Geometric Wavelets for Image Processing: Metric Curvature of Wavelets

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Abstract:

We introduce a semi-discrete version of the Finsler-Haantjes metric curvature to define curvature for wavelets and show that scale and curvature play similar roles with respect to image presentation and analysis. More precisely, we show that there is an inverse relationship between local scale and local curvature in images. This allows us to use curvature as a geometrically motivated automatic scale selection in signal and image processing, this being an incipient bridging of the gap between the methods employed in Computer Graphics and Image Processing.
A natural extension to ridgelets and curvelets is also given. Further directions of study, in particular the development of a curvature transform and the study of its link with wavelet and the scale transforms are also suggested.

1. Introduction

The versatility and adaptability of wavelets for a variety of tasks in Image Processing and related fields is too well established in the scientific community, and the bibliography pertaining to it is far too extensive, to even begin to review it here.
We do, however, stress the fact that the multiresolution property of wavelets has been already applied in determining the curvature of planar curves [1] and to the intelligence and reconstruction of meshed surfaces (see, e.g. [18], [26], amongst many others). Moreover, the intimate relation between scale and differentiability in natural images has also been stressed [10].
We have presented in [24] and other related works, an extension of Shannon's Sampling Theorem when images are viewed as higher dimensional objects (i.e. manifolds), rather than 2-dimensional signals. More precisely, our approach to Shannon's Sampling Theorem is based on sampling the graph of the signal, considered as a manifold, rather than sampling of the domain of the signal, as is customary in both theoretical and applied signal and image processing, motivated by the framework of harmonic analysis. The main tool for proving our geometric sampling theorem, resides in the confluence of Differential Topology and Differential Geometry. More precisely, we consider piecewise-linear (PL) approximations of the manifold, where the geometric feature (i.e. curvature) determines the proper size and shape-ration of the simplices of the constructed triangulation.
Naturally, the question is whether the implementation of the geometric sampling scheme is feasible. We do not address here the purely geometric aspects, that would be highly relevant in Computer Graphics implementation (besides, these were partly addressed in [24]). Instead, we focus on the far more important and popular Image Processing tool of wavelets. The versatility and adaptability of wavelets to a variety of tasks in Image Processing and related fields is too well established in the scientific community, and the bibliography pertaining to it is far to extensive, to even begin to review it here.
Unfortunately, in contrast to Computer Graphics experts, for many investigators concerned with wavelets applications, piecewise-linear approximations are not necessarily among their most familiar tools. It is, therefore, a challenge to consider the integration of tools practiced by both communities. Although it may appear to be a surprising result to those primarily familiar with classical wavelets, the *Strömberg wavelets* [27], are based on piecewise-linear functions. Another, more intriguing issue is whether one can replace the intuitive trade-off between scale and curvature, by a formal concept of *wavelet curvature*, in particular in cases such as those of the Strömberg wavelets, or, in the more difficult case of Haar wavelets that are not even piecewise linear.
Interestingly enough, this can be done by using *metric curvatures* [2] (and [21] for a short presentation). It turns out that the best candidate, for the desired metric curvature is the *Finsler-Haantjes curvature*, due to its adaptability to both continuous and discrete settings.
A more suitable approach to surface reconstruction could, for example, implement *ridgelets* [5], or the more generalized, *curvelets* [6].

2. Mathematical Background

The central mathematical concept of the present paper is the following metric notion of curvature suggested by Finsler and developed by Haantjes [12]:

Definition 1 Let (M,d) be a metric space, let $c : I = [0,1] \xrightarrow{\sim} M$ be a homeomorphism, and let $p,q,r \in c(I)$, $q,r \neq p$. Denote by $\widehat{qr}$ the arc of $c(I)$ between q and r, and by qr segment from q to r. We say that c has

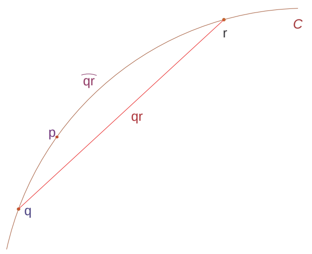

Figure 1: A metric arc and a metric segment.

Finsler-Haantjes curvature $\kappa_{FH}(p)$ at the point p iff:

$$\kappa^2_{FH}(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\big(d(q,r)\big)^3}\;; \tag{1}$$

where "$l(\widehat{qr})$" denotes the length, in intrinsic metric induced by d, of $\widehat{qr}$ – see Figure 1. (Here we assume that the arc $\widehat{qr}$ has finite length.)

Note that, while highly intuitive and definable for a very large class of curves in general rather metric spaces, this definition of curvature would remain some esoteric notion, without the following theorem (see [2]):

Theorem 2 *Let $c \in \mathcal{C}^3(I)$ be a smooth curve in $\mathbb{R}^3$, and let $p \in c$ be a regular point. Then $\kappa_{FH}(p)$ exists and, moreover, $\kappa_{FH}(p) = k(p)$ – the classical (differential) curvature of c at p.*

3. Finsler-Haantjes Curvature of Wavelets

In [23] we have introduced, in the context of both vertex and edge weighted graphs, a discretization of the Finsler-Haantjes curvature, (for applications in DNA analysis). Here we consider a semi-discrete (or semi-continuous) version, as follows:

Let φ be the typical piecewise-linear wavelet depicted in Figure 2, let $\widehat{AE}$ be the arc of curve between the points A and E, and let $d(A,E)$ is the length of the line-segment AE. Then $l(\widehat{AE}) = a+b+c+d$ and $d(A,E) = e+f$. Then $\kappa^2_{FH}(\varphi) = 24[(a+b+c+d)-(e+f)]/(a+b+c+d)^3$. Note that, in addition to the "total" curvature of φ, one can also compute the "local" curvatures at the "peaks" B and D: $\kappa^2_{FH}(B) = 24(a+c-e)/(a+b)^3$ and $\kappa^2_{FH}(D) = 24(c+d-f)/(a+b)^3$, as well as the mean curvature of these peaks: $\kappa = [\kappa_{FH}(B)+\kappa_{FH}(B)]/2$. Even if these variations may prove to be useful in certain applications, we believe that the correct approach, in the sense that it best corresponds to the scale of the wavelet, would be to compute the total curvature of φ.

Let us compare the relationship between curvature and scale, for a concrete piecewise-linear wavelet – the *Meyer wavelet* [19] – see Figure 3. The results indicating the relationship between scale and curvature, for this wavelet, can be seen in the graph in Figure 4.

However, had the definition of Finsler-Haantjes curvature been limited solely to piecewise-linear wavelets, its applicability would have also been diminished. We show,

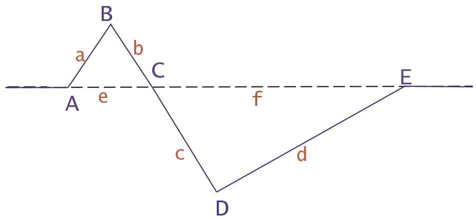

Figure 2: A piecewise-linear wavelet.

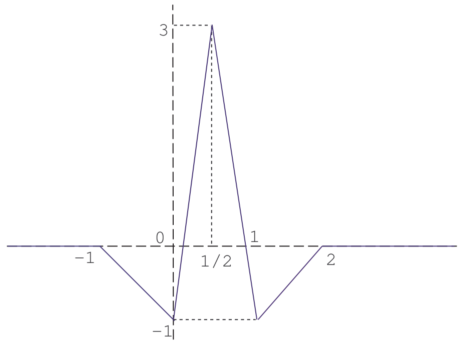

Figure 3: The Meyer wavelet.

however, that it is also definable for the "classical" Haar wavelets, in a rather straightforward manner. For example, consider the basic Haar wavelet and Haar scaling function, illustrated in Figure 5. Then for the scaling function we have: $l(\widehat{AE}) = d(A,B)+d(B,C)+d(C,D) = 3$, while $d(A,D) = 1$. Analogously, for the Haar wavelet we get: $l(\widehat{AE}) = d(M,N,)+d(N,P)+d(P,R)+d(R,S)+d(S,T) = 5$ and $d(M,T) = 1$. The expression for κ_{HF} follow easily in both cases and we present the results for the first 10 scales in Figure 6 and Figure 7, respectively. Moreover, while perhaps of lesser interest, it should be mentioned that $\kappa_{HF}(\varphi)$ can also be computed for smooth wavelets, using the classical formula for the arc-length: $l(\widehat{AE}) = \int_{\mathrm{Supp}\varphi}\sqrt{1+(\varphi')^2}\,.$

4. Ridgelets and beyond

The wavelet curvature definition introduced above is applicable, through standard methods, for image processing goals, by using separable 2-dimensional wavelets. However, while practical in many cases, this presumption contravenes to real geometric structure of images, as emphasized, for instance, in [24]. In addition, as it has already been pointed out by Candès [5], "that wavelets can efficiently represent only a small range of the full diversity of interesting behavior", since wavelets can cope well with pointlike singularities, but they are not fitted for the analysis and reconstruction of singularities of dimension greater that 0, that are distributed along lines (and more general curves), planes (and other surfaces), etc. It is therefore natural to ask whether the notion of curvature defined for wavelets can be extended to ridgelets as well.

The perhaps somewhat surprising answer is that such an extension is not only possible, it is in fact more straight-

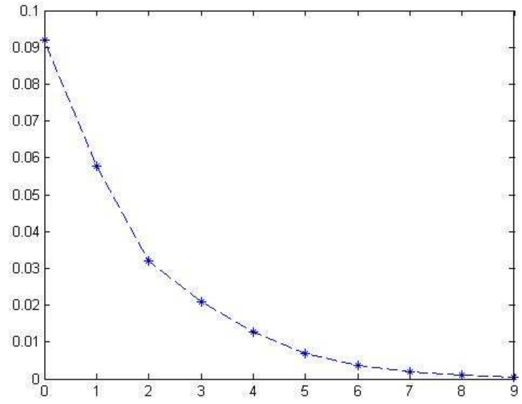

Figure 4: Curvature as a function of scale: Meyer wavelets.

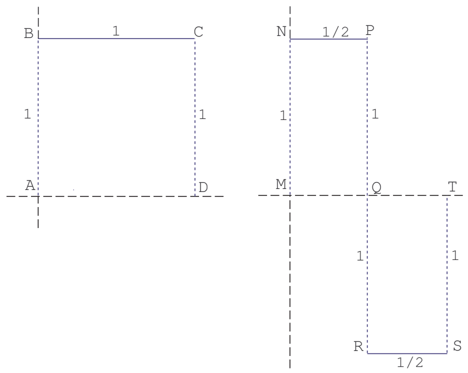

Figure 5: The Harr scaling function and wavelet.

forward and canonical. Indeed, 2-dimensional ridgelets are, in fact, piecewise $\mathcal{C}^2$ surfaces (with line singularities). For these geometrical objects an almost standard notion of curvature exists: the *principal curvatures* (i.e maximal and minimal *normal sectional curvatures* – see [8]) at any point of the surfaces. For ridgelets, we consider only the maximal absolute curvature at points on the ridges (since, along the ridge-line, curvature is 0 (cf. [8]) – see Figure 8. The sectional curvature of curves normal to the ridge is then computed using the method described in the previous section. (See also [22] for the application of the this method to piecewise-flat surfaces.)

Note that similar consideration apply with regard to curvelets (and, evidently, to nonseparable 2-dimensional wavelets as well). However, as far as curvature is concerned, there exists a basic difference between curvelets and ridgelets, which is a direct consequence of the difference between the geometric models employed. Namely, as already noted above, the principal curvature associated with the feature of interest (i.e. the ridge) vanishes. In consequence, Gaussian curvature, being the product of the principal curvatures, will also equal 0 for any point on the ridge (see Figure 8). In contrast, curvelets, being modeled on more flexible types of surfaces, can – and will – exhibit Gaussian curvatures different from 0, both positive and negative.

This geometric analysis can also be applied to shear-

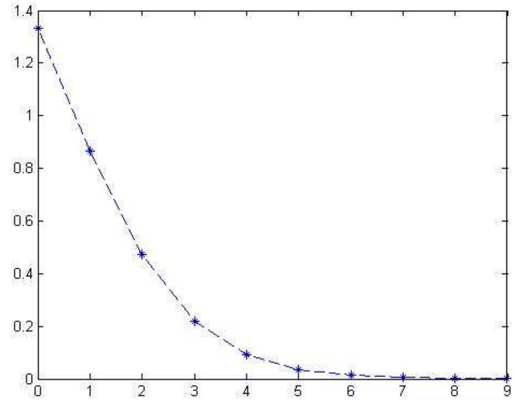

Figure 6: Curvature as a function of scale: The Haar scaling functions.

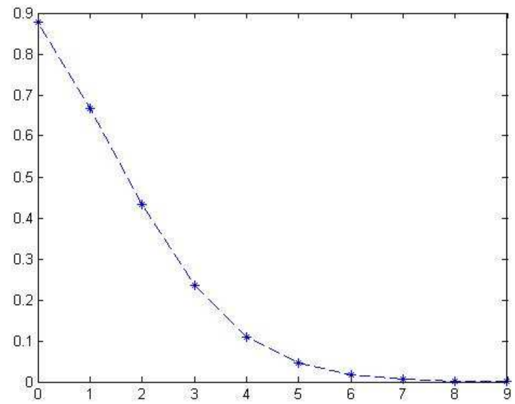

Figure 7: Curvature as a function of scale: The Haar wavelets.

lets. As Figure 9 illustrates, shearlets display "peaks" of high positive Gauss curvature. In consequence, they are ideally suited for modeling phenomena which, in geometric terms, are characterized by positive curvature concentrated at specific points. In view of this, shearlets may be viewed, in the context of our geometric approach, as a complementary tool to ridgelets. Indeed, recall that ridgelets were developed as an extension of wavelets, befitting the modeling of line-type singularities. Point type singularities can still occur in conjunction to 1-dimensional singularities (not least as noise), hence a combination of both type of tools, in a common, integrated "dictionary" is, indeed, required. The geometric approach presented above enables us to build such a "dictionary" in natural manner.

5. Future work – Theory and Applications

As we have seen, curvature can serve as a local scale estimator that is natural, i.e. intrinsic to the geometry of the image. Moreover, it can be easily calculated and used for image analysis and enhancement, especially in edge detection and texture discrimination (since in both cases curvature either large and/or exhibits a large variation). Results

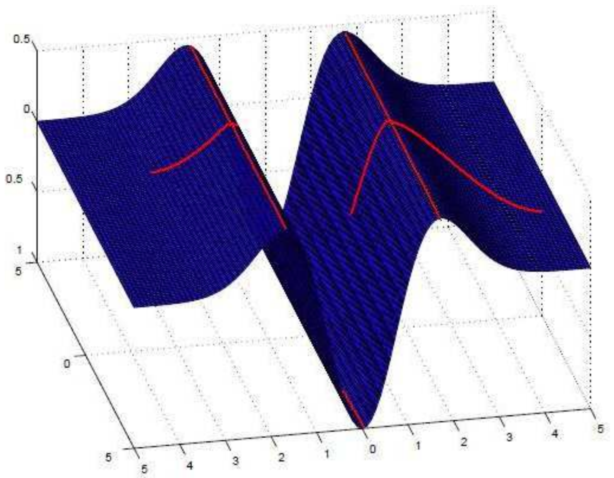

Figure 8: Lines of curvatures on a ridgelet (after [9]).

should be validated using previous work of Brox & Weickert [3] and Lindenberg [17]. It's extension to ridgelets (and curvelets) should be compared with such benchmark works as [6]. Moreover, in view of such works as [4], [15], [16] (to cite only a few), further applications to image compression also impose themselves as naturally stemming from our curvature analysis. In addition, feature extraction is also a natural application for our method, since it allows for a better correlation between the internal scale of he image (i.e. curvature) and wavelets' scale. (In fact, experiments in this direction are currently in progress.)

On the theoretical end of the spectrum, one would like to develop a full multi-curvature analysis framework, where images are constructed using basis functions that are curve-related to one another. This is not an impossible task as it seems, since, as we have already mentioned, we have shown in [24] that image sampling and reconstruction based on their curvature is possible. In fact, in the said paper, we have proven that, in the geometric approach, the *radius of curvature* (see [8]) substitutes for the condition of the Nyquist rate, even in the 1-dimensional case. Since (sectional) curvature is defined as $1/(curvature\ radius)$, the relationship between scale and curvature becomes even clearer, in the light of the results presented herein. Therefore, we aim at presenting a *curvature transform*, akin to the *wavelet transform* and to the *scale transform* of [7]. Of course, in the context of curvatures of ridgelets and curvelets one should consider the appropriate types of transforms.

We conclude with a further natural application of metric curvatures, lying at the confluence of theory and practice, namely to the fractals and their use, in conjunction with wavelets or independent of them, to image processing (see, e.g. [11], [13]). While a metric curvature – namely Menger's metric curvature (see [2], [21]) – was already applied in a purely theoretical context to fractal analysis [20], our geometric method allows for a more flexible and coherent approach, that provides a unified treatment of wavelets (including their extensions mentioned above) and fractals.

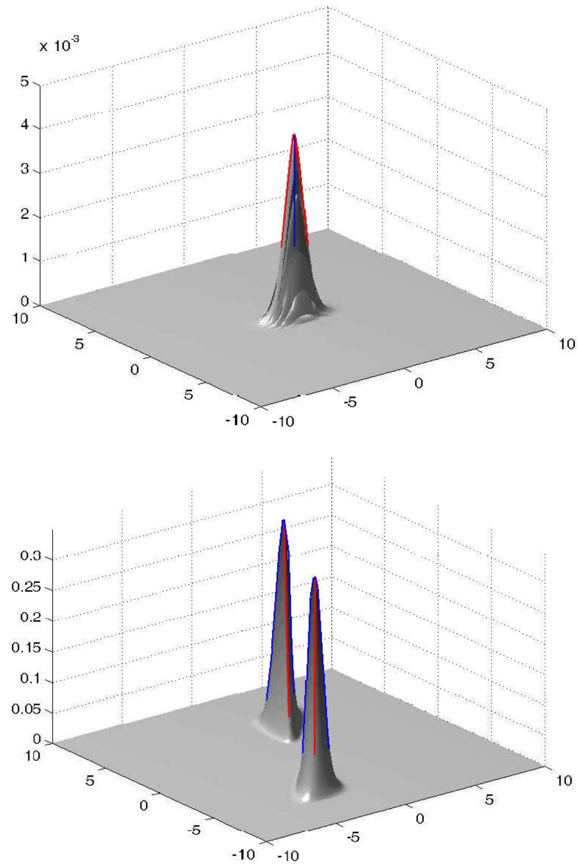

Figure 9: Lines of curvatures on shearlets (after [14]). Note the high positive curvature concentrated at the "apex".

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