Geometric Sampling For Signals With Applications to Images[1]

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Abstract

We present a non uniform version of the sampling theorems introduced in [15], [16]. In addition, we show that the geometric sampling scheme produces sparse sampling for gray scale images.

1. Introduction

Sampling is an essential preliminary step in processing of any continuous signal by a digital computer. This step lies at heart of any digital processing of any (presumably continuous) data/signal. Along with that, in recent years it became common amongst the signal processing community, to consider images and other signals as well, as Riemannian manifolds embedded in higher dimensional spaces. Usually, the embedding manifold is taken to be $\mathbb{R}^n$ yet, other possibilities are also considered.

For instance, in [19] images are considered as surfaces embedded in higher dimensional manifolds, where a gray scale image is a surface in $\mathbb{R}^3$, and a color image is a surface embedded in $\mathbb{R}^5$, each color channel representing a coordinate of the ambient space. In both cases the intensity, either gray scale or color, is considered as a function of the two spatial coordinates (x, y) and thus the surface may be equipped with a metric induced by this function. The question of smoothness of the function is in general omitted, if numerical schemes are used for the approximations of derivatives, whenever this is necessary. A major advantage of such a viewpoint of signals is the ability to apply mathematical tools traditionally used in the study of Riemannian manifolds, for image/signal processing as well. For example, in medical imaging it is often convenient to treat CT/MRI scans, as Riemannian surfaces in $\mathbb{R}^3$. One can then borrow techniques from differential topology and geometry and geometric analysis in the representation and analysis of the considered images.

Along this route, the authors presented in [15], [16], sampling and reconstruction theorems for the family of signals that posses the structure of Riemannain manifolds. While making use of earlier works in geometric analysis as well as differential topology such as [6], [11], [14], the authors show that the sampling resolution is determined by the principal curvatures of the manifold instead of a Nyquist frequency rate. In the works [15] and [16], sampling of a manifold is given by a set of points in the manifold that will serve as the vertices of a triangulation of the manifold. The triangulation is supposed to satisfy a fatness condition (see below). Given this, reconstruction of the manifold can be achieved by linear interpolation of the sampled data. In the sequel we assume the reader is familiar with the concept of triangulation of manifolds.

In Section 2 we briefly states the main definition and geometric sampling theorem that where developed by the authors. In Section 3 we demonstrate the use of the theorem on images. We show that the sampling resolution achieved by the method of geometric sampling is no higher and in fact can be significantly lower than the resolution usually achieved by Fourier analysis. In Section 4 we summarize the paper and pose some open questions.

2. Background

In this section we will briefly present the geometric sampling scheme that where developed. We have omitted the proofs. For further reading as well as detailed proofs we refer the reader to [15].

Definition 1 *Let $f : K \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, and let $\delta : K \to \mathbb{R}^*_+$ be a continuous function. Then $g : |K| \to \mathbb{R}^n$ is called a δ-*approximation *to f iff:*
(i) There exists a subdivision *K' of K such that $g \in \mathcal{C}^r(K', \mathbb{R}^n)$;*
(ii) $d_{eucl}\big(f(x), g(x)\big) < \delta(x)$, for any $x \in |K|$;
(iii) $d_{eucl}\big(df_a(x), dg_a(x)\big) \leq \delta(a) \cdot d_{eucl}(x, a)$, for any $a \in |K|$ and for all $x \in \overline{St}(a, K')$.
(Here and below $|K|$ denotes the underlying polyhedron *of K.)*

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Definition 2 *Let K' be a subdivision of K, $U = \overset{\circ}{U} \subset |K|$, and let $f \in \mathcal{C}^r(K, \mathbb{R}^n)$, $g \in \mathcal{C}^r(K', \mathbb{R}^n)$. g is called a* δ-approximation *of f on U, iff conditions (ii) and (iii) of Definition 2.6. hold for any $a \in U$.*

The most natural and intuitive δ-approximation to a given mapping f is the *secant map induced by f*:

Definition 3 *Let $f \in \mathcal{C}^r(K)$ and let s be a simplex, $s < \sigma \in K$. Then the linear map: $L_s : s \rightarrow \mathbb{R}^n$, defined by $L_s(v) = f(v)$, where v is a vertex of s, is called the* secant map induced by f.

Definition 4 *A k-simplex $\tau \subset \mathbb{R}^n$, $2 \leq k \leq n$, is φ*-fat *if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$; where r denotes the radius of the inscribed sphere of τ (*inradius*) and R denotes the radius of the circumscribed sphere of τ (*circumradius*). A triangulation of a submanifold of $\mathbb{R}^n$, $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is φ-fat if all its simplices are φ*-fat. *A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$* is fat *if there exists $\varphi \geq 0$ such that all its simplices are φ-fat; for any $i \in \mathbf{I}$.*

In [6], [13] and [14], existence of fat triangulations is guaranteed for a wide range of Riemannian manifolds. Much wider range than what we actually need.

2.1. Smooth case

Theorem 1 *Let $\Sigma^n, n \geq 2$ be a connected, not necessarily compact, smooth manifold, with finitely many compact boundary components. Then there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\Big(\frac{1}{k(p)}\Big)$, where $k(p) = \max\{|k_1|, ..., |k_n|\}$, and where $k_1, ..., k_n$ are the principal (normal) curvatures of Σ^n, at the point $p \in \Sigma^n$.*

Remark 1 *The principal curvatures are usually very hard to compute if possible at all. For that reason it would be beneficial if these curvatures could be replaced by or approximated to some extent by more accessible intrinsic curvature measures such as the sectional/Ricci/scalar curvatures. It is still an open question, to what extant the principal curvature can be replaced by the sectional curvatures. It seems reasonable, yet requiring a fully comprehensive proof, that full knowledge of the sectional curvature along with the injectivity radius and volume will yield similar results.*

Corollary 1 *Let $\Sigma^n, \mathcal{D}$ be as above. If there exists $k_0 > 0$, such that $k(p) \leq k_0$, for all $p \in \Sigma^n$, then there exists a sampling of Σ^n of finite density everywhere.*

Corollary 2 *Let Σ^2 be a smooth surface. In the following cases there exist k_0 as in Corollary 1 above:*

1. *There exist H_1, H_2, K_1, K_2, such that $H_1 \leq H(p) \leq H_2$ and $K_1 \leq K(p) \leq K_2$, for any $p \in \Sigma^2$, where H, K denote the mean, respective Gauss curvature. (That is both mean and Gauss curvatures are* pinched.*)*
2. *The Willmore integrand $W(p) = H^2(p) - K(p)$ and K (or H) are pinched.*

In particular we have:

Corollary 3 *If Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density.*

2.2. Non-smooth case

In [15] there are also variations of the above theorem applicable to manifolds which are not smooth (at least on a compact subset). Before introducing this version, we will mention few necessary preliminaries about smoothing of manifolds.

Lemma 1 *For every $0 < \epsilon < 1$ there exists a $\mathcal{C}^\infty$ function $\psi_1 : \mathbb{R} \rightarrow [0,\ 1]$ such that, $\psi_1 \equiv 0$ for $|x| \geq 1$ and $\psi_1 = 1$ for $|x| \leq (1 - \epsilon)$. Such a function is called* partition of unity.

Let $c^n(\epsilon)$ be the ϵ cube around the origin in $\mathbb{R}^n$ (i.e. $X \in \mathbb{R}^n$; $-\epsilon \leq x_i \leq \epsilon$, $i = 1, ..., n$). We can use the above partition of unity in order to obtain a non-negative $\mathcal{C}^\infty$ function, ψ, on $\mathbb{R}^n$, such that $\psi = 1$ on $c^n(\epsilon)$ and $\psi \equiv 0$ outside $c^n(1)$. Define $\psi(x_1, ..., x_n) = \psi_1(x_1) \cdot \psi_1(x_2) \cdots \psi_1(x_n)$.

Theorem 2 ([11]) *Let M be a $\mathcal{C}^r$ manifold, $0 \leq r < \infty$, and $f_0 : M \to \mathbb{R}^k$ a $\mathcal{C}^r$ embedding. Then, there exists a $\mathcal{C}^\infty$ embedding $f_1 : M \to \mathbb{R}^k$ which is a δ-approximation of f_0.*

In light of the above theorem we can now proceed and present our method for sampling a signal which while considered as a manifold, fails to be smooth.

Definition 5 *Let $\Sigma^n, n \geq 2$ be a (connected) manifold of class $\mathcal{C}^0$, and let Σ^n_δ be a δ-approximation to Σ^n. A sampling of Σ^n_δ is called a δ-*sampling of Σ^n.

Theorem 3 *Let Σ^n be a connected, non-necessarily compact manifold of class $\mathcal{C}^0$. Then, for any $\delta > 0$, there exists a δ-sampling of Σ^n, such that if $\Sigma^n_\delta \to \Sigma^n$ uniformly, then $\mathcal{D}_\delta \to \mathcal{D}$ in the sense of measures, where $\mathcal{D}_\delta$ and $\mathcal{D}$ denote the densities of Σ^n_δ and Σ^n, respectively.*

Corollary 4 *Let Σ^n be a $\mathcal{C}^0$ manifold with finitely many points at which Σ^n fails to be smooth. Then every δ-sampling of a smooth δ-approximation of Σ^n is in fact, a sampling of Σ^n apart of finitely many small neighborhoods of the points where Σ^n is not smooth.*

2.3. Reconstruction

We use the secant map as defined in the beginning of this section in order to reproduce a PL-manifold as a δ-approximation for the sampled manifold.

3. Experimental Results

The geometric sampling that is discussed herein was tested in various ways in dimension 2. In [15] some synthetical results are given. Following this, the method was tested on gray scale images with results illustrated below.

One can consider a gray scale image as a surface embedded in $\mathbb{R}^3$. Having done this the image was divided into blocks of about 100×100 pixels each and the maximal principal curvature at each block was assessed using finite element techniques. Afterwards, the image was resampled according to the geometric sampling described in Section 2. The image was farther reconstructed by the secant reconstruction scheme, therein. The result are shown in Figure 1 obtaining a rather faithful reconstruction. The sampling resolution in most blocks is lower than the original resolution while being 20 times lower in about 50% of the blocks. Thus, the geometric sampling scheme gives a sparse sampling relative to the original one, whilst maintaining good reproducing. Note that the reconstructed image was not rescaled with respect to the original one, meaning that we obtained images of the same scale yet, one was much sparsely sampled regarding to the other.

Remark 2 *Recall that a digital image is far from being a smooth signal however, the described testing of the geometric sampling over images was done while no smoothing was applied prior to resampling. In the future such tests will be done while taking some smoothing of the images. Contrarily to the usual Gaussian smoothing commonly used in signal processing tasks, the authors proposed in [15], [16] to use partition of unity as a smoothing kernel instead. Another example shows in Figure* 2 *the original image of a Mandrill monkey and its reconstructed image after resampling using our geometric method. Most of the monkey faces is covered with fair which from image processing point of view is almost a white noise. Again the method achieves good approximation.*

Remark 3 *In order to get better results it is also reasonable to divide the image according to its content rather than dividing it into blocks of some, arbitrarily, predefined size. For that it is essential to first segment the image to areas of rough/mild texture and then sample each area according to the maximal curvature inside it. The rough the texture the higher the curvature should be expected.*

When accounting for color images, it is common in recent years to address color images as surfaces in $\mathbb{R}^5$ in a way similar to gray scale images with the only difference that there are three luminosity coordinates rather than one. One coordinate for each color channel. Since the geometric sampling scheme presented herein is valid in any dimension we should be able to use it also for color images. Caution should be taken here since it is known that for images the three color channels are not independent of one another and a suitable account for

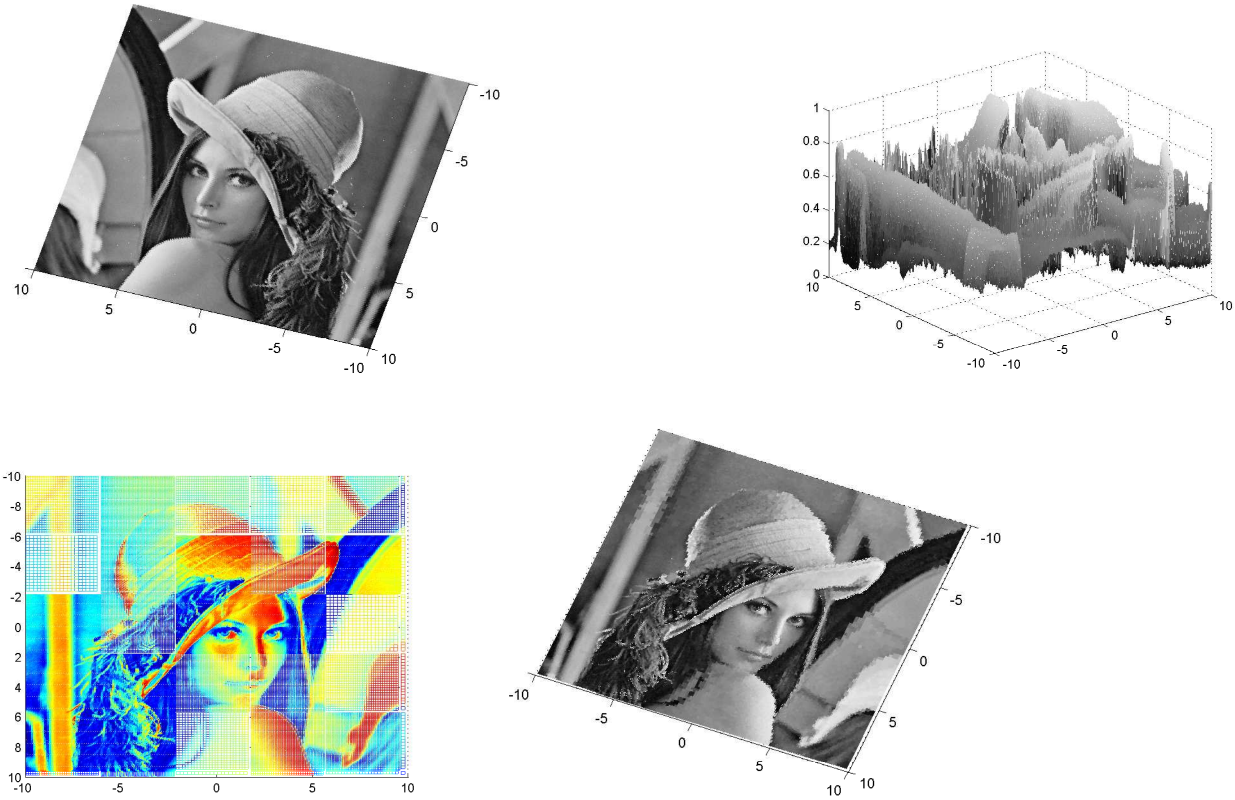

Figure 1: Geometric sampling of a gray scale image. **Top right** - original Lena, **Top left** - The original image as a surface in $\mathbb{R}^3$, **Bottom right** - Lena resampled. The white dots are the new sampling points. One can see the sparseness w.r.t the original. **Bottom left** - Lena reconstructed. Reconstruction using linear interpolation over the sampling points. No smoothing was done.

the crosstalk between the different channels is necessary. The dependant of color channels can be viewed in the following Figure 3 where computations of curvatures was done for two different channels yet, yielding similar results indicating correlations between them. Moreover, in this computations we also considered the dilatation of the two channels from being flat surfaces. Dilatations in both cases where almost the same. We refer the reader for a deep discussion on dilatation of surface mappings, to [16].

4. Summary

In this paper we presented sampling theorem for signals that can be presented as Riemannian manifolds. We also presented a simple piecewise linear method of reproducing of these signals based on the secant map. Further, we observed the implementations of these theorems for images and showed that this method can deliver a mean to sample such signals and moreover, that this geometric scheme yields a sparse sampling with respect to the usual scheme of sampling by Fourier analysis.

Few issues in this context of geometric sampling and reproducing of signals some of which where already mentioned between the lines. We will refer to some of these issues which are basically under study these days.

- As mentioned in Section 2 it would be desirable to account for other geometric measures rather than the principal curvatures. Most probable, combining together the sectional curvature with the ingectivity radius and volume of the manifold will do [2].
- Current reproducing of the signals is done based on the secant map and produces a piecewise linear approximation of the original signal. An alternative version would be to introduce geometric reproducing kernels in the sense of Shannon sampling theorem. Some kernels are being developed these days [1].

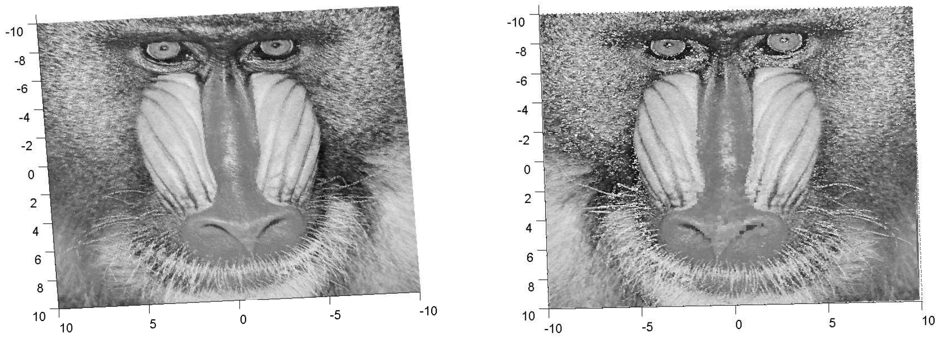

Figure 2: Mandrill: **Left** - original, **Right** - Mandrill resampled. Again, no smoothing was done.

Figure 3: Curvatures Maps: **Left** - Green channel, **Right** - Red channel. The images above show the Gaussian curvature in two different channels yet they look almost the same.

- In this paper we referred also to sampling of non-smooth signals (manifolds). The approach for sampling of these was done via smoothing. Alternatively, it should be possible to use comparison (metric) geometry in order to account for curvature measures also for non-smooth manifolds. Such measures are already known in the cases of sectional and Ricci curvatures so in order to adopt these to the context of signal processing we have to fully accomplish the replacement of principal curvatures with the sectionals [2].

- In terms of image sampling two main aspects are left for future study. One is the question of smoothing the given image before and after the geometric resampling. In our study we intend to use partition of unity as a smoothing kernel. The second question to address is, as said in Section 3, an adaptive process while performing segmentation of the image according to texture (homogenity) before sampling is done. Namely, study the variational statistics of the image so more homogeneous areas are sampled differently from less homogeneous ones.

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