Combinatorial Ricci Curvature for Image Processing

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Abstract. A new Combinatorial Ricci curvature and Laplacian operators for grayscale images are introduced and tested on $2D$ medical images. These notions are based upon more general concepts developed by R. Forman. Further applications are also suggested.

1 Introduction

Curvature analysis plays a major role in Image Processing, Computer Graphics, Computer Vision and their related fields, for many applications, such as reconstruction, segmentation and recognition, to list only a few (see, e.g. [12], [13]). Traditionally, the curvature estimation is that of a polygonal (polyhedral) mesh, approximating the ideally smooth ($\mathcal{C}^2$) image under study, such that the curvature measures of the mesh converge to the classical, differential, curvature measure of the investigated surface. For surfaces, by far the most important curvature is the *intrinsic* Gaussian (or total) curvature.

Recently, partly as an offshoot of the great interest generated by G. Perelman's seminal work on the Ricci flow and its application in the proof of Thurston's Geometrization Conjecture, and, implicitly of the Poincaré Conjecture (see, e.g. [9] for a comprehensive exposition), a flourishing of the study of various discrete versions of the Ricci flow (and similar related flows) occurred (see [2], [4], [6], [8]).

Ricci curvature measures the defect of the manifold from being locally Euclidean in various tangential directions. This is done directionally at the n-dimensional level, by appearing in the second term of the formula for the $(n-1)$-volume $\Omega(\varepsilon)$ generated within a solid angle (i.e. it controls the growth of measured angles). While sectional curvature generalizes Gaussian curvature, Ricci curvature represents an extension of mean curvature:

$$\mathbf{v} \cdot Ricci(\mathbf{v}) = \frac{n-1}{vol\left(\mathbb{S}^{n-2}\right)} \int_{\mathbf{w} \in T_p(M^n),\ \mathbf{w} \perp \mathbf{v}} K(<\mathbf{v},\mathbf{w}>)\,,$$

where $<\mathbf{v},\mathbf{w}>$ denote the plane spanned by $\mathbf{v}$ and $\mathbf{w}$, i.e. Ricci curvature represents an average of sectional curvatures. The analogy with mean curvature is further emphasized by the following remark: Ricci curvature behaves as the

Laplacian of the metric g (see, e.g. [1]). It is also important to note that in dimension $n = 2$, that is in the case that is the most relevant for classical Image Processing and its related fields, Ricci curvature reduces to sectional (and scalar!) curvature, i.e. to the classical Gauss curvature.

However, both in the more classical context, as well as in the new directions mentioned above, smooth surfaces and/or their polygonal approximations considered. Unfortunately, smooth surfaces are at best a crude model (and usually nothing but a polite fiction), as far as digital and grayscale images, i.e. the standard objects of study in Image Processing, are considered, in particular for those objects that are not "natural images", such as images produced using ultrasound imaging, MRI or CT. It would be of practical interest to define a proper notion of curvature for digital objects, in the spirit of [5]. By "proper" we mean discrete, intrinsic to the nature of the spaces under investigation, and not an approximation or rough discretization of a differential notion.

We are fortunate in our quest to be able to rely on the work of R. Forman [3] on Combinatorial Ricci curvature and the so called "Bochner Method", where he addressed this very problem in the far more general setting of weighted *cell complexes*, which represent an abstractization both of polygonal meshes and of weighted graphs. While we succinctly present some of the more general facts residing in Forman's work, in this paper we concentrate solely on the case of grayscale images with their very special combinatorics and weights, and defer the study of higher dimensional images and their curvatures and Laplacians for further study [11].

2 Forman's Combinatorial Ricci Curvature

We sketch below some of the main ideas [3]. While not wishing to become overly technical, we do have to use some technical (yet standard) definitions and notations. Due to obvious space restrictions and to avoid spuriousness we do not introduce here the basic technical notions in Algebraic Topology and Differential Geometry, and refer the reader to [10] for the former and to [1] for the later.

To generalize the notion of Ricci curvature, in a manner that would include weighted cell complexes, one starts from the following form of the *Bochner-Weitzenböck* formula (see, e.g. [1]) for the *Riemann-Laplace operator* $\Box_p$ on p-forms on (compact) Riemannian manifolds:

$$\Box_p = dd^* + d^*d = \nabla_p^*\nabla_p + Curv(R)\,, \tag{1}$$

where $\nabla_p^*\nabla_p$ is the *Bochner* (or *rough*) *Laplacian* and where $Curv(R)$ is a complicated expression with linear coefficients of the *curvature tensor* (Here ∇_p is the *covariant derivative* operator.) Of course, for cell-complexes one cannot expect such differentiable operators. However, a *formal* differential exists: in our combinatorial context (the operator) "d" being replaced by "∂" – the boundary operator of the cellular chain complex (see [10]), cells playing in this setting the role of the forms in the classical (i.e. Riemannian) one. The following definition of the combinatorial Laplacian is now natural:

$$\Box_p = \partial\partial^* + \partial^*\partial\,, \tag{2}$$

where ∂^* is the *adjoint* (or *coboundary*) operator of ∂, defined by: $< \partial_{p+1}c_{p+1}, c_p > = < c_{p+1}, \partial_p^* c_p >_{p+1}$, where $< \cdot,\cdot > = < \cdot,\cdot >_p$ is a (positive definite) inner product on $C_p(M,\mathbb{R})$, i.e. satisfying: (i) $< \alpha, \beta > = 0, \forall \alpha \neq \beta$ and (ii) $< \alpha, \alpha > = w_\alpha > 0$ – the weight of cell α.

Forman [3] shows that an analogue of the Bochner-Weitzenböck formula holds in this setting, i.e. that there exists a canonical decomposition of the form:

$$\Box_p = B_p + F_p \tag{3}$$

where B_p is a *non-negative operator* and F_p is a certain diagonal matrix. B_p and F_p are called, in analogy with the classical Bochner-Weitzenböck formula, the *combinatorial Laplacian* and *combinatorial curvature function*, respectively. Moreover, if $\alpha = \alpha^p$ is a p-dimensional cell (or p-cell, for short), then we can define the *curvature functions*:

$$\mathcal{F}_p = < F_p(\alpha), \alpha >, \tag{4}$$

F_p being regarded as a linear function on p-chains. For dimension $p = 1$ we obtain, by analogy with classical case, the following definition of discrete (weighted) *Ricci curvature*:

Definition 1. *Let $\alpha = \alpha^1$ be a 1-cell (i.e. an edge). Then the* Ricci curvature *of α is defined as:*

$$\mathrm{Ric}(\alpha) = \mathcal{F}_1(\alpha). \tag{5}$$

While general weights are possible, making the combinatorial Ricci curvature extremely versatile, it turns out (see [3], Theorem 2.5 and Theorem 3.9) that it is possible to restrict oneself only to so called *standard weights*:

Definition 2. *The set of weights $\{w_\alpha\}$ is called a* standard set of weights *iff there exist $w_1, w_2 > 0$ such that given a p-cell α^p, the following holds:*

$$w(\alpha^p) = w_1 \cdot w_2^p$$

(Note that the combinatorial weights $w_\alpha \equiv 1$ represent a set of standard weights, with $w_1 = w_2 = 1$.) Using standard weights we obtain the following formula:

$$\mathrm{Ric}(\alpha^p) = w(\alpha^p)\left[\left(\sum_{\beta^{p+1}>\alpha^p}\frac{w(\alpha^p)}{w(\beta^{p+1})} + \sum_{\gamma^{p-1}<e_2}\frac{w(\gamma^{p-1})}{w(\alpha^p)}\right)\right. \tag{6}$$

$$\left. - \sum_{\alpha_1^p\|\alpha^p,\alpha_1^p\neq\alpha^p}\left|\sum_{\beta^{p+1}>\alpha_1^p,\beta^{p+1}>\alpha^p}\frac{\sqrt{w(\alpha^p)w(\alpha_1^p)}}{w(\beta^{p+1})} - \sum_{\gamma^{p-1}<\alpha_1^p,\gamma^{p-1}<\alpha^p}\frac{w(\gamma^{p-1})}{\sqrt{w(\alpha^p)w(\alpha_1^p)}}\right|\right],$$

where $\alpha < \beta$ means that α is a face of β, and the notation $\alpha_1 \parallel \alpha_2$ signifies that the simplices α_1 and α_2 are *parallel*, the notion of parallelism being defined as follows:

Definition 3. *Let $\alpha_1 = \alpha_1^p$ and $\alpha_2 = \alpha_2^p$ be two p-cells. α_1 and α_2 are said to be* parallel *($\alpha_1 \parallel \alpha_2$) iff either: (i) there exists $\beta = \beta^{p+1}$, such that $\alpha_1, \alpha_2 < \beta$; or (ii) there exists $\gamma = \beta^{p-1}$, such that $\alpha_1, \alpha_2 > \gamma$ holds, but not both simultaneously.*

For example, in Fig. 1, e_1, e_2, e_3, e_4 are all the edges parallel to e_0.

Together with the formula above, the (dual) formula for the combinatorial Laplacian (see [3]) is also obtained to be:

$$\Box_p(\alpha_1^p, \alpha_2^p) = \sum_{\beta^{p+1} > \alpha_1^p, \beta^{p+1} > \alpha_2^p} \epsilon_{\alpha_1,\alpha_2,\beta} \frac{\sqrt{w(\alpha_1^p) w(\alpha_2^p)}}{w(\beta^{p+1})} + \sum_{\gamma^{p-1} < \alpha_1^p, \gamma^{p-1} < \alpha_2^p} \epsilon_{\alpha_1,\alpha_2,\gamma} \frac{w(\gamma^{p-1})}{\sqrt{w(\alpha_1^p) w(\alpha_2^p)}}, \tag{7}$$

where $\epsilon_{\alpha_1,\alpha_2,\beta}, \epsilon_{\alpha_1,\alpha_2,\gamma} \in \{-1, +1\}$ depend on the relative orientations of the cells.

3 Combinatorial Ricci Curvature of Images

Before developing the relevant formulae in the special combinatorial setting of the tilling by squares of the plane, as it is usually considered in (Discrete) Image Processing, let us first underline that it is advantageous to use standard weights. The natural such weights are proportional to the geometric content (s.a. length and area). It follows that the weight of any vertex is $w(v) = 0$. Bearing this in mind, and considering the combinatorics of the square tilling (see Fig. 1) the specific form of Combinatorial Ricci curvature for $2D$ images easily follows:

$$\mathrm{Ric}(e_0) = w(e_0) \left[\left(\frac{w(e_0)}{w(c_1)} + \frac{w(e_0)}{w(c_2)} \right) - \left(\frac{\sqrt{w(e_0) w(e_1)}}{w(c_1)} + \frac{\sqrt{w(e_0) w(e_2)}}{w(c_2)} \right) \right]. \tag{8}$$

For the Laplacian there exists more than one possible choice, depending upon the dimension p. The simplest, and operating on cells of the same dimensionality as the Discrete Ricci curvature, is $\Box_1$. Because vertices have weight 0 and adjacent cells have opposite orientations, Equation (7) becomes, in this case (using the notation of Fig. 1):

$$\Box_1(e_0) = \Box_1(e_0, e_0) = \frac{w(e_0)}{w(c_1)} - \frac{w(e_0)}{w(c_2)}. \tag{9}$$

Instead of computing a Laplacian *along* the edge e_0, one can compute a Laplacian operating *across* the edge, namely $\Box_2(c_1, c_2)$. Since no 3-dimensional cells exist, the first sum in Equation (7) vanishes, hence we have (up to sign):

$$\Box_2(c_1, c_2) = \frac{w(e_0)}{\sqrt{w(c_1) w(c_2)}}. \tag{10}$$

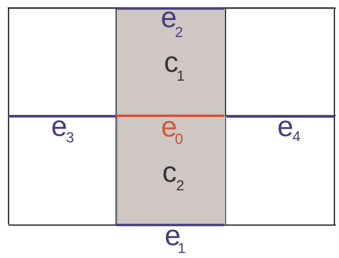

Fig. 1. The elements appearing in the computation of the Combinatorial Ricci curvature of edge e_0 in the geometry of pixels.

4 Experimental Results

Before commencing any experiments with the combinatorial Ricci curvature in the context of images, we had to choose a set of weights for the 2-, 1- and 0-dimensional cells of the picture, that is for squares (pixels), their common edges and the vertices of the tilling of the image by the pixels. Any such choice should, obviously, be as natural and expressive as possible for image analysis. The choice of weights was motivated by two factors: the context of Image Processing, where a natural choice for weights imposes itself (see below) and the desire (and, indeed, sufficiency, see Section 2) to employ solely standard weights.

Since natural weights have to be proportional to the dimension of the cell, it follows immediately that the weight of any vertex (0-cell) has to be 0. Moreover, in the beginning, it is natural to choose $w_1 = 1$ and $w_2 =$ length of a cell. A somewhat less arbitrary choice for the length (i.e. basic weight) of an edge, would be Length(e) = (dimension of the picture)$^{-1}$, hence that for the area (i.e. basic weight) of a pixel α being $\mathrm{Area}(\alpha) = (\text{dimension of the picture})^{-2}$. The proper weight for a cell α should, however, take into account the gray-scale level (or height) h_α of the pixel in question, i.e. $w_\alpha = h_\alpha \cdot \mathrm{Area}(\alpha)$. This will become, we hope, clearer in the following paragraph. The natural weight for an edge e common to the pixels α and β is $|h_\alpha - h_\beta|$. (A less "purely" combinatorial choice of cells and weights is discussed in [11].) Note that, as given an edge e, the Ricci curvature $\mathrm{Ric}(e)$ represents a kind of generalized mean of the weights the cells *parallel* to e. Therefore, it represents a measure of flow in the direction *transversal* to e. It follows, that, contrary perhaps to intuition, this type of Ricci curvature (and the Bochner Laplacian associated to it) in direction, say, parallel to the x-axis, is suited for the detection of edges and ridges in the y-direction. On the other hand, since scalar (i.e. Gauss) curvature, is associated to each pixel, that is to each square of the tessellation, to compute the Gaussian curvature one has to compute the arithmetic mean of the Ricci curvatures of edges of the square under consideration – see Fig. 2. (A similar argument holds if one wishes to compute the 1-Laplacians, $\Box_1$ and B_1, of a given pixel.)

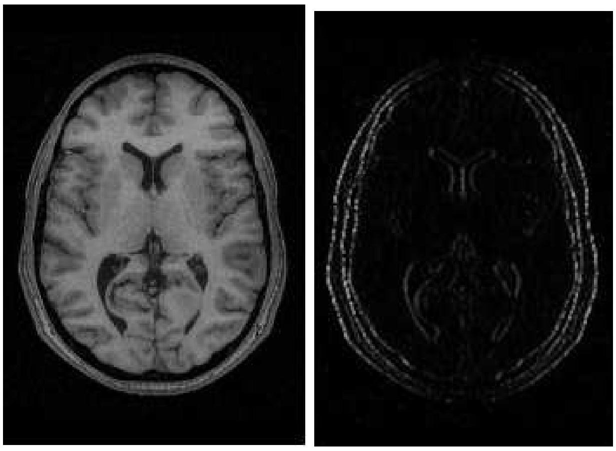

Fig. 2. Horizontal brain scan image (left) and its Combinatorial Ricci curvature (right).

As Fig. 3 illustrates, the combinatorial Ricci curvature we introduce allows, even for a non-optimal choice of weights, a very good approximation of Gaussian curvature of surfaces (i.e. for gray-scale images). Here, classical Gaussian curvature was computed using finite element methods standard in Image Processing – see [12]. (In order to obtain a clearer, more detailed comparison of the two curvatures, we illustrate the performance of the algorithm on a non-medical image.) In contrast, both the Bochner (and Riemann) Laplacian sharply diverge

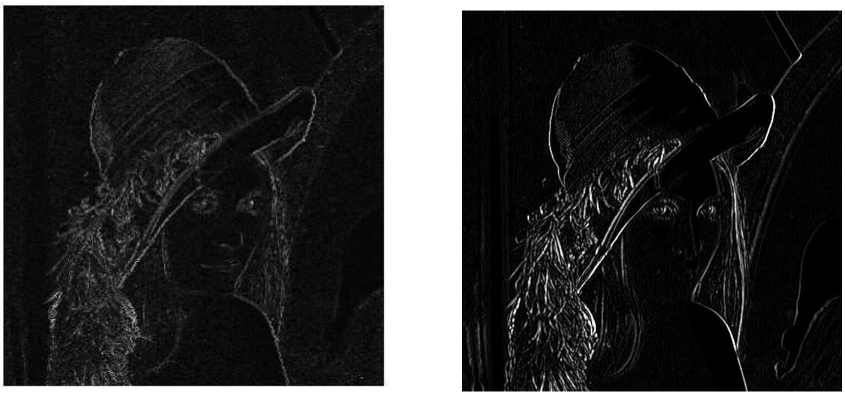

Fig. 3. Gauss (left) and Combinatorial Ricci (right) comparison.

from the classical one (see Fig. 4). This is not too surprising, given the combinatorial definition considered and the somewhat schematic weights employed. However, it is probable that the more geometric scheme suggested in Section 2 will produce better results. However, the Bochner Laplacian proves to be an excellent detector of "sharp" edges, therefore it may prove to be useful for contour detection and for segmentation.

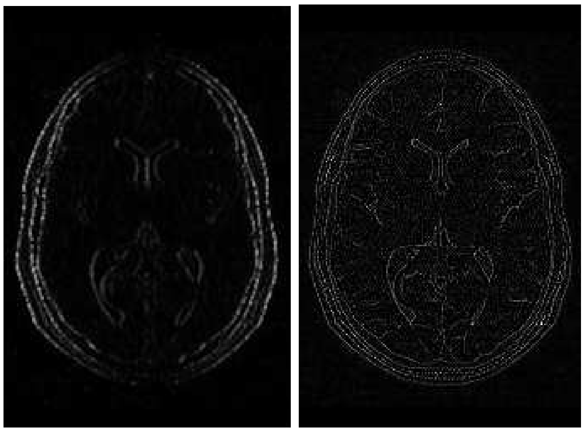

Fig. 4. Comparison of Different Laplacians: From left to right: the Bochner (rough) Laplacian $B_1(e_0)$ (left), and the Matlab Laplacian (right). Note that the combinatorial Laplacian is a better detector of "sharp" edges.

5 Future Work

We briefly discuss below some of the natural and/or seemingly required directions of further study:

1. Evidently, the first task in testing the efficiency of the combinatorial Ricci curvature and Laplacian in medical imaging is to experiment with voxels, that is to apply the apparatus introduced herein to the analysis of volumetric data. Such 3- (and even 4-) dimensional manifolds and their evolution in time is most relevant, for instance, in the analysis of cardiac MRI. This brings us to the following point:
2. As already mentioned in the introduction, the full power of the Ricci curvature reveals itself in the general heat-type diffusion setting and discrete versions of the Ricci flow (and other related flows) were introduced and experimented with ([2], [4], [6], [8])). It is only natural to strive to develop and experiment with a discrete version of the Ricci flow corresponding to the combinatorial Ricci curvature introduced herein. Indeed, such work is actually in progress [11].

3. As already noted, while we prefer, both for theoretical as well as for practical reasons, to work with standard weights, Forman's combinatorial version of Ricci curvature is extremely versatile. Even if restricting oneself to using standard weights, i.e. proportional to the p-dimensional geometric content (p-volume), one still has freedom in choosing the weights w_2 and especially w_1 (see Section 2). Hence, to obtain best result, one can experiment in order to empirically determine, by using, e.g. variational methods, the optimal standard weights for a given application of the method.
4. The Combinatorial Laplacian is closely connected (by its very definition) to the cohomology groups of the cellular complex on which it operates (see [3]). It is natural, therefore, to seek to apply the results and methods of [3] for the estimation of the dimension, and in some cases even the computation, of the cohomology groups (and by duality, of homology groups) of images. (See [7] on this direction of study in Image Processing.)

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