ON THE CLASSICAL – AND NOT SO CLASSICAL – SHANNON SAMPLING THEOREM

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1. Introduction and First Remarks

1.1. **Introduction.** In [19] we have proved a sampling theorem for differentiable manifolds and applied it in the context image processing ([18], [19]). The gist of our constructive proof is the existence of so called fat triangulations (see [21]). The density of the vertices of the triangulation (i.e. of the sampling) is given by the inverse of the maximal principal curvature. Moreover we have showed that our sampling scheme coincides with the one provided by classical Shannon theorem, at least for the large class of (band-limited) signals that are also $\mathcal{C}^2$ curves. In this geometric approach, the role of the role of the Nyquist rate is played by the radius of curvature.

It is the goal of this paper to further investigate the extent and power of this analogy and of the geometric approach in general. We begin by making a few observations regarding the extent of our results, i.e. of finding the largest space of signals on which our results may be applied effectively.

Next, we establish the proper analogies of the basic notions in classical sampling and coding theory of the Gaussian channel. In doing this, we hope to construct a "dictionary" of classical and geometric sampling notions, in the tradition of Shannon's seminal paper [16].

In the following section we focus on classical band-limited signals and we determine the connection between such signals and curves of curvature bounded from above.

Finally, in the last section, we extend our investigation to (a class of) infinitely dimensional manifolds. Such manifolds, and the need for a sampling theory for this class of geometrical objects naturally arise, for instance, in the context of continuous variations and deformations (e.g video) of classical signals, perceived as infinite series (of trigonometric functions).

1.2. **General geometric signals.** We begin our investigation by noting that, by the Paley-Wiener(?) Theorem, any band limited signal is of class $\mathcal{C}^\infty$. We have already shown in [18] that our geometric sampling method applies not only to band limited signals but also to more general "blackboard

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signals", i.e. L^2 functions who's graphs are smooth $\mathcal{C}^2$ curves (not necessarily planar). (We shall further investigate the connections between band limited signals and functions of bounded second derivative in Section 3.) In fact, the geometric sampling approach can be extended to a far larger class of manifolds. Indeed, every piecewise linear (PL) manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) *smoothing* (see e.g. [22]), and every topological manifold of dimension $n \leq 3$ admits a PL structure (cf., e.g., [22]). In particular, for curves (and surfaces) one can first consider a smoothing of class $\geq \mathcal{C}^2$ (so that curvature can be defined properly), which can be then sampled with sampling rate given by the maximal curvature radius. Since the given manifold and its smoothing are arbitrarily close (see [14]) one obtains the desired sampling result. (This very scheme is developed and applied for gray-scale images in [18].)

Now, while numerical schemes for practical computation of smoothing exist, they are not also necessarily computationally desirable. For practical applications, one can circumvent this problem and avoid smoothing by applying such numerical schemes as provided by the finite element method ([20]). However, both for the sake of mathematical correctness and for being able to tackle more general applications, one would like to consider more general curvature measures (see, e.g. [23]) and avoid smoothing altogether (see [20] for the full details of this approach and Section 2 below, for a brief discussion of this topic in a slightly different context).

1.3. **Pulse code modulation for images.** Passingly, we note that our sampling result offers, as a direct application, a new PCM (*pulse code modulation*) method for images (that is, considered as such, and not as 1-dimensional signals). This has as an intrinsic advantage the fact that the sampling points are associated to relevant geometric features (via curvature) and are not chosen randomly via the Nyquist rate. Moreover, such a sampling is adaptive and, indeed, compressive (see discussion above), with the obvious consequent technological benefits.

2. Sampling and Codes

2.1. **Packings, Coverings and Lattice codes.** Recall that in classical signal processing, $W = \eta/2$, where W is the frequency of the signal and η represents the *Nyquist rate.* This admits an immediate (and rather trivial) generalization to periodic signals, or, in geometric terms, for signals over a *lattice*: $\Lambda = \{\lambda_i\}$. In this case, one can even interpret the sides of the lattice as the various coordinates in a multi-dimensional (warped) time (see, e.g. [21], [11]). Note that such signals can be viewed as distributions on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. In this interpretation, the (n-dimensional!) period is the *fundamental cell* λ of the lattice. Two scalars are naturally associated with this cell: its diameter $\text{diam}(\lambda)$ (or, alternatively, the length of the longest edge) and its volume $\text{Vol}(\lambda)$. Either of them can be used as a measure of the n-dimensional period. However, they are both interrelated into one geometric feature, the so called *fatness*:

Definition 2.1. Let $\gamma = \gamma^k$ be an k-dimensional cell. The *fatness* (or *aspect-ratio*) of γ is defined as:

$$\varphi(\gamma) = \min_{\lambda} \frac{\mathrm{Vol}(\lambda)}{\mathrm{diam}^l(\lambda)}\,,$$

where the minimum is taken over all the l-dimensional faces of γ, $0 \leq k$. (If $\dim \lambda = 0$, then $\mathrm{Vol}(\lambda) = 1$, by convention.)

This definition of fatness is equivalent (see [15]) to the following one:

Definition 2.2. A k-dimensional cell $\gamma \subset \mathbb{R}^n$, $2 \leq k \leq n$, is *φ-fat* if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$; where r denotes the radius of the inscribed sphere of γ and R denotes the radius of the circumscribed sphere of γ. A cell-complex $\Gamma = \{\gamma_i\}_{i\in\mathbf{I}}$ is *fat* if there exists $\varphi \geq 0$ such that all its cells are *φ-fat*.

Recall that the in- and circumradius are relevant(important) in lattice problems: given a lattice Λ with (dual) Voronoi cell Π (of volume 1), one has to minimize the inradius to solve the packing problem, and to minimize the circumradius for solving the covering problem (see [4]). Note that Λ and Π are simultaneously fat. It follows that fat cell-complexes (and, in particular, fat triangulations) represent a (mini-max) optimization for both the packing and the covering problem. Moreover, since fat triangulations are essential for the sampling theorem for manifolds, it appears that there exists an intrinsic relation between the sampling problem for manifolds and the covering and packing problems.

2.2. **Average Power, Rate of code and Channel Capacity.** Note that in the context of lattices (with fundamental cell) λ it is natural to extend the classical definitions of *average power* in the signal:

$$P = \frac{1}{T}\int_0^T f^2(t)dt\,,$$

and the *rate* of the code:

$$R = \frac{1}{T}\log_2 N\,,$$

(where N represents the number of code points) in the following manner:

$$P = \frac{1}{\mathrm{Vol}(\Lambda)}\int_{\lambda} f^2(t)dt = \frac{1}{\mathrm{Vol}(N_1\lambda)}\int_{\lambda} f^2(t)dt\,,$$

and

$$R = \frac{1}{\mathrm{Vol}(\Lambda)}\log_2 N = \frac{1}{N_1\mathrm{Vol}(\lambda)}\log_2 N,$$

respectively, N_1 being the number of cells.

Similarly, one can adapt the classical definition of the *channel capacity*:

$$C = \lim_{T\to\infty} R = \lim_{T\to\infty} \frac{\log_2 N}{T}\,,$$

as:

$$C = \lim_{T\to\infty} \frac{\log_2 N}{\mathrm{Vol}(\Lambda)} = \lim_{T\to\infty} \frac{1}{N_1 \mathrm{Vol}(\lambda)} \log_2 N.$$

Since the numbers N and N_1 are related by $N_1 = \alpha(N)$, where α is the *growth function* of the manifold, the expression of C becomes:

$$C = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(\lambda)} \frac{\log_2 N}{\alpha(N)}.$$

It follows immediately that $C = \infty$ for non-compact Euclidean and Hyperbolic manifolds, and $C = 0$ for their Elliptic counterparts. Unfortunately, no such immediate estimates can be readily produced for manifolds of variable curvature.

Note that by putting $1/T = \mathrm{Vol}(M)$, the definitions above apply for any sampling scheme of any manifold of finite volume, not just for lattices. In this case N and N_1 represent the number of vertices, respective simplices, of the triangulation.

The interpretation of frequency considered above does not extend, however, to general geometric signals. For a proper generalization we have to look into the geometric analogue of W: By [19], Theorem 5.2 on curves, i.e. 1-dimensional (geometric) signals, W equals the *curvature rate* $k/2$, were k represents the maximal absolute curvature of the curve. This, and the sampling Theorem 4.11. of [19] naturally conducts us to the following definition of T for general geometric signals:

Definition 2.3. Let $M = M^n$ be an n-dimensional manifold $n \geq 2$. $W = W_M = 1/k_M$, where $k_M = \max k_i$ and $k_i, i = 1, \dots, n$ are the principal curvatures of M.

or $k_M/2$?!?!

Classically, the energy f the signal f is taken to be as his L^2 norm:

$$E = E(f) = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2W} \sum_{-\infty}^{+\infty} f^2\Big(\frac{k}{2W}\Big)\,.$$

One would like, of course, to find proper generalizations of the notion of energy for more general (geometric) signals. In view of the discussion above, it is clear that a first step it is to replace $2W$ by its proper generalization. However, when considering more general function spaces of specific relevance (s.a. *bounded variation* (*BV*), *bounded oscillation* (*BO*), *bounded mean oscillation* (*BMO*)), one should consider energies befitting the specific norm of the space under consideration. Of course, this discussion is also valid with regard to the best way to define average power P, and rate R, of a geometric signal.

We can now look into the first definition of code efficiency: the (*nominal*) *coding gain* of a code over another:

$$\mathrm{ncd}(C1, C2) = 10 \lg \frac{\frac{\mu_1}{E_1}}{\frac{\mu_2}{E_2}} \,,$$

where μ is the square of the minimal square distance between coding points. For geometric codes of bounded curvature (hence compact ones), the expression for μ is particularly simple: $\mu = 1/\min k$ (k denoting again principal curvature).

2.3. **The Channel Codding Problem.** It is only natural to attack the problem of the *Gaussian white noise channel* in the context of "geometric signals", i.e. manifolds. Recall that in the classical context, a *received signal* is represented by a vector $X = F + Y$, where $F = (f_1, \ldots, f_N)$ is the *transmitted signal*, and $Y = (y_1, \ldots, y_n)$ represents the noise, whose components y_i are independent Gaussian random variables, of *mean* 0 and *average power* (*mean*) σ^2. The main, classical result for the Gaussian channel is the following:

Theorem 2.4 (Shannon's Second Theorem, [16])**.** *For any rate R not exceeding the* capacity C_0*:*

$$C_0 = \frac{1}{T} \log_2 \left(1 + \frac{P}{\sigma^2}\right),$$

there exists T sufficiently large, such that there exists a code of rate R and average power $\leq P$, and such that the probability of a decoding error is arbitrarily small. Conversely, it is not possible to obtain arbitrarily small errors for rates $R > C_0$.

In the case of geometric signals, F is given by the sampling (code) points on the manifolds and, since the mean equals 0, the nosy transmitted signal $F + Y$ lies in the *tube* $\mathrm{Tub}_\sigma(M)$. Recall that $\mathrm{Tub}_\varepsilon(S) = \bigcup_{p \in S} I_{p,\varepsilon_p}$, where I_{p,σ_p} is the open symmetric interval through p, in the direction of unit normal of M at p, of length $2\sigma_p$, where ε_p is chosen to be small enough such that $I_{p,\varepsilon_p} \cap I_{q,\varepsilon_q} = \emptyset$, for any $p, q \in S$ such that $||p - q|| > \xi \in \mathbb{R}_+$. Here Shannon gives $\leq C_0$ and Sloane $<$!...?...

While not evident, the existence of tubular neighborhoods is assured both locally, for any regular, orientable manifold, and globally for regular, compact, orientable manifold (see, [7]). In addition, the regularity of the manifolds $\mathrm{Tub}^-_\sigma(M)$, $\mathrm{Tub}^+_\sigma(M)$, $\mathrm{Tub}^-_\sigma(M) \cup \mathrm{Tub}^+_\sigma(M) = \partial\mathrm{Tub}_\sigma(M)$ is at least as high as that of M: If M is convex, then $\mathrm{Tub}^-_\sigma(M)$, $\mathrm{Tub}^+_\sigma(M)$ are piecewise $\mathcal{C}^{1,1}$ manifolds (i.e., they admit parameterizations with continuous and bounded derivatives), for all $\varepsilon > 0$. Also, if M is a smooth enough manifold with a boundary (that is, at least piecewise $\mathcal{C}^2$), then $\mathrm{Tub}^-_\sigma(M)$, $\mathrm{Tub}^+_\sigma(M)$ are piecewise $\mathcal{C}^2$ manifolds, for all small enough σ (see [6]).

In the geometric setting, σ can be taken, of course, to be the maximal Euclidean deviation. However, a better deviation measure is, at least for compact manifolds, the Haussdorf Distance (between M and $\mathrm{Tub}^-_\sigma(M)$, $\mathrm{Tub}^+_\sigma(M)$):

Definition 2.5. Let (X, d) be a metric space and let $A, B \subseteq (X, d)$. The *Hausdorff distance* between A and B is defined as:

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}\,.$$

For non-compact manifolds one has to consider the more general *Gromov-Hausdorff distance* (see, e.g., [1]).

Since, by the remarks above, for σ small enough, both the distance between M and $\mathrm{Tub}_{-}\sigma^{+}(M), \mathrm{Tub}_{+}\sigma^{+}(M)$ and the deviations of their curvature measures are arbitrarily small, we can state a first "soft" geometric version of Shannon's Theorem for the Gaussian channel. While a perfect analogy is not available(possible), we can nevertheless formulate the following theorem:

Theorem 2.6 (Shannon's Second Theorem, qualitative version)**.** *Let M^n be a smooth geometric signal (manifold) and let σ be small enough, such that* $\mathrm{Tub}_\sigma(M)$ *is a submanifold of* $\mathbb{R}^{n+1}$*. Then, given any nosy signal $M + Y$, such that the average noise power σ_Y is at most σ, there exists a sampling of $M + Y$ with probability of decoding error arbitrarily small.*

Remark 2.7. For geometric codes, the analogue of the capacity is $C_0 = C_0(k, \sigma, r)$, where r represents the differentiability class of M.

Remark 2.8. For compact manifolds, the existence of tube $\mathrm{Tub}\sigma^{+}(M)$ is, as we have already remarked, assured globally. Hence it follows that the sampling scheme is also global and necessitates at $O(N)$ points ($N = N_M$). However, for non-compact manifolds (in particular non-band limited geometric signals), the existence of $\mathrm{Tub}\sigma^{+}(M)$ is guaranteed only locally. Therefore "glueing "of the patches is needed, operation which requires the insertion of additional vertices (i.e. sampling points), their number being a function of the dimension of M. hence, in this case, $N_{M+Y} = O(N_M^n)$.

It is important to remark that, again, this result is not restricted to smooth manifolds, but rather extends to much more general signals: Indeed, for any compact set $M \in \mathbb{R}^n$, the $(n-1)$-dimensional sets $\mathrm{Tub}_\sigma^{-}(M)$, $\mathrm{Tub}_\sigma^{+}(M)$, are *Lipschitz manifolds* for almost any ε (see [8]). Moreover, the generalized curvatures measures of $\mathrm{Tub}_\sigma^{-}(M)$, $\mathrm{Tub}_\sigma^{+}(M)$ are arbitrarily close to the curvature of M, for small enough σ ([3], [8]). It follows that the generalization above befits not only the case of the Gaussian noise, but to more general types of noise, as well (see, [16], [9], [10]).

The full details of a *quantitative* version, including the general case, are laborious and warrant a separate discussion (see [20]).

3. Band-limited signals and bounded curvature

Given the geometric sampling algorithm one is naturally conducted to pose the following question: "Are band-limited function of bounded curvature?" The answer to this question is both "Yes" and "No".

On the positive side, we can state the following proposition:

Proposition 3.1. *Let $f \in L^2(\mathbb{R})$ be a band-limited function. Than $f'' \in L^\infty(\mathbb{R})$.*

Proof. Since f is band-limited we have $\text{supp}\hat{f} \subset [-B, B]$. Since $f \in L^2(\mathbb{R})$ it is clear that $\hat{f} \in L^2([-B, B])$. The interval $[-B, B]$ is bounded and this implies $\hat{f} \in L^1([-B, B])$.

The mapping properties of the Fourier transform now shows that $f \in L^\infty(\mathbb{R})$. This just shows that band-limited functions are bounded.

The second derivative of a band-limited function is also band-limited (because $\widehat{f''}(w) = (-2\pi i w)^2 \hat{f}(w)$). Hence, $f'' \in L^\infty(\mathbb{R})$. □

On the other hand, the second derivative of a band-limited function can be arbitrarily large just because λf is again band-limited for every $\lambda > 0$. This shows, that the maximal frequency of a function does not imply a bound on the second derivative of the function.

The classical formula (see, e.g. [5]) for the curvature of the function f is given by:

$$\kappa_f(x) = \frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}} \,,$$

while the scaling $g_\lambda(x) = \lambda f(x)$, has curvature

$$\kappa_{g_\lambda}(x) = \frac{\lambda f''(x)}{(1 + \lambda^2 f'(x)^2)^{\frac{3}{2}}} \overset{\lambda\to\infty}{\longrightarrow} \begin{cases} 0 & , \ f'(x) \neq 0 \\ \infty & , \ f'(x) = 0 \end{cases} .$$

In other words, the curvature goes to infinity at the maximum points of f and to 0 everywhere else.

However, one can give bounds on the derivatives of f:

Proposition 3.2. *For f with* $\text{supp}\hat{f} \subset [-B, B]$ *we have the following estimates on the derivative of f:*

$$|f^{(n)}(x)| \leq (2\pi B)^n \|\hat{f}\|_{L^1}. \tag{3.1}$$

If we assume $\hat{f} \in L^\infty$ (which implies $\hat{f} \in L^p$ for every p) we have

$$|f^{(n)}(x)| \leq \frac{2(2\pi)^n B^{n+1}}{n+1} \|\hat{f}\|_{L^\infty} \leq \frac{2(2\pi)^n B^{n+1}}{n+1} \|f\|_{L^1} \tag{3.2}$$

(while the second inequality only holds if $f \in L^1$, of course). Another estimate is

$$|f^{(n)}(x)| \leq \left(\frac{2(2\pi B)^{pn+1}}{pn+1} \right)^{1/p} \|\hat{f}\|_{L^q} \tag{3.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. All inequalities base of the formula

$$|f^{(n)}(x)| = |\int (-2\pi i \omega)^n \hat{f}(\omega) d\omega| \leq \int_{-B}^{B} |2\pi\omega|^n |\hat{f}(\omega)| d\omega.$$

The equation (3.1) follows from

$$\sup_{\omega\in[-B,B]} |2\pi\omega|^n = (2\pi B)^n$$

equation (3.2) from

$$\int_{-B}^{B} |2\pi\omega|^n |\hat{f}(\omega)| d\omega \leq \|\hat{f}\|_{L^\infty} \int_{-B}^{B} |2\pi\omega|^n d\omega$$

and the fact that $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$. Finally equation (3.3) follows by applying Hölders inequality:

$$\int_{-B}^{B} |2\pi\omega|^n |\hat{f}(\omega)| d\omega \leq \left(\int_{-B}^{B} |2\pi\omega|^{pn} d\omega\right)^{1/p} \|\hat{f}\|_q.$$

□

4. Geometric Sampling of Infinitely Dimensional Signals

Since in the classical context band-limited signals are viewed as elements f of $L^2(\mathbb{R})$, such that *supp* $(\hat{f}) \subseteq [-\pi, \pi]$, where $\hat{f}$ denotes the Fourier transform of f, one would is conducted naturally to the following question: can one extend the sampling theorem proven in [18] to infinitely dimensional manifolds? Using an example developed in [12], we show not only that this question is far from naive, but rather that the answer is positive and that our geometric sampling method translates directly in to the context of infinitely dimensional manifolds, at least for a class of functions that naturally arise in the the context of signal and image processing. However, since the full proofs required in the example below are rather technical, we refer for them to the original paper [12], and limit ourselves here solely to a brief presentation

Consider the following spaces:

$$C_1^\infty = \left\{e \in \mathcal{C}^\infty(\mathbb{R}) \,\middle|\, e(x+1) = e(x)\right\}$$

$$C_+^\infty = \left\{e \in \mathcal{C}^\infty(\mathbb{R}) \,\middle|\, e(x+1) = e(x), \int_0^1 e^2 = 1\right\}$$

$$M \subset C_+^\infty, M = \left\{\lambda_0 = 0 \,|\, \lambda_0 \text{ first eigenvalue of} Q\right\},$$

where Q denotes the Hill operator: $Q = -D^2 + q\,, q(x+1) = q(x)$.

Then M is a smooth, co-dimension one hyper-surface in C_1^∞.

Moreover, exactly like in the finite-dimensional case, for any 2-dimensional section determined by unit tangent vectors to M at q, one can define (and compute) the maximal principal curvature (of the section).

Moreover, since a normal to M at q is (of course) also defined, one can use the same method as in the finitely dimensional case to find a sampling of M.

It follows that a sampling scheme identical to that developed for the finitely dimensional case can be applied for the manifold M, as well. Unfortunately, no uniform sampling is possible for the entire manifold: the maximal curvature associated to function approximating "saw-tooth" functions can be made as large as desired (see [12]).

Of course, one would like to extend these considerations, in a systematic manner, to general infinitely-dimensional manifolds (e.g. l_2 and Hilbert cube manifolds). However, even if the appropriate geometric differential notions are defined and computed, the fundamental problem of constructing fat triangulations for infinitely dimensional manifolds still has to be solved. On the positive side is the fact that triangulations of such manifolds exist (see [2]). However, even finding a notion analogous to that of fatness in the ∞-dimensional case represents a challenge.

Remark 4.1. For a different path towards a differential geometry of (some) infinitely dimensional spaces see e.g. [13].

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