Geometric Flow Methods for Image Processing

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Geometric flow methods are based upon the study of thee physically motivated PDE's: the heat, wave and Schrödinger equations.

In turn, they all involve the **Laplacian** operator: $f \mapsto \triangle f$

$$\triangle f = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}$$

Here we consider first the classical, simple context of $f : D \longrightarrow \mathbb{R}$, where D is a bounded planar domain.

The Heat Equation

$$\triangle f = -\frac{1}{\nu}\frac{\partial f}{\partial t}$$

where $f(x, y, t)$ is the temperature and ν represents the conductivity of the material.

It models the evolution (spreading) of heat, at time t, given an initial distribution at $t_0 = 0$.

The Wave Equation

Obtained by considering a cylinder over D and pouring a thin layer of water over D. Then the wave equation at hight $f(x,y,t)$, after the time t, over the point $(x,y) \in D$ is:

$$\triangle f = -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Where c represents the speed of sound in the fluid.

Remark 1 *The wave equation is the same as the vibrating membrane equation, which describes the normal motions of the membrane (or "drum") D.*

The Schrödinger Equation

This is the equation of a free particle

$$\frac{\hbar^2}{2m} = i\hbar\frac{\partial f}{\partial t}$$

where $I^2 = -1$, $\hbar$ is the Planck constant and m is the mass of the particle.

Remark 2 *For the mathematical study we may presume all the constants in the equations above are* 1 *(by rescaling* t*).*

We shall concentrate on the heat equation.

However, while we shall not discuss Schrödinger equation, we briefly compare the heat and wave equations (especially from the Image Processing viewpoint):

Heat Equation vs. Wave Equation

• The wave equation **much harder** to study, in any dimension.

Moreover:

• Drastic difference of waves behavior, depending on the **parity** of the dimension of the space.

The heat equation has also the following great advantage:

- It **smoothes** the curves.

In contrast with this behaviour

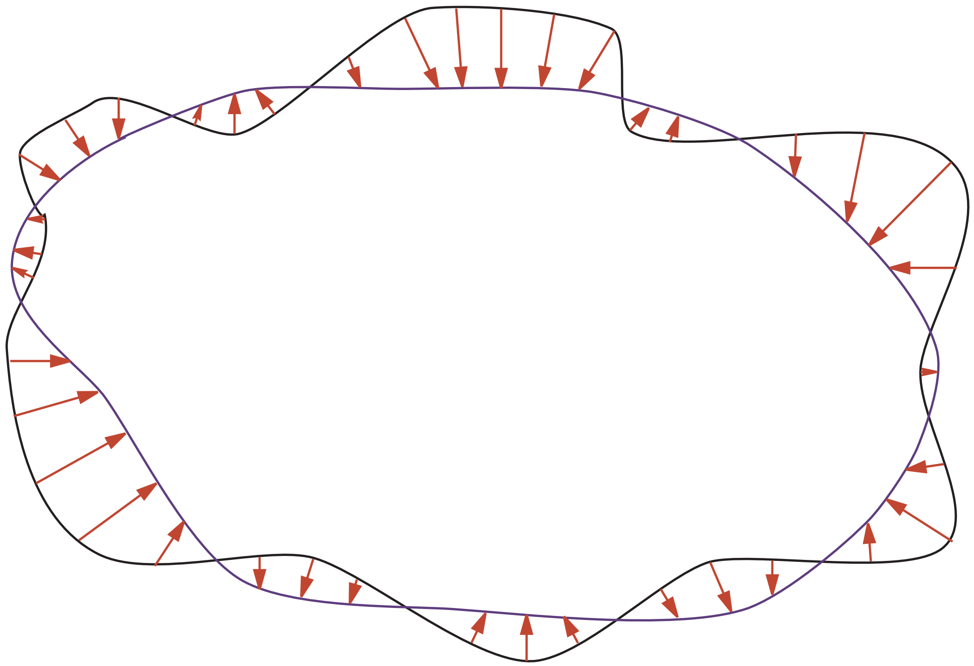
- The wave equation (solution) preserves the discontinuities (of initial data and parameters).

As we already stated, we concentrate on the heat equation, and we start by underlining its fundamental geometric nature.

Start with the following practical, day-to-day phenomenon: if one drops liquid on a peace of hot metal plate, it shrinks (while evaporating) and its evolution is governed – of course – by the heat equation.

However, it can be shown that heat flow is in fact equivalent to the *curvature flow* or *curve-shortening flow*:

Given a simple closed smooth* curve in the plane, we let each point of the curve move in normal direction with velocity† equal to the curvature at that point.



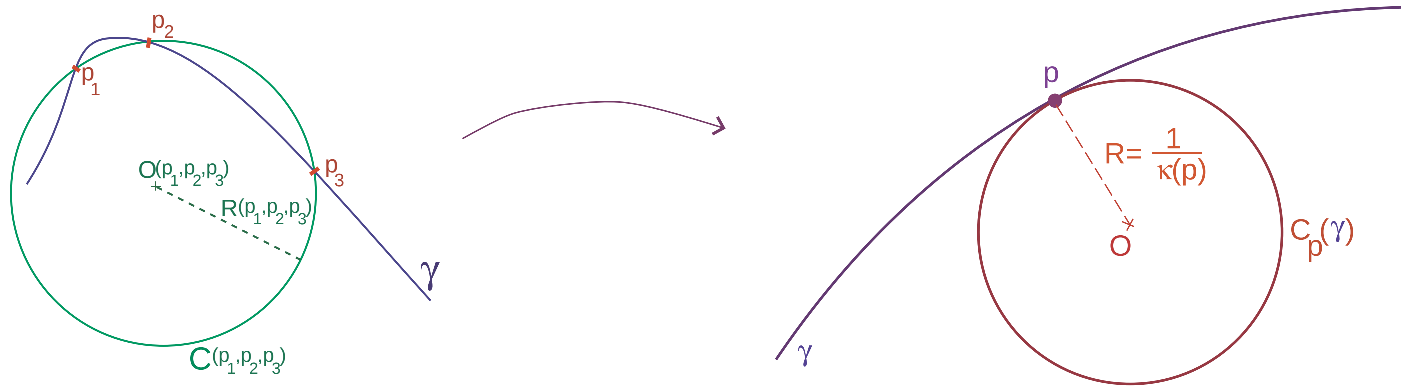
*i.e. at least $\mathcal{C}^2$

†meaning direction, too!

We shall shortly explore the proprieties of the curvature flow, but first we have to define curvature:

Informally, the curvature of the curve at the point p is the curvature of the "best fitting" circle to c at p.

More precisely, we define the **osculatory circle** as the limit of circles that have 3 common points with the curve.



Formally, if $\gamma \subset \mathbb{R}^2$ is the image of the function $c : I = [0,1] \to \mathbb{R}^2$, then the osculatory circle at $\gamma_0 = c(t_0)$ is defined as:

$$C(\gamma_0) = C_\gamma(\gamma_0) = \lim_{\gamma_1,\gamma_2 \to \gamma_0} C(\gamma_0,\gamma_1,\gamma_2) = \lim_{t_1,t_2 \to t_0} C(t_0,t_1,t_2)\,;\gamma_i = \gamma(t_i)\,, i = 1,2\,.$$

The *curvature* $\kappa_\gamma(\gamma_0)$ of γ at γ_0 is defined to be as $1/R\big(C(\gamma_0)\big)$, where $R\big(C(\gamma_0)\big)$ is the radius of $C(\gamma_0)$.

It can be shown that if the parameterization $c : I \to \mathbb{R}^2$ is of class $\mathcal{C}^2$ (or greater), then one can compute the curvature by the following formula:

$$\kappa_\gamma(t) = \frac{||c'(t) \times c''(t)||}{||c'(t)||^3}.$$

Moreover, if c is parameterized by arc-length (e.g it has unit speed), then:

$$\kappa_\gamma(t) = ||c''(t)||.$$

Remark 3 *Even if important for proofs in Differential Geometry this last formula (and even the previous one) has no relevance whatsoever in Image Processing!...*

We can now return and present some properties of the curvature flow. We begin with the one promised by the name:

- Shortening

This is a natural idea: the regions of higher curvature have to be “adjusted” more.

It follows immediately from the equation describing the evolution of curves under this flow is:

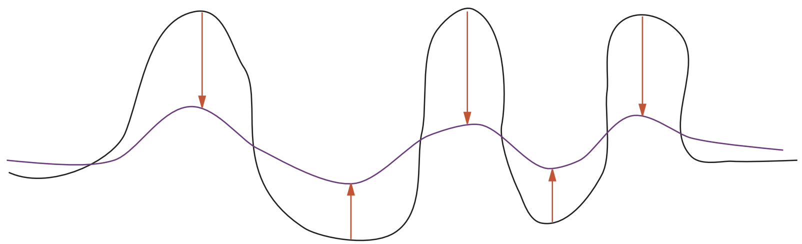
$$\frac{d\text{length}}{dt} = -\int \kappa^2 ds$$

Here ds denotes arclength.

Remark 4 *The same equation holds for the evolution of curves on surfaces (since no term depending on the geometry of the surface appears in the equation above).*

- Smoothing

Even if the initial curve is only $\mathcal{C}^2$, it instantaneously becomes $\mathcal{C}^\infty$ (in fact even real analytic).



This is a consequence of the fact that we are face with a parabolic PDE.

Remark 5 *The time of smoothness may be short. Indeed, singularities may – and* **will** *develop.*

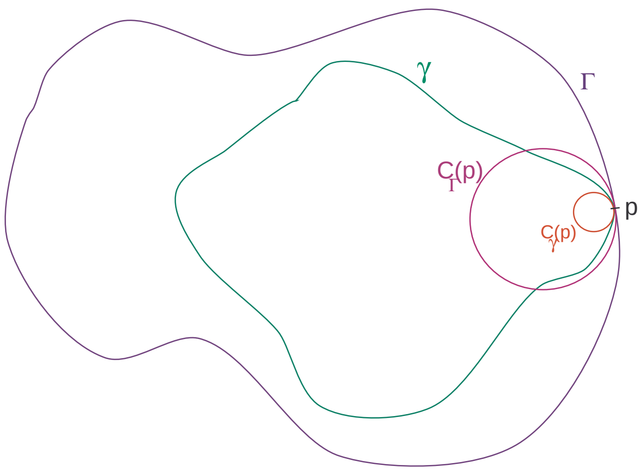
- Collision-freeness

i.e. two initially disjoint curves remain disjoint.

Proof

First, note that is sufficient to study the case when the curves are one inside the other.

Note that at the first time of contact t_0 the curves must be tangential. Then, the curvature of the iner curve is grater then of the exterior one.



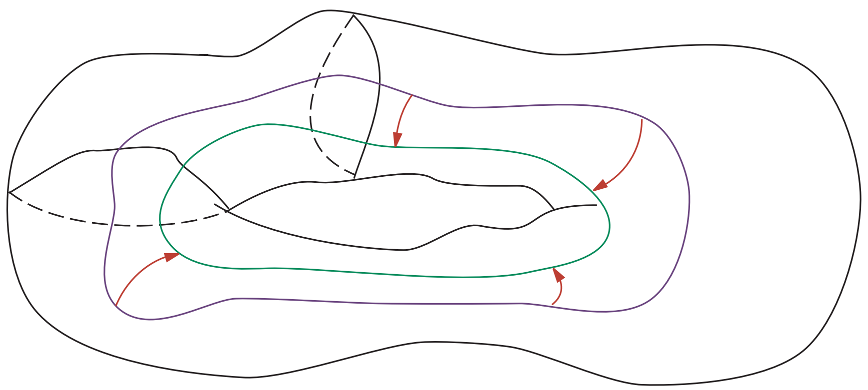
It follows that the inner curve is moving faster than the exterior one.

Hence, at some $t_0 - \varepsilon$ the curves should have intersected, in contradiction to the choice of t_0.

- Embedded curves remain embedded

i.e. planar curves with no inflection points will develop no new inflections under the curvature flow.

(and, given a curve on a surface, it will remain on the surface while evolving by the curvature flow.)



- Finite lifespan

Proof

Every (simple) closed planar curve is contained in the interior of a circle, hence it evolves faster then the circle (see argument above!)

But the circle collapses in fine time, therefore so does the curve.

- “Convex curves shrink to round points”

Formally, this represents the following deep theorem:

Theorem 6 (Gage-Hamilton, 1986) *Under the curvature flow a convex curve remains convex and shrinks to a point. Moreover, it becomes asymptotically circular, i.e. if the evolving curve is rescaled such that the enclosed area is constant, then the rescaled curve converges to a circle.*

Note that this (hard to prove) theorem justifies our intuition regarding the evolution of a liquid drop on a hot plate.

- “Embedded curves become convex”

This is theorem of Matt Greyson:

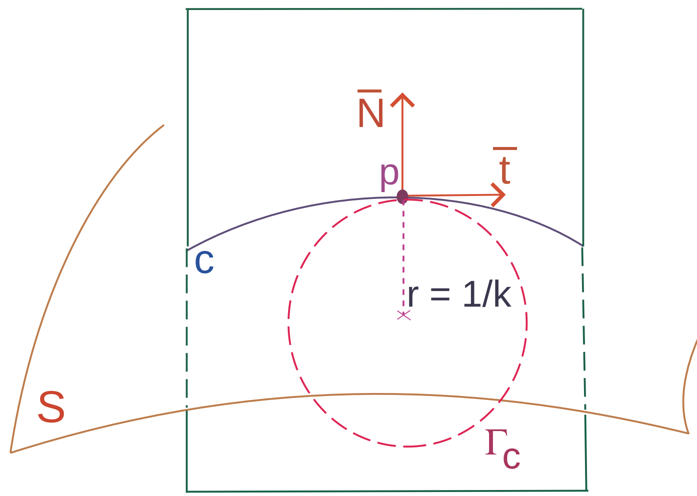
Theorem 7 (Greyson, 1987) *Under the curvature flow embedded curves become convex.*

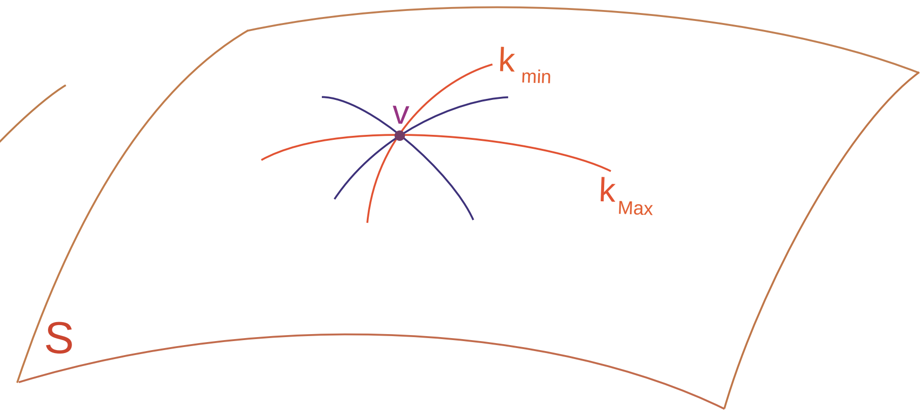
Corollary 8 *Under the curvature flow embedded curves shrink to round points*

We now pass to the next dimension, i.e. we study the geometric version of heat flow on surfaces.

The role of curvature is taken by the mean curvature H (or by the mean curvature vector $\vec{H} = H\vec{n}$ (where $\vec{n}$ denotes the normal to the surface) and where:

Definition 9 *The mean curvature of the surface S at the point p is defined as: $H = H(p) = \frac{1}{2}\Big(k_{min}(p) + k_{Max}(p)\Big)$, where principal curvatures of S at p.*





Almost all the properties of the curvature flow of planar curves extend to the case of mean curvature flow of surfaces:

- Surfaces become smoother (for a short time)

- Area decreases

Remark 10 *The mean curvature flow may be regarded as gradient flow for the area functional.*

- Disjoint surfaces remain disjoint

- Embedded surfaces remain embedded

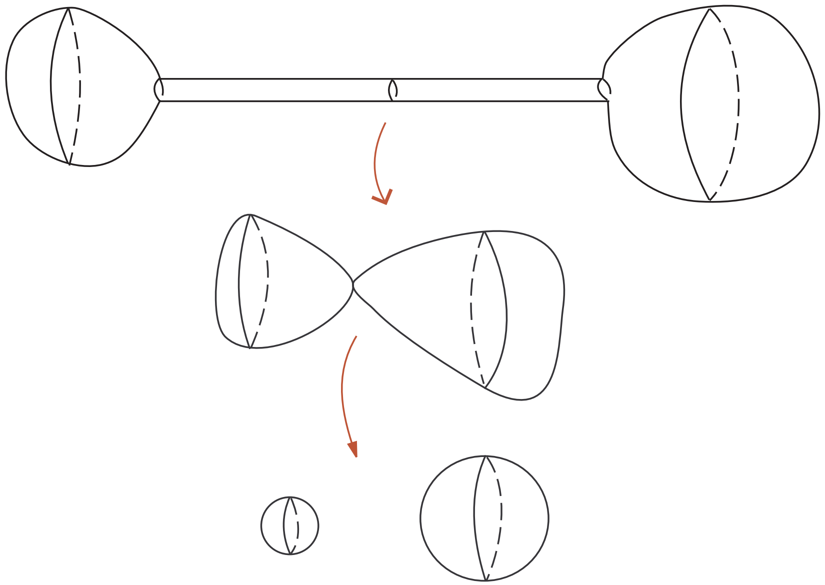
- Compact surfaces have finite lifespans

- The analog of the Gage-Hamilton theorem holds

More precisely, we have the following

Theorem 11 (Huisken, 1984) *Under the mean curvature flow, compact convex surfaces shrink to round points.*

However, the analog of Grayson's theorem is **false**! − See counterexample below:



All these results automatically extend to n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$.

In the general case we have n principal curvatures $k_1, \ldots, k_n$ and the mean curvature is defined (of course) as $H = \frac{1}{n}(k_1 + \cdots + k_n)$.

However, hypersurfaces represent a very particular case and the geometric meaning of the mean curvature is rather limited.

So ... what to do?

Answer: Back to Surfaces for some more insight!...

Note that since for a surface there exist two principal curvatures (at any point) one can construct another symmetric polynomes in these curvatures:

Definition 12 $K = K(p) = k_{min}k_{Max}$ *is called the Gaussian (or total) curvature (of* S *at the point* p*).*

Remark 13 *Gaussian curvature is intrinsic, i.e. it depend solely upon the inner geometry of the surface and not on its embedding (i.e. "drawing") in* $\mathbb{R}^3$*. This is in contrast to mean curvature which is extrinsic, since it depend upon the normals, hence on the embedding.*

Remark 14 *The original definition given by Gauss was much more geometrical. we shall not bring it here, instead we give other geometric characterizations, based upon the circumference (respective area) of a small circle on a surface (or geodesic circle of* K*, that will help us further on:*

Theorem 15 (Bertrand-Diguet-Puiseaux – 1848) Let S be a surface in $\mathbb{R}^3$, $p \in S$ and let $\varepsilon > 0$. Denote by $C(p, \varepsilon)$, $B(p, \varepsilon)$ the geodesic circle, respective the geodesic ball of center p and radius $\varepsilon > 0$. Then:

$$length\, C(p,\varepsilon) = 2\pi\varepsilon - \frac{\pi}{3} K(p)\varepsilon^3 + o(\varepsilon^3)\,,$$

and

$$area\, B(p,\varepsilon) = \pi\varepsilon^2 - \frac{\pi}{12} K(p)\varepsilon^4 + o(\varepsilon^4)\,.$$

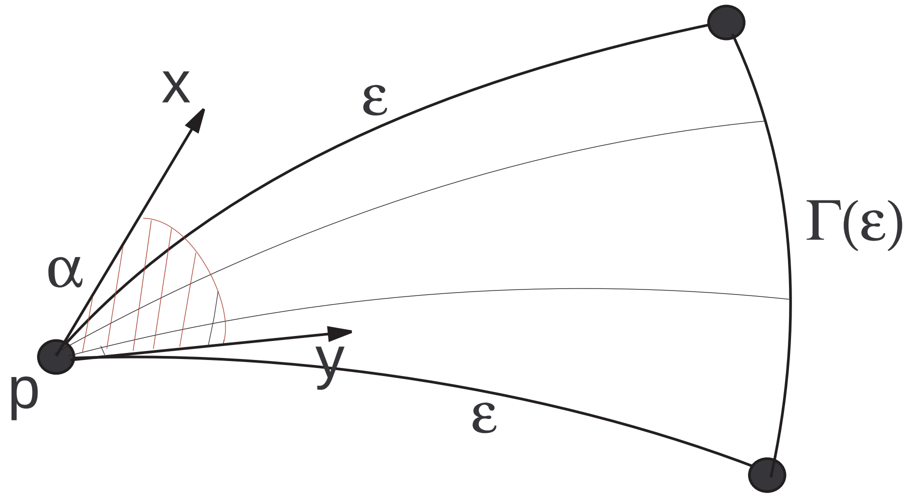
Hence:

$$K(p) = \lim_{\varepsilon\to 0} \frac{3}{\pi} \frac{2\pi\varepsilon - length\, C(p,\varepsilon)}{\varepsilon^3} = \lim_{\varepsilon\to 0} \frac{12}{\pi} \frac{\pi\varepsilon^2 - area\, B(p,\varepsilon)}{\varepsilon^4}$$

However, in higher dimension there are n principal curvatures, and the symmetric polynomes do not convey enough (and simple enough) geometric information

The basic idea is to look at the Gaussian curvatures of **all** 2-dimensional sections. Even after symmetries reduce this to "only" $n^2(n^2-1)/12$, there are clearly to many numbers to deal with and a simple, geometric interpretation is highly needed. Such an interpretation is given by the Bertrand-Diguet-Puiseaux formula (on every 2-dimensional section).

Thus **sectional curvature** $K = K(\mathbf{x}, \mathbf{y})$ measures the defect of M^n from being locally Euclidean. This is done at the 2-dimensional level, by appearing in the second term of the formula for the arc length of an infinitesimal circle.



Remark 16 *In dimension $n = 2$ sectional curvature reduces to the "ordinary" Gaussian curvature (as the notation emphasizes!...)*

Remark 17 K *behaves like a second derivative (or as a Hessian) of the metric* g *of the manifold.*

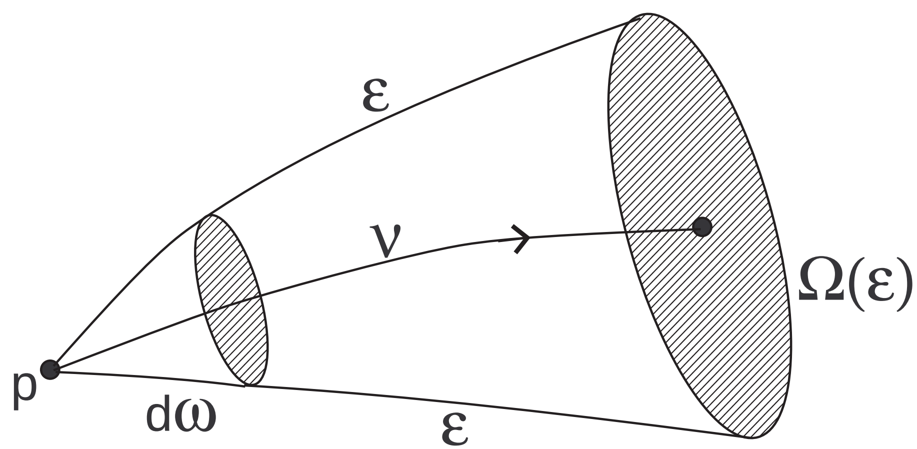
Remark 18 *Sectional curvature is intrinsic (as expected from a generalization of Gaussian curvature).*

Precisely like in the 2-dimensional case the problem resides in the abundance of directions (hence sectional curvatures). In fact, for $n > 4$ one has no way of even determining the 2-dimensional directions (planes) for which the **minimal** and **maximal** sectional curvatures are attained.

However, if one is willing to restrict himself to the scope of "testing", another extremely useful notion of curvature presents itself:

Ricci Curvature

The Ricci curvature measures the defect of the manifold from being locally Euclidean in various tangential directions. This is done **directionally** at at the n-dimensional level, by appearing in the second term of the formula for the $(n-1)$-volume $\Omega(\varepsilon)$ generated within a solid angle (i.e. it controls the growth of measured angles).



More precisely, we have the following analogue of of the (first) Bertrand-Diguet-Puiseaux formula:

Theorem 19

$$\mathrm{Vol}\Big(\Omega(\alpha)\Big) = d\omega \cdot \varepsilon^{n-1}\left(1 - \frac{Ricci(\mathbf{v})}{3}\varepsilon^2 + o(\varepsilon^2)\right).$$

Here $d\omega$ denotes the n-dimensional solid angle in the direction of the vector $\mathbf{v}$, $\Omega(\alpha)$ the $(n-1)$-volume generated by geodesics of length ε in $d\alpha$, and $Ricci(\mathbf{v})$ the Ricci curvature in the direction $\mathbf{v}$.

While sectional curvature generalizes Gaussian curvature, Ricci curvature represents an extension of **mean curvature**:

$$\mathbf{v} \cdot Ricci(\mathbf{v}) = \frac{n-1}{vol\left(\mathbb{S}^{n-2}\right)} \int_{\mathbf{w} \in T_p(M^n),\ \mathbf{w} \perp \mathbf{v}} K(< \mathbf{v}, \mathbf{w} >)\,,$$

where $< \mathbf{v}, \mathbf{w} >$ denote the plane spanned by $\mathbf{v}$ and $\mathbf{w}$, that is Ricci curvature represents an average of sectional curvatures.

The analogy with mean curvature is further emphasized by the following remark:

Remark 20 *The Ricci curvature behaves as the Laplacian of the metric g.*

Moreover:

Remark 21 *A generalization of Ricci curvature to the k-dimensional case, ($2 \leq k < n$) is also feasible.*

If one is willing to restrict even more the number of "testing directions", yet relevant another notion of curvature is available: Scalar Curvature $\mathrm{scal}(p)$. It measures the defect of the manifold from being locally Euclidean at the level of volumes of small geodesic balls.

Again, this can be formulated precisely, as yet another suitable version of the Bertrand-Diguet-Puiseaux formula:

$$\mathrm{Vol}\, B(p,\varepsilon) = \omega_n \cdot \varepsilon^n \left(1 - \frac{1}{6(n+2)}\mathrm{scal}(p)\varepsilon^2 + o(\varepsilon^2)\right) .$$

(Here ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.)

The following mean property of scalar curvature also holds:

$$\text{scal} = 2 \sum_{1 \le i < j \le n} K(< \mathbf{x}_i, \mathbf{x}_j >)$$

Remark 22 *As suggested by the name, scalar curvature is a* **scalar**, *not a* **tensor**, *like sectional and Ricci curvatures.*

We can now return to the heat equation and explore more of its properties, especially those connected to curvature*. But first we have to recall some basic facts (that generalize classical, 2-dimensional ones):

Let M^n be a compact Riemannian manifold. Given any initial data $f : M^n \to \mathbb{R}^n$, the solution of the heat equation:

$$\triangle F = -\frac{\partial F}{\partial t},$$

such that

$$F(x, 0) = f(x),$$

*Or rather **curvatures**!...

there exists a $\mathcal{C}^\infty$ function $K : M^n \times M^n \times \mathbb{R}^*_+ \rightarrow \mathbb{R}$, such that

$$F(x) = \int_M K(x, y, t) f(y) dy \,.$$

The function K is given by the convergent series

$$K(x, y, t) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

Where the functions ϕ_i are eigenfunctions of the Laplace operator $\boldsymbol{\Delta}$, chose such that they form an **orthonormal basis** of $L^2(M^n)$.

Moreover, for every $x \in M^n$, there exists an asymptotic expansion as $t \to \infty$, of the form:

$$K(x, y, t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} u_k(x) t^k \,,$$

where $u_k : M^n \to \mathbb{R}$ are (loosely speaking) functions of the **geometry** of M^n at the point x.

Definition 23 *The function K is called the fundamental solution of the heat equation or (more commonly)* ***the heat kernel*** *of M^n.*

The physical interpretation of the heat kernel is as follows: $K(x, y, t)$ is the temperature a time t and at point y, when a unit of heat (i.e. a Dirac δ function) is placed at the point x.

Surprisingly, the following symmetry property holds:

$$K(x, y, t) = K(y, x, t)$$

Remark 24 *The boundary does not have any influence (i.e. "the particles are not aware of the could outside").*

For the special case $M^n = R^n$ we have the following explicit formula:

$$K(x,y,t) = \frac{1}{(4\pi t)^{d/2}} e^{-(d(x,y))^2/4t}.$$

Remark 25 *The equation above implies the highly non-intuitive fact that heat diffuses* **instantaneously** *(i.e. with infinite speed) from any point.*

We are now ready to explore the connections between the heat kernel and the geometry of the manifold M^n. We begin with the following immediate observation:

By integrating $K(x,y,t) = \sum e^{-\lambda_i}\varphi_i(x)\varphi_i(y)$, over the the manifold M^n and using the asymptotic expression of K we obtain the following basic formula (the Herman Weyl estimate):

$$\sum e^{-\lambda_i} \sim \frac{1}{(4\pi t)^{d/2}} \mathrm{Vol}(M^n)$$

when $t \to \infty$.

However, we can do better and obtain geometric expressions for the terms in the asymptotic expression of K (at least for the first ones). We begin with the more “human” of these:

$$u_1(x) = \frac{1}{6}\mathsf{scal}(x)\,.$$

By integrating one obtains:

$$\int_{M^n} = \frac{1}{6}\int_{M^n}\mathsf{scal}(x)\,.$$

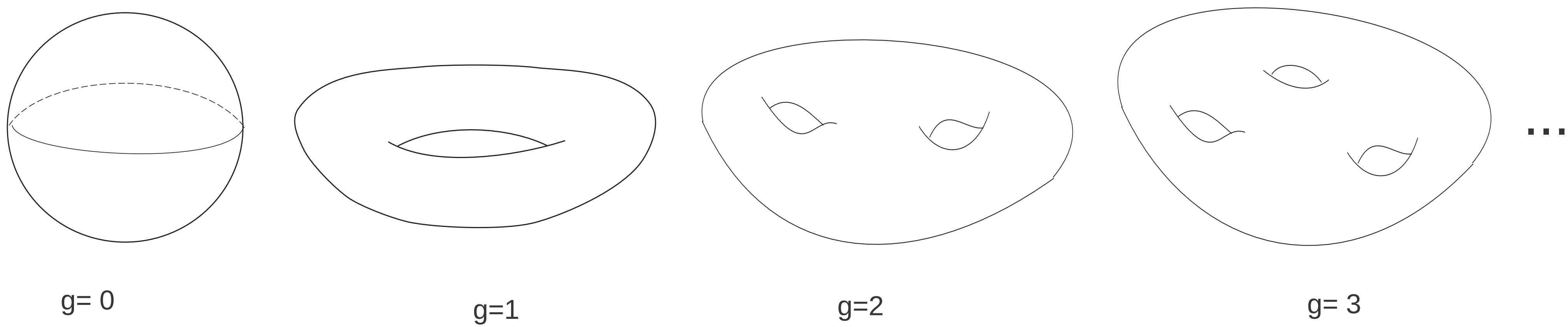
In the special case of surfaces (i.e. $n = 2$) scalar curvature reduces to Gaussian curvature, hence, by the glorified Gauss-Bonnet theorem:

$$\int_{M^2} \mathrm{scal}(x) = 2\pi\chi(M^2)\,.$$

where $\chi(M^2)$ denotes (as usual) the Euler characteristic of M^2:

$$\chi(M^2) = 2 - 2g\,,$$

where g denotes (as always) the genus of M^2.


g= 0
g=1
g=2
g= 3
...

For surfaces a higher order approximation (Kac, 1966)is also available:

$$\sum_{i=1}^{\infty} e^{\lambda_i t} \sim \frac{\text{Area(D)}}{2\pi t} - \frac{\text{length}(\partial D)}{\sqrt{2\pi t}} + \frac{1}{6}(1 - r)\,,$$

where r denotes the number of holes inside D.

Corollary 26 *"One can hear the number of holes."*

One can also obtain a "passable" similar formula in dimension 3:

$$u_2(x) = \frac{1}{360}\Big(2||R||^2 - 2||\mathrm{Ricci}||^2 + 5\,\mathrm{scal}^2\Big)$$

Remark 27 *The formula above seems to suggest that* U_2 *is* **intrinsic**, *since*

$$\chi(M^n) = \frac{1}{32\pi^2}\int_{M^n}\Big(2||R||^2 - 2||\mathrm{Ricci}||^2 + 5\,\mathrm{scal}^2\Big)$$

Unfortunately, this does not prove to be the case!...

Remark 28 *Similar – yet far more complicated! – formulas exist for* $u_k, k \geq 4$.