SURFACE TRIANGULATION – THE METRIC APPROACH

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ABSTRACT. We consider the problem of better approximating surfaces by triangular meshes. The approximating triangulations are regarded as finite metric spaces and the approximated smooth surfaces are viewed as their Haussdorff-Gromov limit. This allows us to define in a more natural way the relevant elements, constants and invariants s.a. principal directions and Gauss curvature, etc. By a "natural way" we mean an intrinsic, discrete, metric definitions as opposed to approximating or paraphrasing the differentiable notions. Here we consider the problem of determining the Gauss curvature of a polyhedral surface, by using the *embedding curvature* in the sense of Wald and Menger. We present two modalities of employing these definitions for the computation of Gauss curvature.

CONTENTS

1. INTRODUCTION

The paramount importance of triangulations of surfaces and their ubiquity in various implementations s.a. numerous algorithms applied in robot and computer vision, computer graphics and geometric modelling, has hardly to be underlined here. Applications range from industrial ones, to biomedical engineering to cartography and astrography – to name just a few.

In consequence, determining the intrinsic proprieties of the surfaces under study, and especially computing their Gaussian curvature is essential. However Gaussian curvature is a notion that is defined for smooth surfaces only, and usually attacked

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with differential tools. A common approach for dealing with non-smooth surfaces is to use discretizations of these differential tools such as numerical schemes for first and second order derivatives. Such an approach though effective in various problem can hardly can hardly represent good approximations for curvature for PL-surfaces (i.e. sampled surfaces) which are the actual objects under study in all real life aspects.

Moreover, since considering triangulations, one is faced with finite graphs, or, in many cases (when given just the vertices of the triangulation) only with finite – thus discrete – metric spaces. Therefore, the following natural questions arise

(1) Can one find discrete, metric equivalents of the differentiable notions, notions that are intrinsically more apt to describe the properties of the finite spaces under investigations?
(2) Is one fully justified in employing discrete metric spaces when evaluating numerical invariants of continuous surfaces?

In this paper we review some fundamental studies that concerned these and other similar questions and show that the answers for them are affirmative. The outcome of this study is a fully rigorous mathematical theory of metric geometry which is, in a one sentence summery, the ability to apply differential geometry in metric spaces, the metric of whom may not be smooth. It is shown that their role is not restricted to that of being yet another discrete version of Gaussian Curvature, but permits one to attach a meaningful notion of curvature to points where the surface fails to be smooth, such as *cone points* and *critical lines*.

This metric method has already been successfully used in the such diverse fields as Geometric Group Theory, Geometric Topology and Hyperbolic Manifolds, and Geometric Measure Theory. Their relevance to Computer Graphics in particular and Applied Mathematics in general is made even more poignant by the study of *Clouds of Points* (see [18], [19]) and also in applications in Chemistry (see [27]) and Biology (see [26]).

Further in this paper, we propose to employ metric geometry in all applicative aspects mentioned above. We follow this root suggest various ways by which such employment can be done for say, surfaces with "folds", "ridges" and "facets". The idea of using metric geometry in "real life problems" is where the novelty of this paper lies.

The paper is organized as follows: in Section 2 we concentrate our efforts on the theoretical level and study the Lipschitz and Gromov-Hausdorff distances between metric spaces, and show that approximating smooth surfaces by nets and triangulations is not only permissible, but is, in a way, the natural thing to do. In particular we show that every compact surface is the Gromov-Hausdorff limit of a sequence of finite graphs. In Section 3 we introduce two metric analogues for the curvature of curves, namely the Menger, and Haantjes curvatures and study their mutual relationship. Furthermore we show how to relate to these notions as metric analogues of sectional curvature and how to employ them in the evaluation of Gauss curvature of triangulated surfaces. Further in this section we present a metric version of curvature for, not necessarily smooth, surfaces. This is the embedding, or Wald curvature. We study its proprieties and investigate the relationship between Wald and Gauss curvatures, and show that for smooth surfaces they coincide. Hence, the Wald curvature represents a legitimate discrete candidate for approximating the Gaussian curvature for triangulated surfaces. Section 4 is dedicated to developing formulas that allow the computation of Wald curvature: first the precise ones, based upon the Cayley-Menger determinants, and then we develop (after Robinson) elementary formulas that approximate well the embedding curvature. Last but not least, is Section 5 in which we present some preliminary numerical results obtained by employing metric geometry tools for sampled surfaces and approximating Gaussian curvature by metric curvatures.

2. The Haussdorff-Gromov limits

In this section we give a brief review of metric geometry and mention the most relevant, to the latter part of the paper, results and theorems exist. Some of the detailed and lengthy proofs are omitted, and in general we try to give a flavor of the theory and its techniques. However, the most essential theorems will be stated along with their proofs.

2.1. **Lipschitz Distance.** This definition is based upon a very simple idea, based upon physical measurements: It measures the *relative* difference between metrics, more precisely it evaluates their ratio; i.e.: The metric spaces (X, d_X), (Y, d_Y) are *close* iff there exists $f : X \xrightarrow{\sim} Y$ such that $\frac{d_Y(fx,fy)}{d_X(x,y)} \approx 1$, for all $x, y \in X$. Here and in the sequel "fx" stands as a short-hand version of "$f(x)$", etc.. Technically, we give the following:

Definition 2.1. The map $f : (X, d_X) \to (Y, d_Y)$ is *bi-Lipschitz* iff there exist $c, C > 0$ such that:

$$c \cdot d_X(x, y) \le d_Y(fx, fy) \le C \cdot d_X(x, y) .$$

Definition 2.2. Given a Lipschitz map $f : X \to Y$, we define the *dilatation* of f by:

$$dil\, f = \sup_{x \neq y \in X} \frac{d_Y(fx, fy)}{d_X(x, y)} .$$

Remark 2.3.
(1) The dilatation represents the *minimal* Lipschitz constant of maps between X and Y.
(2) If f is not Lipschitz, then $dil\, f = \infty$.
(3) (a) If f is Lipschitz, then f is continuous;
    (b) f is bi-Lipschitz, then f is a homeomorphism on its image.

We have the following results:

Proposition 2.4. *Let $f, g : X \to Y$ be Lipschitz maps. Then:*
(1) *$g \circ f$ is Lipschitz;*
(2) *$dil\,(g \circ f) \le dil\, f \cdot dil\, g$.*

Proposition 2.5. *The set $\{f : (X, d) \to \mathbb{R} \,|\, f Lipschitz\}$ is a vector space.*

Now we can return to our main interest and define the following notion:

Definition 2.6. Let (X, d_X), (Y, d_Y) be metric spaces. Then the *Lipschitz distance* between (X, d_X) and (Y, d_Y) is defined as:

$$d_L(X, Y) = \inf_{\substack{f:X \xrightarrow{\sim} Y \\ f\; bi-Lip.}} \log\max\,(dil\, f, dil\, f^{-1}) .$$

Remark 2.7. If there is no bi-Lipschitz map between X and Y, then we put $d_L(X,Y) = \infty$ (i.e. d_L is not suited for pairs of spaces that are not bi-Lipschitz equivalent.)

Lemma 2.8. *d_L satisfies the following conditions:*

1. *$d_L \geq 0$;*
2. *d_L is symmetric;*
3. *d_L satisfies the triangle inequality;*
   Moreover, if X, Y are compact, then:
4. *(d) $d_L(X,Y) = 0 \Leftrightarrow X \cong Y$ (i.e. X is* isometric *to Y);*
   that is d_L is a metric on the space of isometry classes of compact metric spaces

Having defined the distance between two metric spaces we now can define the *convergence* in this metric in the following natural way:

Definition 2.9. The sequence of metric spaces $\{(X_n, d_n)\}$ converge to the metric space $\{(X,d)\}$ iff

$$\lim_n d_L(X_n, X) = 0\,.$$

In this case we write: $(X_n, d_n) \underset{L}{\longrightarrow} 0$.

Example 2.10. Let S_t be a family of regular surfaces, $S_t = f_t(U)$; where $U \subseteq \mathbb{R}^2$ is an open set, such that, the family $\{f_t\}$ of parameterizations is smooth (i.e. $F : U \times \mathbb{R} \to \mathbb{R}^3 \in \mathcal{C}^1$; where $F((u,v),t) = f_t(u,v)$). Then $S_t \underset{t \to 0}{\longrightarrow} S_0$, where the convergence is in the Lipschitz metric.

Remark 2.11. If F is not smooth, only continuous, it does not necessarily hold that $S_t \underset{t \to 0}{\longrightarrow} S_0$.

Let us recall the following definition of uniform convergence,

Definition 2.12. (X_n, d_n) converge uniformly to (X,d) iff d_n converge uniformly to d *as a real function*; i.e.

$$\sup_{x,y \in X} |d_n(x,y) - d(x,y)| \underset{u}{\longrightarrow} 0\,;$$

(where "u" denotes uniform convergence.)

Then $(X_n \underset{u}{\longrightarrow} X) \;\Rightarrow\; (X_n \underset{L}{\longrightarrow} X)$ but $(X_n \underset{L}{\longrightarrow} X) \not\Rightarrow (X_n \underset{u}{\longrightarrow} X)$. However, for *finite* spaces indeed $(X_n \underset{u}{\longrightarrow} X) \;\Leftrightarrow\; (X_n \underset{L}{\longrightarrow} X)$.

2.2. **Gromov-Hausdorff distance.** This is also a distance between compact metric spaces distinguished up to isometry. However, it gives a weaker topology (In particular, it is always finite, even for pairs of non-homeomorphic spaces.

We start by first introducing the classical

2.2.1. *Hausdorff distance.*

Definition 2.13. Let $A, B \subseteq (X,d)$. We define the *Hausdorff distance* between A and B as:

$$d_H(A,B) = \inf\{r > 0 \,|\, A \subset U_r(B),\; B \subset U_r(A)\}$$

(see Fig. 1); where $U_r(A)$ is the r-neighborhood of A, $U_r(A) \stackrel{\triangle}{=} \bigcup_{a \in A} B_r(a)$.

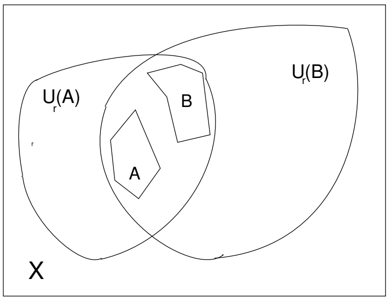

FIGURE 1

Another equivalent way of defining the Hausdorff distance is as follows:

$$d_H(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\}\,.$$

We have the following

Proposition 2.14. *Let (X,d) be a metric space. Then:*

1. *d_H is a semi-metric (on 2^X). (i.e. $A = B \Rightarrow d_H(A,B) = 0$.)*
2. *$d_H(A,\bar{A}) = 0$, for all $A \subseteq X$.*
3. *If A and B are closed then $\big(d_H(A,B) = 0 \Leftrightarrow A = B\big)$.*
   i.e. d_H is a metric on the set of closed subsets of X.

Notation We put: $\mathcal{M}(X) = \big(\{A \subseteq X \mid A = \bar{A}\}, d_H\big)$.

Lemma 2.15. (1) *Let $\{A_n\}_{n\geq 1} \subseteq X$, such that $A_n \xrightarrow[H]{} A \in \mathcal{M}(X)$. Then:*
   (a) *$A = \{\lim_n a_n \mid a_n \in A_n\,; n \geq 1\}$;*
   (b) *$A = \xrightarrow[H]{} \bigcap_{n\geq 1}\big(\overline{\bigcup_{m=n}^{\infty} A_m}\big)$.*

(2) *If X is compact and if $\{A_n\}_{n\geq 1} \subseteq X$ is a sequence of compact subsets of X, then:*
   (a) *$A_{n+1} \subseteq A_n \Rightarrow A_n \xrightarrow[H]{} \bigcap_{n\geq 1} A_n$;*
   (b) *$A_n \subseteq A_{n+1} \Rightarrow A_n \xrightarrow[H]{} \bigcup_{n\geq 1} A_n$.*

(3) *If $A_n \xrightarrow[H]{} A$, and if the sets A_n are all convex, then A is convex sets.*

We have the following two important results, which we present without their respective lengthy proofs (see [10], pp. 253-254):

Proposition 2.16. *X complete $\Rightarrow$ $\mathcal{M}(X)$ complete.*

Theorem 2.17. (Blaschke) *X compact $\Rightarrow$ $\mathcal{M}(X)$ compact.*

2.2.2. *Gromov-Hausdorff Distance.* We are now able to define the Gromov-Hausdorff distance using the following basic guide-lines: we want to get the *maximum* distance that satisfies the two conditions below:

1. $d_{GH}(A,B) \leq d_H(A,B)$, for any $A,B \subset X$ (i.e. sets that are close as subsets of X will still be close as abstract metric spaces);
2. X is isometric to Y iff $d_{GH}(X,Y) = 0$.

Definition 2.18. Let X,Y be metric spaces. Then the *Gromov-Hausdorff distance* between X and Y is defined by:

$$d_{GH}(X,Y) = \inf d_H^Z(f(X), g(Y))\,;$$

where the infimum is taken over all metric spaces Z in which both X and Y can be isometrically embedded and over all such isometric embeddings. (See Fig. 2).

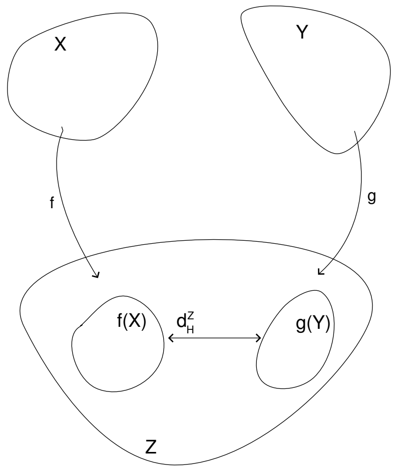

FIGURE 2

Recall the following definition:

Definition Let (X, d) be a metric space, and let $A \subset X$. A is called an *ε-net* iff $d(x, A) \leq \varepsilon$, for all $x \in X$.

Lemma 2.19. *Let Y be an ε-net in X. Then $d_{GH}(X, Y) \leq \varepsilon$.*

Proof Take $Z = X = X'$, $Y = Y'$.

□

Remark 2.20. It is sufficient to consider embeddings f into the disjoint union of the spaces X and Y, $X \coprod Y$. $X \coprod Y$ is made into a metric space by defining

$$d(x, y) = \begin{cases} inf_{z \in A \cap B}(d_A(x, z) + d_B(z, y)) & (x \in A) \wedge (y \in B) \\ \infty & A \cap B = \emptyset \end{cases}$$

Remark 2.21.
(1) X, Y bounded $\Longrightarrow d_{GH}(X, Y) < \infty$.
(2) If $diamX, diamY < \infty$, then $d_{GH}(X, Y) \geq \frac{1}{2}|diamX - diamY|$; where $diamX = \sup_{x,y \in X} d(x, y)$.

However, the straightforward definition of d_{GH} may be difficult to implement. Therefore we would like to compute d_{GH} by comparing distances in X with distances in Y, as done in the cases of uniform and Lipschitz metrics. We start by defining a *correspondence* between metric spaces: $X \longleftrightarrow Y$, given by correspondences $x \leftrightarrow y$ between points $x \in X, y \in Y$.

We shall prove that $d_{GH}(X, Y) < r$ iff there exists a correspondence $X \longleftrightarrow Y$, such that $(x \leftrightarrow y, x' \leftrightarrow y') \Longrightarrow |d_X(x, x') - d_Y(y, y')| < 2r$. Formally, we have:

Definition 2.22. Let X, Y denote sets. A *correspondence* $X \longleftrightarrow Y$ is a subset of the Cartesian product of X and Y: $\mathcal{R} \subset X \times Y$ such that

(1) for all $x \in X$, there exists $y \in Y$, such that $(x,y) \in \mathcal{R}$;
(2) For all $y \in Y$, there exists $x \in X$, such that $(x,y) \in \mathcal{R}$.

Remark 2.23. A correspondence is not necessarily a function, that is to a single "x" may correspond to several "y"-s.

Example 2.24. Any surjective function $f : X \to Y$ represents correspondence a $\mathcal{R} = \{(x, f(x))\}$.

Remark 2.25. $\mathcal{R}$ is a correspondence $\Longleftrightarrow$ there exists Z and there exist $f : Z \to X$, and $g : Z \to Y$; f, g surjective, such that $\mathcal{R} = \{(f(z), g(z)) \mid z \in Z\}$.

Definition 2.26. Let $\mathcal{R}$ be a correspondence between X and Y, where X, Y are metric spaces. We define the *distortion* of $\mathcal{R}$ by:

$$dis\,\mathcal{R} = \sup\left\{ |d_X(x,x') - d_Y(y,y')| \,\middle|\, (x,y)\,,(x',y') \in \mathcal{R}\right\}.$$

Remark 2.27. (1) If $\mathcal{R} = \{(x, f(x))\}$ is a correspondence induced by a surjective function $f : X \to Y$, then $dis\,\mathcal{R} = dis\,f$, where:

$$dis\,f = \sup_{a,b\in X} |\,d_Y(fa, fb) - d_X(a,b)\,|\,.$$

(2) If $\mathcal{R} = \{(f(z), g(z))\}$, where $f : X \to Z,\ g : Y \to Z$ are surjective functions, then:

$$dis\,\mathcal{R} = \sup_{z,z'\in Z}\{\,|\,d_X(fz, fz') - d_Y(gz, gz')\,|\,\}\,.$$

(3) $dis\,\mathcal{R} = 0$ iff $\mathcal{R}$ is associated to an isometry.

We bring, without proof, the following theorem:

Theorem 2.28. *Let X, Y be metric spaces. Then:*

$$d_{GH}(X,Y) = \frac{1}{2}\inf_{\mathcal{R}}(dis\,\mathcal{R})\,;$$

where the infimum is taken over all the correspondences $X \overset{\mathcal{R}}{\longleftrightarrow} Y$.

Before bringing the next result which is very important in determining the topology, we first introduce one more notion:

Definition 2.29. $f : X \to Y$ is called an ε-*isometry* ($\varepsilon > 0$), iff
(1) $dis\,f \le \varepsilon$;
(2) $f(x)$ is an ε-net in Y.

Note that if f is an ε-isometry, then f is continuous.

Corollary 2.30. *Let X, Y be metric spaces and let $\varepsilon > 0$. Then:*
(1) *If $d_{GH}(X,Y) < \varepsilon$, then there exists a 2ε-isometry $f : X \to Y$.*
(2) *If there exists an ε-isometry $f : X \to Y$, then $d_{GH}(X,Y) < 2\varepsilon$.*

Proof (1) Let $X \overset{\mathcal{R}}{\longleftrightarrow} Y$ such that $dis\,\mathcal{R} < 2\varepsilon$. For any $x \in X$ and $f(x) \in Y$, choose $y = f(x)$ s.t. $(x, f(x)) \in \mathcal{R}$. Then $x \mapsto f(x)$ defines a map $f : X \to Y$. Moreover: $dil\,f \le dis\,\mathcal{R} < \varepsilon$.
We shall prove that $f(X)$ is a 2ε-net in Y. Indeed, let $x \in X$ and $y \in Y$ s.t. $(x,y) \in \mathcal{R}$. Then $d(y, fx) \le d(x,x) + dis\,\mathcal{R} < 2r$, thence $d(y, f(X)) < 2r$.
□
(2) Let f be an 2ε-isometry. Define $\mathcal{R} \subset X \times Y$, $\mathcal{R} = \{(x,y) \mid d(y, fx) \le \varepsilon\}$.

Since $f(X)$ is an ε-net, it follows that $\mathcal{R}$ is a correspondence. Then, for all $(x,y),(x',y') \in \mathcal{R}$, we have: $|\, d_Y(y,y') - d_X(x,x')\,| \leq |\, d(fx,fx') - d(x,x')\,| + d(y,fx) + d(y',fx') \leq dis\, f + \varepsilon + \varepsilon \leq 3\varepsilon$. It follows that $dis\,\mathcal{R} \leq 3\varepsilon \implies d_{GH}(x,y) \leq 3r/2 < 2r$.
□

The next result is of great importance in particular in our context:

Theorem 2.31. *d_{GH} is a finite metric on the set of isometry classes of compact metric spaces.*

Proof It suffices to prove that if $d_{GH}(X,Y) = 0$, then X is isometric to Y. Indeed, let X, Y be compact spaces such that $d_{GH} = 0$. Then it follows from the previous corollary (for $\varepsilon = 1/n$) that there exists $(f_n)_{n \geq 1}$, $f_n : X \to Y$ such that $dis\, f_n \underset{n}{\longrightarrow} 0$.

Let $S \subset X$, $S = \bar{S}$, $|S| = \aleph_0$. Using a Cantor-diagonal argument one easily shows that there exists $(f_{n_k})_{k \geq 1} \subset (f_n)_{n \geq 1}$ such that $(f_{n_k})_{\geq 1}$ converges in Y, for all $x \in S$. Without loss of generality, we may assume that this happens for $(f_n)_{n \geq 1}$ itself. Thus we can define a function $f : X \to Y$ by putting: $f(x) = \lim_n f_n(x)$.

But $|\, d(f_n x, f_n y) - d(x,y)\,| \leq dis\, f_n \underset{n}{\longrightarrow} 0 \implies d(fx,fy) = \lim d(f_n x, f_n y)$. In other words $f|_S$ is an isometry. But $S = \bar{S}$, therefore this isometry can be extended to an isometry $\tilde{f}$ from X to Y. In a analogous manner one shows the existence of an isometry $\tilde{\tilde{f}} : X \to Y$.

□

Remark 2.32. $(X_n \underset{u}{\longrightarrow} X) \implies (X_n \underset{L}{\longrightarrow} X) \implies (X_n \underset{GH}{\longrightarrow} X)$.

In fact, the following relationship exists between "$\underset{L}{\longrightarrow}$" and "$\underset{GH}{\longrightarrow}$":

Theorem 2.33. $(X_n \underset{GH}{\longrightarrow} X)$ *iff* *ε-nets in $X_n \underset{L}{\longrightarrow}$ ε-nets in X.*

One can formulate this assertion in a more formal manner and prove it directly (see [?], p . 73). However we shall proceed in more "delicate" manner, starting with:

Definition 2.34. Let X, Y be compact metric spaces, and let $\varepsilon, \delta > 0$. X, Y are called *ε-δ-approximations* (of each-other) iff: there exist sequences $\{x_i\}_{i=1}^N \subset X$ and $\{y_i\}_{i=1}^N \subset Y$ such that

(1) $\{x_i\}_{i=1}^N$ is an ε-net in X and $\{y_i\}_{i=1}^N$ is an ε-net in Y;
(2) $|\, d_X(x_i,x_j) - d_(y_i,y_j)\,| < \delta$ for all $i,j \in \{1,...,N\}$.

An $(\varepsilon,\varepsilon)$-approximation is called, for short: an *ε-approximation.*

The relationship between this last definition and the Gromov-Hausdorff distance is first revealed in

Proposition 2.35. *Let X, Y be compact metric spaces. Then:*

(1) *If Y is a (ε,δ)-approximation of X, then $d_{GH}(X,Y) < 2\varepsilon + \delta$.*
(2) *If $d_{GH}(X,Y) < \varepsilon$, then Y is a 5ε-approximation of X.*

A more accessible yet equivalent proposition is the following:

Proposition 2.36. *Let $X, \{X_n\}_1^\infty$ compact metric spaces. Then: $X_n \underset{GH}{\longrightarrow} X$ iff for all $\varepsilon > 0$, there exist a*finite *ε-nets $S \subset X$ and $S_n \subset X_n$, such that $S_n \underset{GH}{\longrightarrow} S$ and, moreover, $|S_n| = |S|$, for large enough n.*

Remark 2.37. The proposition above can be summarized as the convergence of geometric properties of S_n to those of S, as $X_n \underset{GH}{\longrightarrow} X$. This fact represents the reward of the laborious technical steps above.

We will review the convergence of geometric properties via a typical example. As such we consider the *intrinsic metric* i.e. the metric induced by a length structure (i.e. path length) by a metric on a subset of a given metric space. (See Fig. 3 for the classical example of surfaces in $\mathbb{R}^3$.)

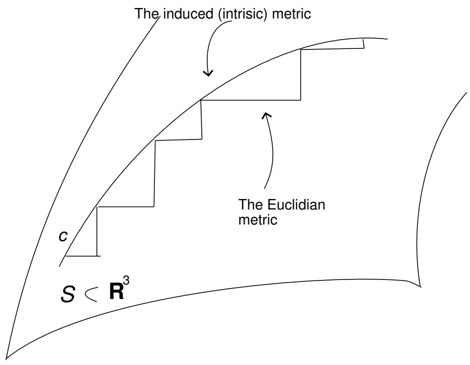

FIGURE 3

On a more formal note, we have the following characterization of intrinsic metrics:

Theorem 2.38. *Let* (X, d) *be a* complete *metric space.*

(1) *If* $\frac{1}{2}xy$ *exists, for any* $x, y \in X$*, then* d *is strictly intrinsic.*
(2) *If the* ε*-middle of* xy *exists, for any* $x, y \in X$*, and for any* $\varepsilon > 0$*, then* d *is intrinsic.*

Here we used the following definitions and notations:

Definition 2.39. (1) Given x, y points in (X, d), *the middle* (or *midpoint*) of the *segment* xy (more correctly: 'a midpoint between "x" and "y" ') is defined as: $\frac{1}{2}xy = z\,,\; d(x,z) = d(z,y)$.
(2) d is called *strictly intrinsic* iff the length structure is associated with is complete.
(3) Let d be an intrinsic metric. z is an ε*-middle* (or an ε*-midpoint*) for xy iff: $|\,2d(x,z) - d(x,y)\,| \leq \varepsilon$ and $|\,2d(y,z) - d(x,y)\,| \leq \varepsilon$.

Remark 2.40. The converse of Theorem 2.38. holds in any metric space, more precisely we have:

Proposition 2.41. *If* d *is an intrinsic metric, then* $\frac{1}{2}xy$ *exists, for any* x, y*.*

The following theorem shows that length spaces are closed in the *GH-topology*:

Theorem 2.42. *Let* $\{X_n\}$ *be length spaces and let* X *be a complete metric space such that* $X_n \underset{GH}{\longrightarrow} X$*. Then* X *is a length space.*

Proof We have already presented the idea of the proof: it is sufficient to show that for every $x, y \in X$ there exist an ε-*midpoint*, for any $\varepsilon > 0$. Indeed, let n be such that $d_{GH} < \frac{\varepsilon}{10}$. Then, from the a preceding result, it follows that there exist a correspondence $X_n \overset{\mathcal{R}}{\longleftrightarrow} X$ such that $dis\,\mathcal{R} < \frac{\varepsilon}{5}$.

Let $\bar{x}, \bar{y} \in X_n$, $x \overset{\mathcal{R}}{\leftrightarrow} \bar{x}$, $y \overset{\mathcal{R}}{\leftrightarrow} \bar{y}$. Since X_n is a length space, it follows that there exists $\bar{z} \in X_n$, such that $\bar{z}$ is a $\frac{\varepsilon}{5}$-midpoint of $x_n y_n$. Consider $z \in X$, $z \overset{\mathcal{R}}{\leftrightarrow} \bar{z}$. Then: $\left|d_X(x,z) - \frac{1}{2}d_X(x,y)\right| \leq \left|d_n(\bar{x},\bar{z}) - \frac{1}{2}d_n(\bar{x},\bar{y})\right| + 2dis\,\mathcal{R} <, \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} < \varepsilon$. (Here d_n, d_X denote the metric on X_n, X, respectively.)

In a similar manner we show that: $\left|d(y,z) - \frac{1}{2}d(x,y)\right| < \varepsilon$; i.e. z is a ε-*midpoint* of xy.

□

The next theorem and its corollary are of paramount importance:

Theorem 2.43 (Gromov)**.** *Any compact length space is the GH-limit of a sequence of finite graphs.*

Proof Let ε, δ ($\delta \ll \varepsilon$) small enough, and let S be a δ-net in X. Also, let $G = (V, E)$ be the graph with $V = S$ and $E = \{(x,y) \,|\, d(x,y) < \varepsilon\}$. We shall prove that G is an ε-approximation of X, for δ small enough (for $\delta < \frac{\varepsilon^2}{4}\, diam(X)$, to be more precise). (See Fig. 4.)

But, since S is an ε-net both in X and in G, and since $d_G(x,y) \geq d_X(x,y)$, it is sufficient to prove that:

$$d_G(x,y) \leq d_X(x,y) + \varepsilon\,.$$

Let γ be the shortest path between x and y, and let $x_1, \ldots, x_n \in \gamma$, such that $n \leq length(\gamma)/\varepsilon$ (and $d_X(x_i, x_{i+1}) \leq \varepsilon/2$). Since for any x_i there exists $y_i \in S$, such that $d_X(x_i, y_i) \leq \delta$, it follows that $d_X(y_i, y_{i+1}) \leq d_X(x_i, x_{i+1}) + 2\delta < \varepsilon$. (See Fig. 4.)

Therefore, (for $\delta < \varepsilon/4$), there exists an edge $e \in G, e = y_i y_{i+1}$. From this we get the following upper bound for $d_G(x,y)$:

$$d_G(x,y) \leq \Sigma_0^n d_X(y_i, y_{i+1}) \leq \Sigma_0^n d_X(x_i, x_{i+1}) + 2\delta n$$

But $n < 2length(\gamma)/\varepsilon \leq 2diam(X)/\varepsilon$. Moreover: $\delta < \varepsilon^2/4diam(X)$. It follows that:

$$d_G(x,y) \leq d_X(x,y) + \delta\frac{4diam(X)}{\varepsilon} < d_X(x,y) + \varepsilon\,.$$

Thus, for any $\varepsilon > 0$, there exists an ε-approximation of X, $G = G_\varepsilon$. Hence $G_n \underset{\varepsilon}{\rightarrow} X$.

□

Corollary 2.44. *Let X be a compact length space. Then X is the Gromov-Hausdorff limit of a sequence $\{G_n\}_{n\geq 1}$ of finite graphs, isometrically embedded in X.*

Remark 2.45.

(1) If $G_n \underset{\varepsilon}{\rightarrow} X$, $G_n = (V_n, E_n)$. If there exists $N_0 \in \mathbb{N}$ such that

$$(\star) \quad |E_n| \leq N_0, \text{ for all } n \in \mathbb{N}\,,$$

then X is a finite graph.

(2) If condition $(\star)$ is replaced by:

$$(\star\star) \quad |V_n| \leq N_0, \text{ for all } n \in \mathbb{N}\,,$$

then X will still be always a graph, but not necessarily finite.

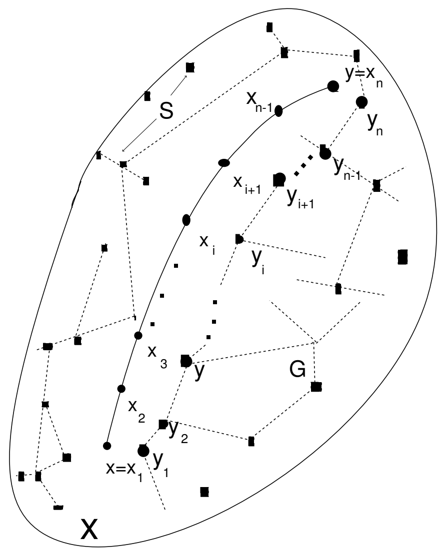

FIGURE 4

As seen by this example indeed, geometric properties of metric spaces are inherited by their Gromov-Hausdorff limits. Thus, we can use the Gromov-Hausdorff limit each and every time the geometric properties of X_n can be expressed in term of a finite number of points, and, by passing to the limit, automatically obtain proprieties of X. This is essentially the affirmative answer for Question 1.

In the next section we will consider Question 2 and discuss the ability of efficient approximation of some geometric properties of a smooth surface by those of a sequence of sampled surfaces, where the smooth surface is considered as the Gromov-Hausdorff limit of the sampled surfaces. The property we will focus on is curvature. Moreover, in view of the preceding theorem and corollary, triangulations are viewed as the finite graphs given by their 1-skeleta.

3. Metric Curvatures

In the light of the discussion at the end of the previous section, we focus in this section on a number of metric versions of curvature for rather general metric spaces. We begin by introducing metric analogues for the curvature of plane curves and in the sequel some metric definitions for Gauss curvature are considered.

3.1. **Menger and Haantjes Curvatures.** The *Menger* and *Haantjes* curvatures are metric definitions of curvature for curves.

We begin by introducing the most elementary of them, the *Menger* curvature: this is a metric expression for the circumradius of a triangle – thus giving in the limit a metric definition of the osculatory circle – and it is based upon some elementary high-school formulas:

Definition 3.1. *Let (M, d) be a metric space, and let $p, q, r \in M$ be three distinct points. Then:*

$$K_M(p,q,r) = \frac{\sqrt{(pq+qr+rp)(pq+qr-rp)(pq-qr+rp)(-pq+qr+rp)}}{pq \cdot qr \cdot rp}\ ;$$

is called the Menger Curvature *of the points p, q, r. Throughout this section the distance between the points p, q is denoted by pq, etc.*

We can now define the Menger curvature at a given point by passing to the limit:

Definition 3.2. *Let (M,d) be a metric space and let $p \in M$ be an accumulation point. We say that M has at p* Menger curvature *$\kappa_M(p)$ iff for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any triple of points p_1, p_2, p_3, satisfying $d(p, p_i) < \delta$, $i = 1, 2, 3$; the following inequality holds: $|K_M(p_1, p_2, p_3) - \kappa_M(p)| < \varepsilon$.*

Applications of Menger curvature include, most notably, estimates (obtained via the Cauchy integral) for the regularity of fractals and the flatness of sets in the plane (see [20]). Also, it was employed, in a more practical implementation, for curve reconstruction (see [13]).

Remark 3.3. *The apparent equivalent notion of* Alt *curvature, $\kappa_A(p)$, in which one uses only two points converging to the third, is in fact a more general notion (see* [5], [7]*).*

Since both $\kappa_M(p)$ and $\kappa_A(p)$ are modelled after a specific geometric property of the Euclidean plane, they convey this Euclidean type of curvature upon the space they are defined on. Therefore they are unsuited for the geometrization of generic metric spaces. There exists another notion of curvature that does not closely mimic $\mathbb{R}^2$, therefore is better fitted for generalizations (e.g. for the metrization of graphs – see [26]):

Definition 3.4. *Let (M,d) be a metric space and let $c : I = [0, 1] \xrightarrow{\sim} M$ be a homeomorphism, and let $p, q, r \in c(I)$, $q, r \neq p$. Denote by $\widehat{qr}$ the arc of $c(I)$ between q and r, and by qr segment from q to r.*

Then c has Haantjes curvature *$\kappa_H(p)$ at the point p iff:*

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\big(l(\widehat{qr})\big)^3}\ ;$$

where "$l(\widehat{qr})$" denotes the length – in intrinsic metric induced by d – of $\widehat{qr}$.

Remark 3.5. *It should be noted that κ_H exists only for rectifiable curves, but if κ_M exists at any point p of c, then c is rectifiable. Also, the existence of κ_M implies that of κ_A, while the existence of κ_A does not automatically imply that of κ_M (see* [5]*, p. 76). However, one can prove the following theorem (see* [7]*, Theorem 10.2):*

Theorem 3.6. Let $c : I \to M$ be a rectifiable curve, and let $p \in M$. If $\kappa_A(p)$ (or $\kappa_M(p)$) exists, then $\kappa_H(p)$ exists and $\kappa_A(p) = \kappa_H(p)$.

Of course, both Menger and Haantjes curvatures (as well as the related Alt curvature) can be employed – as approximations to sectional curvatures – of triangulated surfaces. In Section 5 we include some preliminary results in this direction.

3.2. **Wald Curvature.** In the light of the results above, it is evident that one inherits all the theoretical drawbacks of this approach in the classical case of smooth surfaces and also is prone to additional numerical errors.

A more powerful approach stems from Gauss' original method of comparing surface curvature to a standard, model surface (i.e. the unit sphere in $\mathbb{R}^3$). Wald's idea was to use more types of gauge surfaces and to restrict oneself to the study of the minimal geometric figure that allows this comparison.

Definition 3.7. *Let (M,d) be a metric space, and let $Q = \{p_1,...,p_4\} \subset M$, together with the mutual distances: $d_{ij} = d_{ji} = d(p_i,p_j)$; $1 \leq i,j \leq 4$. The set Q together with the set of distances $\{d_{ij}\}_{1\leq i,j\leq 4}$ is called a* metric quadruple.

Remark 3.8. *One can define metric quadruples in slightly more abstract manner, without the aid of the ambient space: a metric quadruple being a 4 point metric space; i.e. $Q = \big(\{p_1,...,p_4\},\{d_{ij}\}\big)$, where the distances d_{ij} verify the axioms for a metric.*

Before passing to the next definition, let us introduce the following notation: S_κ denotes *the complete, simply connected surface of constant Gauss curvature (or space form) κ*, i.e. $S_\kappa \equiv \mathbb{R}^2$, if $\kappa = 0$; $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$, if $\kappa > 0$; and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$, if $\kappa < 0$. Here $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$ denotes the Sphere of radius $R = 1/\sqrt{\kappa}$, and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$ stands for the Hyperbolic plane of curvature $\sqrt{-\kappa}$, as represented by the Poincaré model of the plane disk of radius $R = 1/\sqrt{-\kappa}$.

Definition 3.9. *The* embedding curvature $\kappa(Q)$ *of the metric quadruple Q is defined to be the curvature κ of the gauge surface S_κ into which Q can be isometrically embedded. (See Figure 3 for embeddings of a metric quadruple in $\mathbb{S}^2_{\sqrt{\kappa}}$ and $\mathbb{H}^2_{\sqrt{-\kappa}}$, respectively.)*

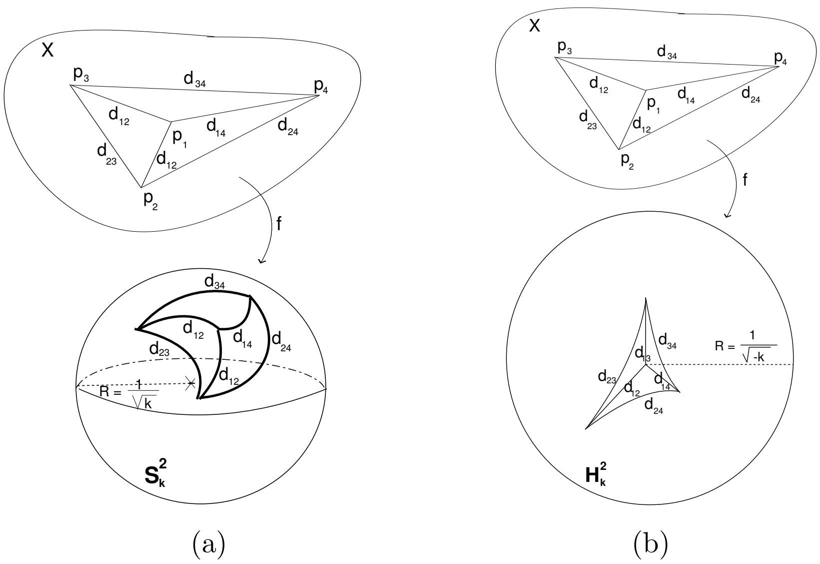

FIGURE 5. Embedding of a Metric Quadruple in (a) $\mathbb{S}^2_{\sqrt{\kappa}}$ and (b) $\mathbb{H}^2_{\sqrt{\kappa}}$

The embedding curvature at a point can now be defined in a natural way as a limit:

Definition 3.10. (M,d) *be a metric space, let* $p \in M$ *and let* N *be a neighbourhood of* p*. Then* N *is called linear iff* N *is contained in a geodesic curve.*

Definition 3.11. *Let* (M,d) *be a metric space, and let* $p \in M$ *be an accumulation point. Then* M *has* Wald curvature $\kappa_W(p)$ *at the point* p *iff*

(1) *Every neighbourhood of* p *is non-linear;*
(2) *For any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $Q = \{p_1,...,p_4\} \subset M$ *and if* $d(p,p_i) < \delta\,, i = 1,...,4$*; then* $|\kappa(Q) - \kappa_W(p)| < \varepsilon$*.*

Note that if one uses the second (abstract) definition of the metric curvature of quadruples, then the very existence of $\kappa(Q)$ is not assured, as it is shown by the following counterexample (see [7]):

Counterexample 3.12. The metric quadruple of lengths $d_{12} = d_{13} = d_{14} = 1,\ d_{23} = d_{24} = d_{34} = 2$ admits no embedding curvature.

Moreover, even if a quadruple has an embedding curvature, it still may be not unique (even if Q is not linear), indeed, one can study the following examples:

Example 3.13. (a) The quadruple Q of distances $d_{ij} = \pi/2, 1 \leq i < j \leq 4$ is isometrically embeddable both in $S_0 = \mathbb{R}^2$ and in $S_1 = \mathbb{S}^2$.

(b) The quadruple Q of distances $d_{13} = d_{14} = d_{23} = d_{24} = \pi,\ d_{12} = d_{34} = 3\pi/2$ admits exactly two embedding curvatures: $\kappa_1 \in (1.5, 2)$ and $\kappa_2 = 3$.

However, for "good" metric spaces (i.e. spaces that are locally "plane like") the embedding curvature exists and it is unique. Moreover, this embedding curvature coincides with the classical Gaussian curvature. The proof of this result is rather involved, hence we shall present here only a brief sketch of it.

We start by introducing the following definition:

Definition 3.14. A metric quadruple $Q = Q(p_1,p_2,p_3,p_4)$, of distances $d_{ij} = dist(p_i,p_j),\ i = 1,...,4$, is called *semi-dependent* (or a *sd-quad*, for brevity), iff 3 of its points are on a common geodesic, i.e. there exist 3 indices – e.g. 1,2,3 – such that: $d_{12} + d_{23} = d_{13}$.

For sd-quads the uniqueness of the embedding curvature is assured:

Proposition 3.15. *An sd-quad admits at most one embedding curvature.*

Proof. See Proof of Theorem 4.4 or [28] for the original proof. □

It turns out that the class of quadruples for which one can ascertain the uniqueness of the embedding curvature it is not confined to that of sd-quads:

Definition 3.16. *Let* $Q = \{p,q,r,s\}$ *be a non-linear and non-degenerate quadruple.* Q *is called* planar *iff* $\measuredangle(q,p,r) + \measuredangle(q,p,s) + \measuredangle(s,p,r) = 2\pi$*.*

Proposition 3.17. Let $Q = \{p,q,r,s\}$ be a non-linear and non-degenerate quadruple in $\mathcal{S}_\kappa$. Then

(1) If Q is planar, then it admits no isometric embedding in $\mathcal{S}_{\kappa_1},\ \kappa_1 > \kappa$.
(2) If Q is not planar, then it admits no isometric embedding in $\mathcal{S}_{\kappa_2},\ \kappa_2 < \kappa$.

Corollary 3.18. Let $Q = \{p,q,r,s\}$ be a non-linear and non-degenerate quadruple. Then Q has at most two different embedding curvatures.

In fact, a classification theorem for embedding curvature possibilities(??) can be given (cf. Berestkovskii [1], – see also [22], Theorem 18). Before enouncing this theorem we need an additional definition:

Definition 3.19. Let M, Q be as above. Then one and only one of the following assertion holds:

(1) Q is linear.
(2) Q has exactly one embedding curvature.
(3) Q can be isometrically embedded in some $\mathcal{S}^m_\kappa$, $m \geq 2$; where $\kappa \in [\kappa_1, \kappa_2]$ or $(-\infty, \kappa_0]$, where $S^m_\kappa \equiv \mathbb{R}^m$, if $\kappa = 0$; $S^m_\kappa \equiv \mathbb{S}^m_{\sqrt{\kappa}}$, if $\kappa > 0$; and $S^m_\kappa \equiv \mathbb{H}^m_{\sqrt{-\kappa}}$, if $\kappa < 0$. (Here $S^m_\kappa \equiv \mathbb{S}^m_{\sqrt{\kappa}}$ denotes the m-dimensional sphere of radius $R = 1/\sqrt{\kappa}$, and $S^m_\kappa \equiv \mathbb{H}^m_{\sqrt{-\kappa}}$ stands for the m-dimensional hyperbolic space of curvature $\sqrt{-\kappa}$, as represented by the Poincaré model of the ball of radius $R = 1/\sqrt{-\kappa}$) .
   Moreover, $m = 2$ iff $\kappa \in \{\kappa_0, \kappa_1, \kappa_2\}$.
(4) There exist no m and k such that Q can be isometrically embedded in $\mathcal{S}^m_\kappa$.

3.3. **Wald and Gauss Curvatures Comparison.** The discussion above would have little relevance in Differential Geometry in general and for the problem of approximating curvatures of triangulated surfaces, in particular, where it not for the fact that the metric (Wald) and the classical (Gauss) curvatures coincide whenever the second notion makes sense, that is for smooth (i.e. of class $\geq \mathcal{C}^2$) surfaces in $\mathbb{R}^3$. More precisely the following theorem holds:

Theorem 3.20 (Wald [28])**.** Let $S \subset \mathbb{R}^3$, $S \in \mathcal{C}^m$, $m \geq 2$ be a smooth surface. Then, given $p \in S$, $\kappa_W(p)$ exists and $\kappa_W(p) = \kappa_G(p)$.

Moreover, Wald also proved the following partial reciprocal theorem:

Theorem 3.21. Let M be a compact and convex metric space. If $\kappa_W(p)$ exists, for all $p \in M$, then M is a smooth surface and $\kappa_W(p) = \kappa_G(p)$, for all $p \in M$.

Note that if one tries to restrict oneself, in the building of Definition 3.11 only to sd-quads, then Theorem 3.21 holds only if the following presumption is added:

Condition 3.22. *M is locally homeomorphic to $\mathbb{R}^2$.*

Unfortunately, the proof of this facts is laborious and, as such, beyond the scope of this paper. Therefore we shall restrict ourselves to a succinct description of the principal steps of the proofs. The main idea is to show that if a metric M space admits a Wald curvature at any point, than M is locally homeomorphic to $\mathbb{R}^2$, thus any metric proprieties of $\mathbb{R}^2$ can be translated to M, (in particular the first fundamental form). The first of these partial results is:

Theorem 3.23. Let M be a convex metric space. Then M admits at most one Wald curvature $\kappa_W(p)$, for any $p \in M$.

Proof. By Corollary 3.15, suffice to prove that any disk neighborhood $B(p\,;\rho) \in \mathcal{N}(p)$ contains a non degenerate sd-quad. Without loss of generality one can assume that $B(p\,;\rho)$ contains three points p_1, p_2, p_3 such that $d(p, p_i) < \rho/2$, $i = 1, 2, 3$ (see [5]). Then, by the convexity of M it follows that there exists $q \in M$ such that $p \neq p_2, p_3$ and $p_2q + p_3q = p_2p_3$. But $p_2p_3 \leq pp_2 + pp_3 < \rho$ implies that $pq < \rho/2$ or $pp_2 < \rho/2$. If the first inequality holds, then $pq \leq pp_2 + p_2q < \rho$, i.e $q \in B(p\,;\rho)$;

and if the second one holds, then $pd \leq pp_3 + p_3q < \rho$, i.e. $q \in B(p\,;\rho)$. But $p \neq q$, therefore p, p_2, p_3, q are not linear. $\square$

Next we analyze those neighborhoods that display "a normal behavior", both from a metric and curvature viewpoint: i.e. precisely those disk neighborhoods on which the Wald curvature is defined and ranges over a small, bounded set of values determined by the radius of the disk:

Definition 3.24. A disk neighborhood $B(p\,;\rho); \rho > 0$ is called *regular* iff for any non-degenerate quadruple $Q \subset B(p\,;\rho)$, $\kappa_W(Q)$ exists and $|\kappa_W(Q)| < \pi^2/16\rho^2$.

Remark 3.25. If $\kappa_W(p)$ exists, then $B(p\,;\rho)$ is regular for any sufficiently small ρ.

It turns out that regular neighborhoods, in *compact, convex* spaces have the following "nice" (i.e. real plane like) proprieties:

Proposition 3.26. *Let M be a compact, convex metric space and let $B(p\,;\rho) \subset M$ be a regular neighborhood. Then:*

1. *If a non-degenerate quadruple $Q \subset B(p\,;\rho)$ contains two linear triples of points, then Q is linear.*
2. *There exist $q, r \in B(p\,;\rho)$ such that p, q, r are not linear.*
3. *$B(p\,;\rho)$ is* strictly convex*, i.e. $q, r \in B(p\,;\rho) \Longrightarrow int(qr) \subset B(p\,;\rho)$.*

We restrict ourselves to the proof of the following corollary:

Corollary 3.27. Let $B(p\,;\rho)$ be a regular neighborhood. Then, for any $q, r \in B(p\,;\rho)$, the geodesic segment qr exists and $int(qr) \subset B(p\,;\rho)$.

Proof. By the convexity of $B(p\,;\rho)$ it follows the existence of at least one geodesic qr, for all $q, r \in B(p\,;\rho)$. If $s \in int(q)r$, then by the proposition above we have that $s \in B(p\,;\rho)$. It follows that $B(p\,;\rho)$ contains all the geodesics with end points q, r. Hence, by Proposition 3.26, the geodesic segment qr is unique. $\square$

We can now begin to prove that a compact, convex metric space is locally homeomorphic to $\mathbb{R}^2$. The classical method of proof is to introduce polar coordinates on regular neighbourhoods, in the same way geodesic polar coordinates are defined on classical surfaces (see, e.g., [5], [25]).

One starts by showing that the *sinus* function is defined on M, thus allowing for angle measure, as follows: First, let M be as before, and let $p \in M$ such that $\kappa_W(p)$ exists. Let $q, r \in B(p\,;\rho), q \neq p \neq r$, where $B(p\,;\rho)$ is a regular neighborhood of p. Then, for any $x \in [0, min\{pq, pr\})$, define $q(x) \in pq,\ r(x) \in pr$ by: $d(p, q(x)) = x = d(p, r(x))$, and let $d(x) = d(q(x), r(x))$ (see Figure 6 bellow).

The next step is to prove the result below:

Proposition 3.28. The following limit exists:

$$\lim_{x \to 0} \frac{d(x)}{x}\,.$$

Now we can define the measure of angles at a point p:

Definition 3.29. *The measure of the angle $\measuredangle(q, p, r))$ is given by:*

$$m(\measuredangle(q, p, r)) = 2arcsin\Big(\frac{1}{2}\lim_{x \to 0}\frac{d(x)}{x}\Big)\,.$$

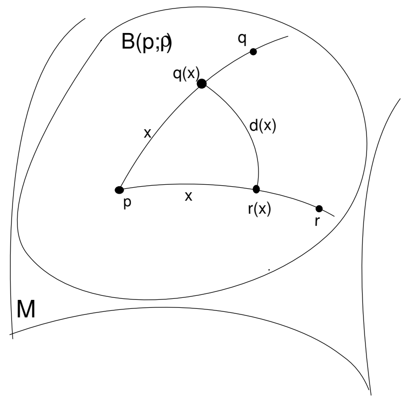

FIGURE 6. Metric Definition of the Sinus Function

The definition above enables us to define polar coordinates on regular neighborhoods in the following manner: Let $p_1, p_2 \in B(p\,;\rho)$ such that p, p_1, p_2 are not collinear. (Such points exist by Proposition 3.26). To every point $q \in B(p\,;\rho)$ we associate the following pair of real numbers (defining the polar coordinates at q relative to the frame determined by p, p_1, p_2): $(r(q), \theta(q)$, where

$$r(q) = d(p, q)$$

and

$$\theta(q)) = \begin{cases} m(\measuredangle(q, p, p_1)) & \text{if } |m(\measuredangle(p_2, p, p_1)) - m(\measuredangle(q, p, p_1))| = m(\measuredangle(q, p, p_1))\,; \\ 2\pi - m(\measuredangle(q, p, p_1)) & \text{if } |m(\measuredangle(p_2, p, p_1)) - m(\measuredangle(q, p, p_1))| \neq m(\measuredangle(q, p, p_1))\,. \end{cases}$$

We can thus conclude that the following holds:

Proposition 3.30. Any convex, compact metric space is locally homeomorphic to the real plane.

For the detailed proofs of the results above, see [5], [7].

4. Computing Wald Curvature

In this section we bring formulas for the computation and approximation of embedding curvature of quadruples. In the beginning follow the classical approach of Wald-Blumenthal (see, e.g., [5], [7]) that employs the so-called *Cayley-Menger determinants* (see below). Unfortunately, the formulas obtained, albeit precise are transcendental, and as such difficult to employ in practical implementations. Therefore we next present the approximate formulas developed by C.V. Robinson ([24]).

4.0.1. *The Cayley-Menger Determinant.* Given a general metric quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j)$, $i = 1, ..., 4$; denote by $D(Q) =$

$D(p_1, p_2, p_3, p_4)$ the following determinant:

$$D(p_1, p_2, p_3, p_4) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 \\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix} \tag{4.1}$$

The determinant $D(Q) = D(p_1, p_2, p_3, p_4)$ is called the *Cayley-Menger determinant* (of the points $p_1, ...p_4$). This definition readily generalizes to any dimension, as do the results bellow. To get some geometric intuition regarding Formula (4.2) we first examine the Euclidean case (see [5], [2] for details).

We start with the following proposition:

Proposition 4.1. The points $p_1, ..., p_4$ are the vertices of a non-degenerate simplex in $\mathbb{R}^3$ iff $D(p_1, p_2, p_3, p_4) \neq 0$.

However, a much strong result can be proven:

Theorem 4.2. Let $d_{ij} > 0\,, 1 \leq 4\,, i \neq j$. Then there exists a simplex $T = T(p_1, ..., p_4) \subseteq \mathbb{R}^3$ such that $dist(x_i, x_j) = d_{ij}\,, i \neq j$; iff $D(p_i, p_j) < 0$, for any $\{i, j\} \subset \{1, ..., 4\}$ and $D(p_i, p_j, p_k) > 0$, for any $\{i, j, k\} \subset \{1, ..., 4\}$; where, for instance,

$$D(p_1, p_2) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & d_{12}^2 \\ 1 & d_{12}^2 & 0 \end{vmatrix}$$

and

$$D(p_1, p_2, p_3) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 \end{vmatrix}\,;$$

etc...

In fact, the necessary and sufficient condition above can be further relaxed. Indeed, one can also show that the following result (which we formulate − for simplicity − for the case $n = 3$ only, even if it immediately generalizes to any dimension) holds:

Proposition 4.3. Let $d_{ij} > 0\,, 1 \leq 4\,, i \neq j$. Then there exists a simplex $T = T(p_1, ..., p_4) \subseteq \mathbb{R}^3$ such that $dist(x_i, x_j) = d_{ij}\,, i \neq j$; iff $D(p_1, p_2, p_3, p_4) \neq 0$ and $sign\, D(p_1, p_2, p_3, p_4) = +1$.

The developments of the expressions of volumes as Cayley-Menger determinants, in the spherical and hyperbolical cases are far too technical for this limited exposition; suffice therefore to add that they essentially reproduce the proof given in the Euclidean case, taking into account the fact that, when performing computations in the spherical (resp. hyperbolic) metric, one has to replace the distances d_{ij} by $\cos d_{ij}$ (resp. $\cosh d_{ij}$) − see [5] for the full details. Now the following formula for the *embedding curvature* $\kappa(Q)$ of Q (and its dependence upon the curvature's sign of the embedding space) is natural:

$$\kappa(Q) = \begin{cases} 0 & \text{if } D(Q) = 0\,; \\ \kappa,\ \kappa < 0 & \text{if } det(\cosh\sqrt{-\kappa}\cdot d_{ij}) = 0\,; \\ \kappa,\ \kappa > 0 & \text{if } det(\cos\sqrt{\kappa}\cdot d_{ij}) \text{ and } \sqrt{\kappa}\cdot d_{ij} \le \pi \\ & \quad \text{and all the principal minors of order 3 are} \ge 0. \end{cases} \tag{4.2}$$

4.0.2. *Approximate Formulas.* We have noted in the preceding section that the formulas we just developed in are not only transcendental, but also the computed curvature may fail to be unique. However, uniqueness is guaranteed for sd-quads. Moreover, the relatively simple geometric setting of sd-quads facilitates the development of simple (i.e. rational) formulas for the approximation of the embedding curvature.

Theorem 4.4 ([24])**.** Given the metric semi-dependent quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j),\ i, j = 1, ..., 4$; the embedding curvature $\kappa(Q)$ admits a rational approximation given by:

$$K(Q) = \frac{6(\cos\measuredangle_0 2 + \cos\measuredangle_0 2')}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2) + d_{23}\sin^2(\measuredangle_0 2')\big)} \tag{4.3}$$

where: $\measuredangle_0 2 = \measuredangle(p_1p_2p_4)\,,\ \measuredangle_0 2' = \measuredangle(p_3p_2p_4)$ represent the angles of the Euclidian triangles of sides d_{12}, d_{14}, d_{24} and d_{23}, d_{24}, d_{34} , respectively.

The *error R estimate* is given by the following inequality:

$$|R| = |R(Q)| = |\kappa(Q) - K(Q)| < 4\kappa^2(Q)diam^2(Q)/\lambda(Q)\,; \tag{4.4}$$

where we put: $\lambda(Q) = d_{24}(d_{12}\sin\measuredangle_0 2 + d_{23}\sin\measuredangle_0 2')/S^2$, and where $S = Max\{p, p'\}$; $2p = d_{12} + d_{14} + d_{24}\,,\ 2p' = d_{32} + d_{34} + d_{24}$.

Proof. The basic idea of the proof is to mimic, in a general metric setting, the Gauss map (see, e.g. [11]) – in this case one measures the curvature by the amount of "bending" one has to apply to a general planar quadruple so that it can be isometrically embedded as a *sd-quad*) in S_κ, for some κ.

Consider two planar (i.e. embedded in $R^2 \equiv \mathcal{S}_0$) triangles $\triangle p_1p_2p_4$ and $\triangle p_2p_3p_4$, and denote by $\triangle p_1^k p_2^k p_4^k$ and $\triangle p_2^k p_3^k p_4^k$ their respective isometric embeddings into $\mathcal{S}_k$. Then $p_{i,k}p_{j,k}$ will denote the geodesic (of $\mathcal{S}_k$) through $p_{i,k}$ and $p_{j,k}$. Also, let $\measuredangle_k 2$ and $\measuredangle_k 2'$ denote, respectively, the following angles of $\triangle p_{1,k}p_{2,k}p_{4,k}$ and $\triangle p_{2,k}p_{3,k}p_{4,k}$: $\measuredangle_k 2 = \measuredangle p_{1,k}p_{2,k}p_{4,k}$ and $\measuredangle_k 2' = \measuredangle p_{2,k}p_{3,k}p_{4,k}$ (see Fig. 7).

But $\measuredangle_k 2$ and $\measuredangle_k 2'$ are strictly increasing as functions of k. Therefore the equation

$$\measuredangle_k 2 + \measuredangle_k 2' = \pi \tag{4.5}$$

has at most one solution k^*, i.e. k^* represents the unique value for which the points p_1, p_2, p_3 are on a geodesic in $\mathcal{S}_k$ (for instance on p_1p_4).

But that means that k^* is precisely the embedding curvature, i.e. $k^* = \kappa(Q)$, where $Q = Q(p_1, p_2, p_3, p_4)$.

Equation (4.5) is equivalent to

$$\cos^2\frac{\measuredangle_{k^*}2}{2} \;+\; \cos^2\frac{\measuredangle_{k^*}2'}{2} \;=\; 1$$

The basic idea being the comparison between metric triangles with equal sides, embedded in $\mathcal{S}_0$ and $\mathcal{S}_k$, respectively, it is natural to consider instead of the previous equation, the following equality:

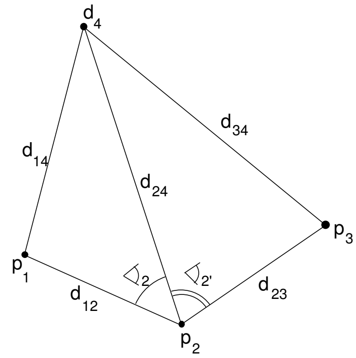

FIGURE 7. An sd-quad

$$\theta(k,2)\cdot\cos^2\frac{\measuredangle_0 2}{2} \;+\; \theta(k,2')\cdot\cos^2\frac{\measuredangle_0 2'}{2} \;=\; 1 \tag{4.6}$$

where we denote:

$$\theta(k,2)=\frac{\cos^2\frac{\measuredangle_{k^*}2}{2}}{\cos^2\frac{\measuredangle_0 2}{2}}\,;\;\;\theta(k,2')=\frac{\cos^2\frac{\measuredangle_{k^*}2'}{2}}{\cos^2\frac{\measuredangle_0 2}{2}}\,.$$

Since we want to approximate $\kappa(Q)$ by $K(Q)$ we shall resort – naturally – to expansion into MacLaurin series. We are able to do this because of the existence of the following classical formulas:

$$\cos^2\frac{\measuredangle_k 2}{2}=\frac{\sin(p\sqrt{k})\cdot\sin(d\sqrt{k})}{\sin(d_{12}\sqrt{k})\cdot\sin(d_{24}\sqrt{k})}\,;\;k>0\,;$$
$$\cos^2\frac{\measuredangle_k 2}{2}=\frac{\sinh(p\sqrt{k})\cdot\sinh(d\sqrt{k})}{\sinh(d_{12}\sqrt{k})\cdot\sinh(d_{24}\sqrt{k})}\,;\;k<0\,;$$

and, of course

$$\cos^2\frac{\measuredangle_0 2}{2}=\frac{pd}{d_{12}d_{24}}\,;$$

were $d=p-d_{14}=(d_{12}+d_{24}-d_{14})/2$ (and the analogous formulas for $\cos^2\frac{\measuredangle_{k'}2}{2}$).

By using the development into series of $f_1(x)=\frac{\sin\sqrt{x}}{\sqrt{x}}$ and $f_2(x)=\frac{\sinh\sqrt{x}}{\sqrt{x}}$; one (easily) gets the desired expansion for $\theta(k,2)$:

$$\theta(k,2)=1+\frac{1}{6}kd_{12}d_{24}\big(\cos(\measuredangle_0 2)-1\big)+r\,; \tag{4.7}$$

where: $|r|<\frac{3}{8}k^2p^4$, for $|kp^2|<1/16$.

By applying (4.7) to (4.6), we receive:

$$\Big[1+\frac{1}{6}k^*d_{12}d_{24}\big(\cos(\measuredangle_0 2)-1\big)+r\Big]\cos^2\frac{\measuredangle_0 2}{2}\;+ \tag{4.8}$$

$$\Big[1+\frac{1}{6}k^*d_{23}d_{24}\big(\cos(\measuredangle_0 2')-1\big)+r'\Big]\cos^2\frac{\measuredangle_0 2'}{2}=1\,;$$

for: $|r|+|r'|<\frac{3}{4}(k^*)^2(Max\{p,p'\})^4=\frac{3}{4}(k^*)^2S^4$.

By solving the linear equation (in variable k^*) (4.8) and using some elementary trigonometric transformation one has:

$$k^*=\frac{6(\cos\measuredangle_0 2+\cos\measuredangle_0 2')}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2)+d_{23}\sin^2(\measuredangle_0 2')\big)}+R\,;$$

where:

$$|R|<\frac{12(|r|+|r'|)}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2)+d_{23}\sin^2(\measuredangle_0 2')\big)}<\frac{9(k^*)^2\max\{p,p'\}}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2)+d_{23}\sin^2(\measuredangle_0 2')\big)}\,.$$

But $k^*\equiv\kappa(Q)$, so we get the desired formula (4.3) .

To prove the correctness of the bound (4.4) one has only to observe that:

$$S=Max\{p,p'\}<2diam(Q),\ \big(diam(Q)=\max_{1\leq i<j\leq 4}\{d_{ij}\}\big),$$

and perform the necessary arithmetic manipulations. □

Example 4.5 ([24]). Let Q_0 be the quadruple of distances $d_{12}=d_{23}=d_{24}=0.15, d_{14}=d_{34}$ and of embedding curvature $\kappa=\kappa(Q_0)=1$. Then $\kappa S^2<1/16$ and $K(Q_0)\approx 1.0030280$, which shows that the actual computed error can be far less then the one given by formula (4.4), which, in this case gives $|R|<0.45$.

Remark 4.6. (a) The function $\lambda=\lambda(Q)$ is continuous and 0-homogenous as a function of the d_{ij}-s. Moreover: $\lambda(Q)\geq 0$ and $\lambda(Q)=0\Leftrightarrow\sin\measuredangle_0 2=\sin\measuredangle_0 2'=0$, i.e. iff Q is linear. (Therefore for sd-quads $\lambda(Q)>0$. Moreover, when $\lambda(Q)$ tends to 0, Q approaches linearity.)

(b) Since $\lambda(Q)\neq 0$ it follows that: $K(Q)\in\mathbb{R}$ for any quadrangle Q. Moreover: $sign(\kappa(Q))=sign(K(Q))$.

(c) If Q is any sd-quad, then $\kappa^2(Q)diam^2(Q)/\lambda(Q)<\infty$. Moreover, $|R|$ is small if Q is not close to linearity. In this case $|R(Q)|\sim diam^2(Q)$ (for any given Q).

Since the Gaussian curvature $K_G(p)$ at a point p is given by:

$$K_G(p)=\lim_{n\to 0}\kappa(Q_n)\,;$$

where $Q_n\to Q=\Box p_1pp_3p_4\,;\ diam(Q_n)\to 0$,, from Remark 4.6(c) we immediately infer that the following holds:

Theorem 4.7. Let S be a differentiable surface. Then, for any point $p\in S$:

$$K_G(p)=\lim_{n\to 0}K(Q_n)\,;$$

for any sequence $\{Q_n\}$ of sd-quads that satisfy the following condition:

$$Q_n\to Q=\Box p_1pp_3p_4\,;\ diam(Q_n)\to 0\,.$$

Remark 4.8. In the following special cases even "nicer" formulas are obtained:

(1) If $d_{12}=d_{32}$, then

$$K(Q)=\frac{12}{d_{13}\cdot d_{24}}\cdot\frac{\cos\measuredangle_0 2+\cos\measuredangle_0 2'}{\sin^2\measuredangle_0 2+\sin^2\measuredangle_0 2'}\,;\tag{4.9}$$

(here we have of course: $d_{13} = 2d_{12} = 2d_{32}$); or, expressed as a function of distances alone:

$$K(Q) = 12\frac{2d_{12}^2 + 2d_{24}^2 - d_{14}^2 - d_{13}^2}{8d_{12}^2 d_{24}^2 - (d_{12}^2 + d_{24}^2 - d_{14}^2)^2 - (d_{12}^2 + d_{24}^2 - d_{34}^2)^2} \tag{4.10}$$

(2) If $d_{12} = d_{32} = d_{24}$ and if the following condition also holds:
(3) $\measuredangle_0 2' = \pi/2$; i.e. if $d_{34}^2 = d_{12}^2 + d_{24}^2$ or, considering 2., also: $d_{34}^2 = 2d_{12}^2$, then

$$K(Q) = \frac{6\cos\measuredangle_0 2}{d_{12}(1+\sin^2\measuredangle_0 2)} = \frac{2d_{12}^2 - d_{14}^2}{4d_{12}^4 + 4d_{14}^2 d_{12}^2 - d_{14}^4}\,. \tag{4.11}$$

5. Experimental Results

In this section we view some preliminary numerical results of approximating Gauss curvature of the Torus by the metric curvature computed on a sequence of sampled tori with increasing resolutions. The precise parametrization of the Torus is known therefore computational error can be precisely assessed. In addition to that the Torus was chosen since it has both positive, zero and negative Gauss curvature. Computations where done using various definitions of metric curvatures. In the graphs below approximation results are given. In both examples Haantjes and Robinson approximations are done while computing sectional curvatures along the mesh edges. Therefore in order to more accurately approximating Gauss curvature one needs to adjust the triangulation so that its edges will best coincide with geodesic lines of the surface. The second graph shows improvement of the results after such an adjustment was done.

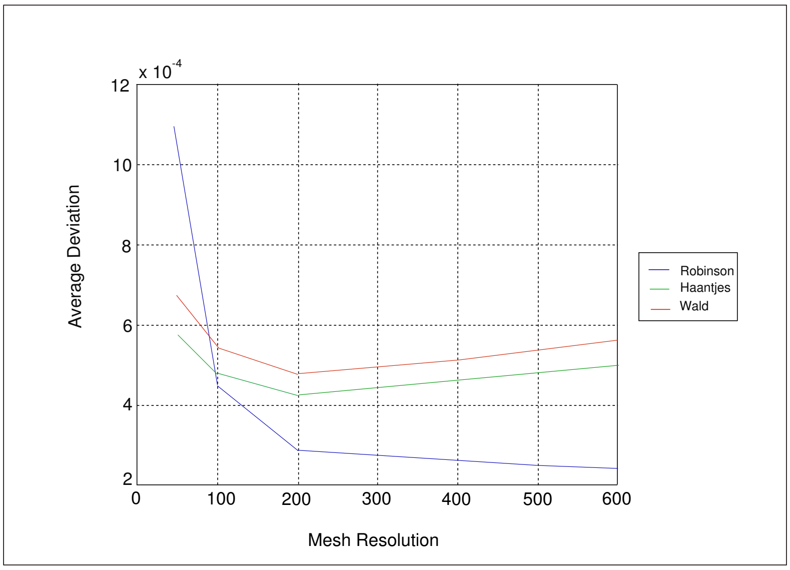

Figure 8. Approximation errors w.r.t mesh resolution (as number of triangles).

We also analyzed the relative performances of the considered algorithms as displayed in the regions of various sign of Gauss curvature, both for the original

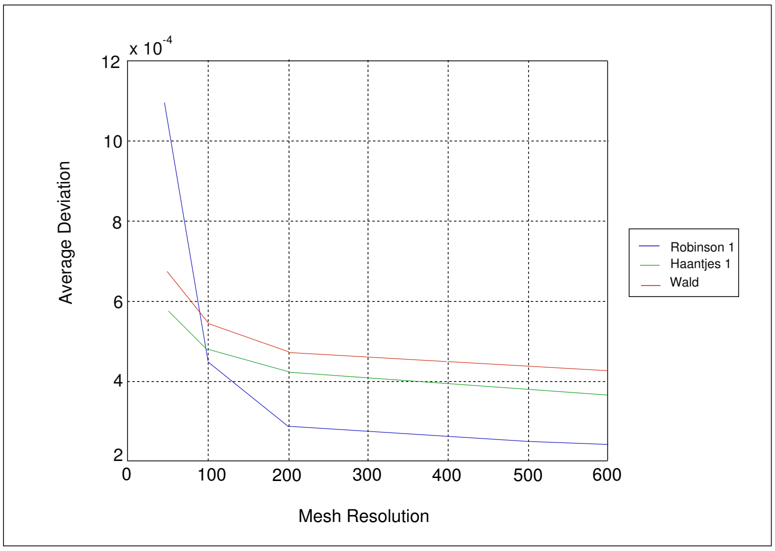

FIGURE 9. Approximation errors w.r.t mesh resolution (as number of triangles).

numerical schemes and for the improved ones. It was observed that both Robinson and Wald based algorithms display a "jump" at the boundary between the elliptic and hyperbolic regions. This step-function behaviour is due to the dichotomy intrinsic to both methods, dichotomy that is induced by the sign of the curvature.

Of course, for a better evaluation of the capabilities and limits of the metric methods, further experiments on more divers surfaces and with finer meshes are to be undertaken.

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