeomorphism, and let \(p, q, r \in c(I)\). Denote \(c qr\) the arc of \(c(I)\) between \(q\) and \(r\), and by segment \(qr\) from \(q\) to \(r\).

Then \(c\) has Haantjes Curvature \(H(p)\) at the point \(p\), given by the intrinsic metric induced by \(d\) of \(c qr\).

\[
\frac{\xi(x,y)}{(x,y)p-\xi(x,y)} d\left(c qr\right) = (d)H(p) \quad \forall p \in I.
\]

\[\text{where } \xi(x,y) \text{ denotes the length of } c\]
Apparently, the Haantjes Curvature is a much more restricted notion than the Menger Curvature, since it applies only to rectifiable curves. However, the two definitions coincide whenever they are both applicable, as the following theorem shows:

**Theorem 3 (Haantjes)**

Let \( \gamma \) be a rectifiable arc in \((M, d)\). Let \( p \in \gamma \) and let \( \gamma_M \) and \( \gamma_H \) exist, then they are equal.

and let \( p \in \gamma \). If \( \gamma_M \) and \( \gamma_H \) exist, then they are equal.
Theorem 4

Let $\gamma \in C^1$ be a smooth curve in $\mathbb{R}^3$ and let $p \in \gamma$ be a regular point. Then the metric curvatures $\kappa_M(p)$ and $\kappa_H(p)$ exist and they both equal the classical curvature of $
abla^d \kappa_H(p)$ at $p$.
And now for some (possible) applications (at last!...)

In the view of the Theorem above, it is clear that one can

Also, an application of Menger curvature to the problem of

reconstruction of curves by a Traveling Salesman Method

(curves) and surfaces). But these cur-}

are ideally fitted for the intelligence of PL-curves Td

(values since you use an approximation of a smooth curve.

It is due to Giesen.² But one can expect obvious errors (since you use an approximation of sectional curvatures for triangulated surfaces.²) But one can expect

And now for some (possible) applications (at last!...)

But the main flaw of applying Menger or Haantjes curvatures to the study of (metric) surfaces, resides in the fact that they - as analogous of sectional curvatures - do not convey an intrinsic measure of surface curvature.

Therefore a proper notion is to be searched for...

And, once again, Gauss' idea of comparing the given surface to a model one provides the answer...

...as we have already remarked in the classical case.
However, we can’t restrict ourselves to the unit sphere $S^2$. As a gauge surface, but we shall compare the given surface to any of the complete, simply connected surface of constant curvature $\kappa$, i.e.,

Here $S^\kappa \equiv S^2_{\kappa}$ denotes the sphere of radius $R = 1/\sqrt{\kappa}$, if $\kappa > 0$.

$\mathbb{H}_2^\kappa \equiv S^\kappa$ stands for the hyperbolic plane of curvature $\kappa$ as represented by the Poincare Model of the plane disk of radius $R = 1/\sqrt{\kappa}$, if $\kappa < 0$.

$\mathbb{H}_2 \equiv \mathbb{H}_2^{0}$, if $\kappa = 0$.

and

\[ \text{disk of radius } R = \frac{1}{\sqrt{-\kappa}}, \]
We can now start towards our goal of defining an intrinsic metric curvature for surfaces. Metric figures that allow the differentiation between metric spaces, since quadruples are classically the "minimal" geo-idea, to those in a gauge surface. It is, in fact, a natural space, to comparing quadruples on the given metric space. We do this by comparing quadruples on the given metric spaces.
Definition 5. Let $(M; d)$ be a metric space, and let $Q = \{p_1, \ldots, p_4\}$ together with the mutual distances:

$$d_{ij} = d_{ji} = d(p_i; p_j), 1 \leq i, j \leq 4.$$

The set $Q$ together with the mutual distances $d_{ij}$ is called a metric quadruple.

Remark 6. One can define metric quadruples in a more abstract manner, without the aid of the ambient space: a metric quadruple being a 4-point metric space, i.e. $Q = \{p_1, \ldots, p_4\}$ such that the distances

$$d_{ij}$$

verify the axioms for a metric.
The following definition is almost obvious:

**Definition 7** The embedding curvature $\kappa (Q)$ of the metric quadruple $Q$ is defined be the curvature $\kappa$ of $S_2^k$, into which $Q$ can be isometrically embedded.
We can now define the embedding curvature at a point in a natural way by passing to the limit (but without neglecting the existence conditions), more precisely:

**Definition 8** Let \((M, d)\) be a metric space, and let \(p \in M\) be an accumulation point. Then \(p\) is said to have Wald curvature \(\kappa_W(p)\) iff

(i) \(\forall N \in N(p), N \text{ linear}^* \)

(ii) \(\forall \varepsilon > 0, \exists \delta > 0\) s.t. \(Q = \{p_1, \ldots, p_4\} \subseteq M\), and s.t. \(d(p, p_i) < \delta (i = 1, \ldots, 4) \implies |\kappa_i(Q) - \kappa_W(p)| < \varepsilon\).

The neighborhood \(N\) of \(p\) is called linear iff \(N\) is contained in a geodesic.
So the notion of **Embedding Curvature**, however interesting, may prove to be either ambiguous or even in some cases — empty!...
More precisely the following Theorem holds:

**Theorem 9 (Wald)** Let $S \subset \mathbb{R}^3$, be a smooth surface. Then $\mathcal{W}(p)$ exists, for all $p \in S$, and $\mathcal{W}(p) = \mathcal{G}(p)$, \( \forall p \in S \).

Moreover, Wald also proved that a partial reciprocal theorem holds:

**Theorem 10 (Wald)** Let $M$ be a complete and convex metric space. If $\mathcal{W}(p)$ exists, for all $p \in M$, then $M$ is a smooth surface and $\mathcal{W}(p) = \mathcal{G}(p)$, \( \forall p \in M \).

\( \therefore \text{i.e. } S \in \mathcal{C}_m, m \geq 2 \)
And now, to an application (in a non-graphic context):

**DNA Microarray Data Analysis**

We start by adapting Haantjes’ curvature to vertex weighted graphs. Let $(G, E, \eta)$ be a connected vertex weighted graph.

For all $v \in V(G)$:

**Definition 11**

Let $(G, E, \eta)$ be a connected vertex weighted graph. We define the curvature at $v$ by:

$$d(v, w) = \begin{cases} 0 = (m)\eta \quad \text{or} \quad 0 = (\alpha)\eta, \quad m \neq \alpha & 0 \\ 0 \neq (m)\eta, (\alpha)\eta, \quad m \neq \alpha & \frac{|(m)\eta(\alpha)\eta|}{|(m)\eta| + |(\alpha)\eta|} \\ 1 & \end{cases} = (m, \alpha)p$$
Remark 12

In our context it is natural to choose positive, integer weights.

Remark 13

The metric just defined may appear arbitrary, but in fact it is rather general, because of the following reasons:

1. Any family of (bounded) metric spaces \( (\mathcal{M}_i; d_i) \) admits an isometric embedding in some (bounded) metric space \( (\mathcal{M}; d) \).
2. The metrics of any finite family of metric spaces are Lipschitz equivalent.
3. One can easily "jiggle" the given metric to obtain an equivalent one by applying a function with certain properties.

\[ \text{s.a. } \langle p, \mid \ln d \rangle \]
Definition 14

Let $G = (V; E; \Delta)$ be as before, let $d$ be the metric on $G$ defined above, and let $\nu \in V$. Let $\nu = u \overset{\nu}{\longrightarrow} v$ be a path through $u$. First we define the curvature of triangles of

\[
\frac{|\{\nu \neq (u, v) \sim (u, v) \sim (u, v) \mid (u, v, u, v, v, u)\}|}{(u, v, u, v, v, u)} H^\nu_{uv} \neq (u, v) \sim (u, v) \sim (u, v)} = (d)^{H \nu}_{uv}
\]

of all the triangles with apex $\nu$.

Then the modified Haantjes curvature

\[
\begin{cases}
\nu \notin (u, v, u) = e \\
\nu \in (u, v, u) = e \left( \frac{(z \nu, \nu, v) + (v \nu, \nu, v)}{|(z \nu, \nu, v) - (z \nu, \nu, v) + (v \nu, \nu, v)|} \right) \end{cases}
\]

with vertex $\nu$ as being:

\[
(\nu, v, u, v) = \left( \frac{(\nu, v, u) p + (v, u, v) p}{|(\nu, v, u) p - (\nu, v, u) p + (v, u, v) p|} \right) = (\nu, v, u, v, u, v)
\]

Definition 14 Let $G = (V, E, \eta)$ be as before, let $d$ be the metric on $G$ defined above, and let $\nu \in V$. Let $\nu = u \overset{\nu}{\longrightarrow} v$ be a path through $u$. First we define the curvature of triangles of
In this variation of the definition, the curvature at each point depends on the curvatures at the points in a \( v \) such that...
We compare the clustering performance of our (metric)
			
curvature to that of the combinatorial curvature.

<table>
<thead>
<tr>
<th>( \frac{2}{(1-(a)d)(a)d} )</th>
<th>( {a \sim n \mid n} )</th>
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Definition: Let \( G \) be a (connected) graph and let \( a \) be a vertex of \( G \), s.t. \( a \) is defined as:

The combinatorial curvature of the combinatorial curvature of \( a \) is defined as:

\[ \text{curv}(a) = 2 - \frac{2}{(1-(a)d)(a)d} \]

Remark: One can show that \( \text{curv}(a) \geq 1 \).

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To perform clustering, one selects a curvature threshold $T_{\text{curv}} \in [0, 1]$ and selects a subgraph $H_{T_{\text{curv}}} \subseteq G$ by removing all nodes of curvature $< T_{\text{curv}}$ together with their adjacent edges.

DNA microarray data taken from http://rana.lbl.gov/EisenData.html is made into a graph by a method of correlation based “edging”. Namely, one computes the correlation between different DNA microarrays and sets an edge between them according to a (correlation) threshold. For that we used the open source Trixy (J. Rougemont and P. Hingamp).

Afterwards the obtained graph undergoes clustering according to curvature. For the metric we used gene length as vertices weights for they were shown to be relevant for the functioning of genes.
We conclude with the words of the great Dutch Geometer N.H. Kuiper.

"Real understanding in mathematics means an intuitive grasp of a fact. Therefore the urge to understand will seek satisfaction in simplicity of stated theorems, simplicity of methods and proofs, and simplicity of structures like metric, ... is influenced by this simplicity as well as by the success of the methods and tools. Thus the specific interest of a geometrically-minded mathematician, who deals with figures like curves, surfaces, with ematician, who deals with figures like curves, surfaces, with..."
In the hope that we have added a bit to your – and our – intuition of curvature, we conclude this presentation.