THE EXISTENCE OF AUTOMORPHIC QUASIMEROMORPHIC MAPPINGS

EMIL SAUCAN

Abstract. We give a complete characterization of all Kleinian groups G, acting on hyperbolic space $\mathbb{H}^n$, that admit non-constant G-automorphic quasimeromorphic mappings, for any $n \geq 2$. We also address the related problem of existence of qm-mappings on manifolds and prove the existence of such mappings on manifolds with boundary, of low differentiability class.

1. Introduction

It is classical that every Riemann surface carries non-constant meromorphic functions, implying that every Fuchsian group G has non-constant G-automorphic meromorphic functions. In higher dimensions $n \geq 3$ the only locally conformal mappings are restrictions of Möbius transformations, and since they are injective, an extension of the classical existence theorem requires to look at quasimeromorphic mappings.

The question whether quasimeromorphic mappings (qm) exist for any $n \geq 3$ was originally posed by Martio and Srebro in [MS1]; subsequently in [MS2] they proved the existence of the above mentioned mappings in the case of finite co-volume groups, i.e. groups such that $Vol_{hyp}(\mathbb{H}^n/G) < \infty$. (Here Vol_{hyp} denotes hyperbolic volume.) Also, it was later proved by Tukia ([Tu]) that the existence of non-constant quasimeromorphic mappings (or qm-maps, in short) is assured in the case when G acts torsionfree on the hyperbolic n-space $\mathbb{H}^n$. Moreover, since for torsionfree Kleinian groups G, $\mathbb{H}^n/G$ is a analytic manifold, the next natural question to ask is whether there exist qm-maps $f : M^n \rightarrow \widehat{\mathbb{R}^n}$; where M^n is an orientable n-manifold. A partial affirmative answer to this question is due to Peltonen (see [Pe]); more precisely, she proved the existence of qm-maps in the case when M^n is a connected, orientable $\mathcal{C}^\infty$-Riemannian manifold.

In contrast with the above results it was proved by Srebro ([Sr]) that, for any $n \geq 3$, there exists a Kleinian group G acting on $\mathbb{H}^n$ such that there exists no non-constant, G-automorphic function $f : \mathbb{H}^n \rightarrow \mathbb{R}^n$. More

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precisely, if G (as above) contains elliptics of unbounded orders with non-degenerate fixed set, then G admits no non-constant G-automorphic qm-mappings.

To obtain a complete answer to the existence problem we consider the case when the orders of all elliptics with non-degenerate fixed set are bounded, and show that such groups do carry non-constant qm automorphic mappings, in any dimension $n \geq 3$. This result, in conjunction with Srebro's non-existence theorem, gives a complete characterization of those Kleinian group which admit non-constant G-automorphic quasimeromorphic mappings.

Since the classical methods employed in proving the existence in the case $n = 2$ do not apply in higher dimensions, other methods are needed. Following other researchers, we shall employ the classical "Alexander trick" (see [Al]). According to the Alexander method, first one constructs a chess-board triangulation (Euclidian or hyperbolic) of $\mathbb{H}^n$, i.e. a triangulation whose simplices satisfy the condition that every $(n-2)$-face is incident to an even number of n-simplices. Then one alternately maps the simplices of the triangulation onto the interior and the exterior of the standard simplex in $\mathbb{R}^n$ using quasiconformal (qc) maps. If the dilatations of the qc-maps constructed above are uniformly bounded, then the resulting map will be quasimeromorphic.

If the simplices are uniformly *fat* (see Definition 2.7 below), than the restrictions of the mapping to the simplices can be made quasiregular (qr), yielding a quasiconformal mapping. (See [Tu] , [MS2]).

Another natural direction of study stems from Tukia's and Peltonen's Theorems: since they proved the existence of quasimeromorphic mappings for complete (analytic) hyperbolic manifolds and $\mathcal{C}^\infty$ complete Riemannian manifolds, respectively, we want to prove the existence of quasimeromorphic mappings for manifolds with boundary, and when the regularity condition is relaxed. To this end we extend a classical theorem of Munkres regarding the existence of triangulation of manifolds with boundary, to the case of fat triangulations (see Theorem 3.4).

As a corollary to Theorem 3.4, our method yields another proof of the existence of automorphic quasimeromorphic mappings for groups with torsion, in the classical case $n = 3$.

The remainder of this paper is structured as follows: in Section 2 we present the necessary background on quasimeromorphic mappings, Kleinian groups, elliptic transformations and fat triangulations, in Section 3 we present our main results and in Section 4 we sketch the main steps of the proofs (details being provided in our papers [S1], [S2], [S3]).

2. Background

Following Gromov ([Gro]), we consider the following definition of quasiregularity, which befits the best our geometric setting. Other, more general and analytic definitions can be found, e.g. in [Ric] , [V].

Definition 2.1. Let M^n, N^n be oriented, Reimannian n-manifolds.

(1) $f : M^n \to N^n$ is called *quasiregular* iff
 (a) f is locally Lipschitz (and thus differentiable a.e.);
 and
 (b) $0 < |f'(x)|^n \leq K J_f(x)\,; \forall x \in M^n$.
 where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup_{|h|\,=\,1}|f'(x)h|$, and where $J_f(x) = det f'(x)$;

(2) *quasiconformal* iff $f : D \to f(D)$ is a quasiregular homeomorphism;

(3) *quasimeromorphic* iff $f : D \to \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is quasiregular, where the condition of quasiregularity at $f^{-1}(\infty)$ is checked by conjugation with auxiliary Möbius transformations.

The smallest number K that satisfies (4.1) is called the *outer dilatation* of f.

Remark 2.2. These notions represent natural generalizations of conformal, analytic and meromorphic functions, respectively.

Recall that a group G of homeomorphisms acts *properly discontinuously* on a locally compact topological space X iff the following conditions hold for any $g \in G_x\,, x \in X$: (a) the stabilizer $G_x = \{g \in G\,|\,g(x) = x\}$ of x is finite; and (b) there exists a neighbourhood V_x of x, such that (b_1) $g(V_x) \cap V_x = \emptyset$, for any $g \in G \setminus G_x$; and (b_2) $g(V_x) \cap V_x = V_x$.

Definition 2.3. A discontinuous group of orientation-preserving isometries of $\mathbb{H}^n$ is called a *Kleinian* group.

It is well known that a discontinuous group is discrete (see [Ms]).

Definition 2.4. Let $f : \mathbb{H}^n \to \widehat{\mathbb{R}^n}$, and let G be a Kleinian group acting upon $\mathbb{H}^n$. The function f is called G-automorphic iff:

$$f(g(x)) = f(x)\,; \quad \text{for any } x \in \mathbb{H}^n \text{ and for all } g \in G\,; \tag{2.1}$$

Recall also the definition of elliptic transformations:

Definition 2.5. A Möbius transformation $f : \mathbb{H}^n \to \mathbb{H}^n$, $f \neq Id$ is called *elliptic* iff f has a fixed point in $\mathbb{H}^n$.

If G is a discrete Möbius group and if $f \in G$, $f \neq Id$ is an elliptic transformation, then there exists $m \geq 2$ such that $f^m = Id$. The smallest m satisfying this condition is called the *order* of f, and it is denoted by $ord(f)$. In the 3-dimensional case the *fixed point set* of f i.e. $Fix(f) = \{x \in \mathbb{H}^3 | f(x) = x\}$, is a hyperbolic line and will be denoted by $A(f)$ – the *axis of f*. In dimension $n \geq 4$ the fixed set (or *axis of f*) of an elliptic transformation is a k-dimensional hyperbolic plane, $0 \leq k \leq n-2$. An axis A is called *degenerate* iff $dim\, A = 0$. In dimensions higher than $n = 3$, different elliptics may have fixed sets of different dimensions.

Remark 2.6. If G is a discrete group, G is countable and so are the sets $\{f_i\}_{i\geq 1}$ of elliptic transformations and the set $\{C_j\}$ of connected components of $Fix(G) = \big\{x \in \mathbb{H}^n \,|\, \text{exists } g \in G \setminus \{Id\}, \text{ s.t. } g(x) = x\big\}$. Moreover, by the discreteness of G, the sets $\mathcal{A} = \{A_i\}_{i\geq 0} = \{A(f_i)\}_{i\geq 0}$ – and hence $\mathcal{S} = \{C_j\}$ – have no accumulation points in $\mathbb{H}^n$.

We conclude this section with the definition of fat triangulations:

Definition 2.7. A k-simplex $\tau \subset \mathbb{R}^n$ (or $\mathbb{H}^n$); $2 \leq k \leq n$ is *f-fat* if there exists $f \geq 0$ such that the ratio $\frac{r}{R} \geq f$; where r denotes the radius of the inscribed sphere of τ (*inradius*) and R denotes the radius of the circumscribed sphere of τ (*circumradius*). A triangulation of a submanifold of $\mathbb{R}^n$ (or $\mathbb{H}^n$) $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is *f-fat* if all its simplices are f-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is *fat* if there exists $f \geq 0$ such that all its simplices σ_i are *f-fat*.

Remark 2.8. There exists a constant $c(k)$ that depends solely upon the dimension k of τ such that

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\sigma<\tau} \measuredangle(\tau,\sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{2.2}$$

and

$$\varphi(\tau) \leq \frac{Vol_j(\sigma)}{diam^j\, \sigma} \leq c(k) \cdot \varphi(\tau)\,; \tag{2.3}$$

where $\measuredangle(\tau,\sigma)$ denotes the (*internal*) *dihedral angle* of $\sigma < \tau$ and $Vol_j(\sigma)$ and $diam\, \sigma$ stand for the Euclidean j-volume and the diameter of σ respectively. (If $dim\, \sigma = 0$, then $Vol_j(\sigma) = 1$, by convention.)

Remark 2.9. The definition above is the one introduced in [Pe] and we employ it mainly for briefness. For other, equivalent definitions of fatness, see [Ca1], [Ca2], [CMS], [Mun], [Tu].

Remark 2.10. Fat triangles are precisely those for which the individual simplices considered in the Alexander method may each be mapped onto a standard n-simplex, by a L-bilipschitz map, followed by a homotety, with a fixed L.

3. Results

3.1. **The Existence of Automorphic Quasimeromorphic Mappings.**

The following existence theorem, that represents a generalization of previous results of Tukia ([Tu]) and Martio and Srebro ([MS2]), is the main result in this topic (for details see [S1] and [S2]):

Theorem 3.1. *Let G be a Kleinian group with torsion acting upon $\mathbb{H}^n, n \geq 3$. If the elliptic elements (i.e. torsion elements) of G have uniformly bounded orders, then there exists a non constant G-automorphic quasimeromorphic mapping $f : \mathbb{H}^n \to \widehat{\mathbb{R}^n}$.*

Remark 3.2. Given any finitely generated Kleinian group acting on $\mathbb{H}^3$ the number of conjugacy classes of elliptic elements is finite (see [FM] and, for an alternative proof, [S4]). Therefore, for such groups, the orders of the elliptics are bounded and Theorem 3.1 holds. It follows that any finitely generated group G acting upon $\mathbb{H}^3$ admits G-automorphic quasimeromorphic mappings. Note that the result above is not true for Kleinian groups acting upon $\mathbb{H}^n$, $n \geq 4$ (for counterexamples, see [FM], [KP1] and [H]).

Note that by Remark 3.2 we have the following corollary:

Corollary 3.3. *Let G be a finitely generated Kleinian group acting upon $\mathbb{H}^3$. Then there exists a non constant G-automorphic qm-mapping $f : \mathbb{H}^3 \rightarrow \widehat{\mathbb{R}^3}$.*

3.2. **The Existence of Fat Triangulations and the Existence of Quasimeromorphic Mappings on Manifolds.** The main results we shall prove in this topic are listed below. For details see [S2]. The following theorem represents a generalization of a classical result of Munkres (see [Mun]):

Theorem 3.4. *Let M^n be an n-dimensional $\mathcal{C}^\infty$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then any uniformly fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Remark 3.5. We prove that the Theorem above also holds when the compactness condition of the boundary components is replaced by the condition that ∂M^n is endowed with a fat triangulation $\mathcal{T}$ such that $\inf_{\sigma \in \mathcal{T}} diam\, \sigma > 0$.

From Theorem 3.4 above, and from Peltonen's Theorem, it follows immediately that the following holds:

Corollary 3.6. *Let M^n be as above. Then M^n admits a fat triangulation.*

Since every PL manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) smoothing (see [Mun1], [Mun], [Th]), and every topological manifold of dimension $n \leq 3$ admits a PL structure (cf. [Moi], [Th]), we obtain from our results the following corollary:

Corollary 3.7. *Let M^n be an n-dimensional, $n \leq 4$ (resp. $n \leq 3$), PL (resp. topological) connected manifold with boundary, having a finite number of compact boundary components. Then any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

By applying Alexander's Trick to Theorem 3.4, we obtain the following Theorem of quasimeromorphic mappings, which represents a generalization of Peltonen's Theorem (see [Pe]):

Theorem 3.8. *Let M^n be a connected, oriented $\mathcal{C}^1$ Riemannian manifold without boundary or having a finite number of compact boundary components. Then there exists a non-constant quasimeromorphic mapping $f : M^n \rightarrow \widehat{\mathbb{R}^n}$.*

Applying again Alexander's Trick we obtain the following result:

Corollary 3.9. *Let M^n be a connected, oriented n-dimensional manifold ($n \geq 2$), without boundary or having a finite number of compact boundary components. Then in the following cases there exists a non-constant quasimeromorphic mapping $f : M^n \to \widehat{\mathbb{R}^n}$:*

(1) *M^n is of class $\mathcal{C}^r$, $1 \leq r \leq \infty$, $n \geq 2$;*
(2) *M^n is a PL manifold and $n \leq 4$;*
(3) *M^n is a topological manifold and $n \leq 3$.*

4. Proofs and Methods

4.1. **The Existence of Automorphic Quasimeromorphic Mappings.** The idea of the proof of Theorem 3.1 is, in a nutshell, as follows: Based upon the geometry of the elliptic transformations construct a fat triangulation $\mathcal{T}_1$ of N_e^*, where N_e^* is a certain closed neighbourhood of the singular set of $\mathbb{H}^n/G$. Since $M_p = \left(\mathbb{H}^n \setminus Fix(G)\right)/G$, is an orientable analytic manifold, we can apply Peltonen's result to gain a triangulation $\mathcal{T}_2$ of M_p. Therefore, if the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are chosen properly, each of them will induce a triangulation of $N_e^* \setminus N_e^{*\prime}$, for a certain $N_e^{*\prime} \subsetneq N_e^*$.

"Mash" $\mathcal{T}_1$ and $\mathcal{T}_2$ (in $N_e^* \setminus N_e^{*\prime}$) i.e. ensure that the given triangulations intersect into a new triangulation $\mathcal{T}_0$ (see [Mun], Theorem 10.4). Modify $\mathcal{T}_0$ to receive a new fat triangulation $\mathcal{T}$ of $\mathbb{H}^n/G$.

In the presence of degenerate components $A_k = A(f_k)$ of the fixed set of G, where the transformations f_k may have arbitrarily large orders, a modification of this construction is needed .

Apply Alexander's trick to receive a quasimeromorphic mapping $f : \mathbb{H}^n/G \to \widehat{\mathbb{R}^n}$. The lift $\widetilde{f}$ of f to $\mathbb{H}^n$ represents the required G-automorphic quasimeromorphic mapping.

The proof of the case $n = 3$ is treated separately in [S1] for several reasons: it develops and uses a technique for meshing distinct fat triangulations while preserving fatness, technique that employs mainly elementary tools. This technique is relevant in Computational Geometry and Mathematical Biology (see [S5]).

In the general case we employ a method for fattening triangulations developed in [CMS], Lemma 6.3.

4.2. **The Existence of Fat Triangulations and the Existence of Quasimeromorphic Mappings on Manifolds.** The idea of the proof of Theorem 3.4 is first to build two fat triangulations: $\mathcal{T}_1$ of a product neighbourhood N of ∂M^n in M^n and $\mathcal{T}_2$ of $int\, M^n$, and then to "mash" the two triangulations into a new triangulation $\mathcal{T}$, while retaining their fatness. While the mashing procedure of the two triangulations is basically that developed in the original proof of Munkres' theorem, the triangulation of $\mathcal{T}_1$ was modified, in order to ensure the fatness of the simplices of $\mathcal{T}_1$, more precisely we prove the following Theorem (see [S3]):

Theorem 4.1. *Let M^n be a $\mathcal{C}^r$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then any fat $\mathcal{C}^r$-triangulation of ∂M^n can be extended to a $\mathcal{C}^r$-triangulation $\mathcal{T}$ of M^n, $1 \leq r \leq \infty$, the restriction of which to a product neighbourhood $\widetilde{K}_0 = \partial M^n \times I_0$ of ∂M^n in M^n is fat.*

The existence of $\mathcal{T}_2$ follows from Peltonen's result. The fattening technique employed here is, again, that of [CMS]. In the proof of Theorem 3.8 we again use Alexander's Trick.

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Department of Mathematics and Electrical Engineering Department, Technion, Haifa,Israel
E-mail address: semil@tx.technion.ac.il, semil@ee.technion.ac.il