On the Existence of Quasimeromorphic Mappings

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Neve Ilan – 25.05.2006.

- The main results presented herein constitute the author's Ph.D. Thesis written under the supervision of Prof. Uri Srebro at The Technion, Haifa.

- Some generalization and problems for further research are also included.

The main problem we address is the following:

[A] The existence of G-automorphic quasimeromorphic mappings (in the sense of Martio and Srebro) $f : \mathbb{H}^n \to \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$, for a given Kleinian group G acting on $\mathbb{H}^n$, i.e. such that

$$f(g(x)) = f(x)\,, \text{ for any } x \in \mathbb{H}^n \text{ and for all } g \in G\,. \qquad (1)$$

where $\mathbb{H}^n$ denotes the hyperbolic n-space,

and

where a Kleinian group is a discontinuous group of orientation preserving isometries of $\mathbb{H}^n$.

Recall $f \in G$ is a torsion element of $G \setminus \{Id\}$ iff there exists $m \geq 2$ s.t. $f^m = Id$ and that the smallest m satisfying the condition above is called the *order* of f.

Recall also that in Kleinian groups the torsion elements are the elliptic transformations, i.e. the hyperbolic isometries f that have (at least) a fixed point in $\mathbb{H}^n$.

In the 3-dimensional case the *fixed point set* of f, i.e. $Fix(f) = \{x \in H^3 | f(x) = x\}$, is a hyperbolic line and will be denoted by $A(f)$ – the *axis of* f. In dimension $n \geq 4$ the fixed set of an elliptic transformation is a k-dimensional hyperbolic plane, $0 \leq k \leq n-2$. An axis A is called *degenerate* iff $dim\, A = 0$. In dimensions higher than $n = 3$, different elliptics may have fixed sets of different dimensions.

Since for torsionless Kleinian groups G, $\mathbb{B}^n/G$ is a (analytic) manifold, the next natural problem to address is that of:

[B] The existence non-constant qm-maps $f : M^n \to \widehat{\mathbb{R}^n}$; where M^n is an orientable n-manifold. A partial affirmative answer to this question is due to Peltonen:

Peltonen (1992) For M^n (open) connected, orientable $\mathcal{C}^\infty$-Riemannian n-manifolds, $n \geq 3$.

In contrast with the positive results above, we have the following

Non-existence Result

Srebro (1998) For any $n \geq 3$, there exists a Kleinian group $G \bowtie \mathbb{H}^n$ such that there exists no non-constant, G-automorphic function $f : \mathbb{H}^n \to \mathbb{R}^n$. More precisely, if G (as above) contains elliptics of unbounded orders (with non-degenerate fixed set), then G admits no non-constant G-automorphic qm-mappings.

This follows from the following facts:

- the local topological index: $i(x,f) = \inf_{U \in \mathcal{N}(x)} \sup_{y} |f^{-1}(y) \cap U|$ of a qm-map f cannot be too big on all the points of a non-degenerate continuum

  and

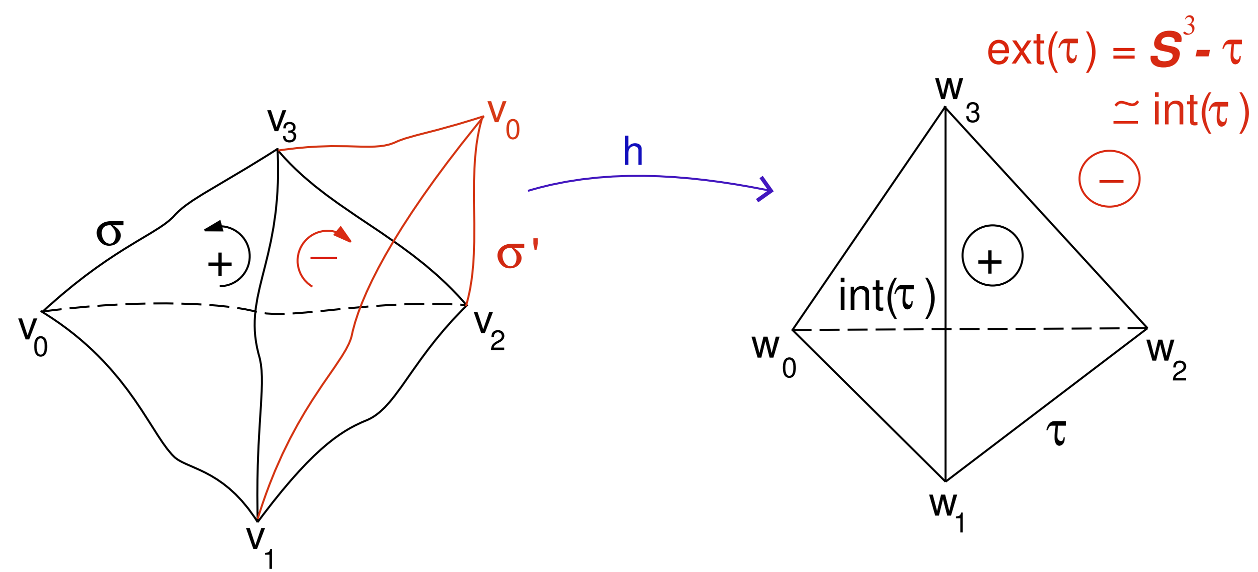
- if g is an elliptic Möbius transformation fixing x and such that $f \circ g = f$, then the order of g divides $i(x,f)$.

All these existence results were obtained by employing

"Alexander's Trick"

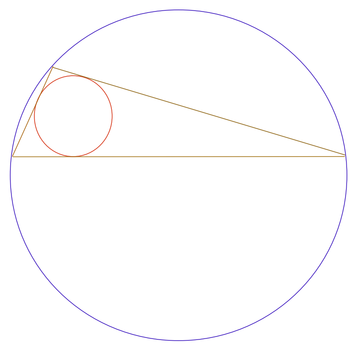
One starts by constructing a chessboard triangulation of M^n, i.e. a triangulation whose simplices satisfy the condition that every $(n-2)$-face is incident to an even number of n-simplices. Since M^n is orientable, a consistent orientation can be chosen for all the simplices of the triangulation (i.e. such that two given n-simplices having a $(n-1)$-dimensional face in common will have opposite orientations).

Then one quasiconformally maps the simplices of the triangulation into $\widehat{\mathbb{R}^n}$ in a chess-table manner: the positively oriented ones onto the interior of the standard simplex in $\mathbb{R}^n$ and the negatively oriented ones onto its exterior. If the dilatations of the qc-maps constructed above are uniformly bounded, then the resulting map will be quasimeromorphic.



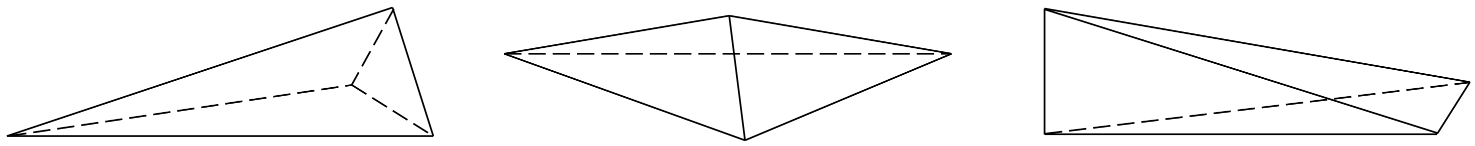
If the simplices are uniformly *fat*, than the restrictions of the mapping to the simplices can be made quasiregular, yielding a quasiconformal mapping. The notion of *fatness* is given in the following definition:

Definition 1 *A k-simplex $\tau \subset \mathbb{R}^n$ (or $\mathbb{H}^n$); $2 \leq k \leq n$ is f-fat if there exists $f \geq 0$ such that the ratio $\frac{r}{R} \geq f$; where r denotes the radius of the inscribed sphere of τ (*inradius*) and R denotes the radius of the circumscribed sphere of τ (*circumradius*). A triangulation of a submanifold of $\mathbb{R}^n$ (or $\mathbb{H}^n$) $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is f-fat if all its simplices are f-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is fat if there exists $f \geq 0$ such that all its simplices are f-fat; $\forall i \in \mathbf{I}$.*



Remark 2 *The definition above is the one introduced by Peltonen and we employ it mainly for briefness. Other, equivalent definitions of fatness, were given by Cairns, Cheeger, Munkres and Tukia.*

Remark 3 *We want to ensure that "slim" or "flat" simplices such as the ones below do not appear in the triangulation.*



Remark 4 *Fat triangles are precisely those for which the individual simplices considered in Alexander's trick may each be mapped onto a standard n-simplex, by a L-bilipschitz map, followed by a homotety, with a fixed L.*

Results

The Existence of Automorphic Quasimeromorphic Mappings

Theorem 5 * *Let G be a Kleinian group (with torsion) acting upon $\mathbb{H}^n, n \geq 3$. If the elliptic elements of G with non-degenerate fixed set have uniformly bounded orders, then there exists a non constant G-automorphic quasimeromorphic mapping $f : \mathbb{H}^n \to \widehat{\mathbb{R}^n}$.*

Corollary 6 *Let G be a finitely generated Kleinian group with torsion acting upon $\mathbb{H}^3$. Then there exists a non constant G-automorphic quasimeromorphic mapping $f : \mathbb{H}^3 \to \widehat{\mathbb{R}^3}$.*

*Annales Academiæ Scientiarum Fennicæ Mathematica, Vol. 31, 2006, 131-142.

This existence theorem, together with Srebro's non-existence result, gives a complete characterization of those Kleinian group which admit G-automorphic quasimeromorphic mappings. Namely:

Theorem 7 *Let G be a Kleinian group acting on $\mathbb{B}^n$. Then G admits non-constant automorphic qm-mappings iff:*

1. $n = 2$;

   or

2. $n \geq 3$, *and the orders of the elliptic elements of G having non-degenerate fixed sets are uniformly bounded.*

The Existence of Fat Triangulations and the Existence of Quasimeromorphic Mappings on Manifolds

Theorem 8 * *Let M^n be an n-dimensional $\mathcal{C}^1$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then any uniformly fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Remark 9 *We prove that the Theorem above also holds when the compactness condition of the boundary components is replaced by the condition that ∂M^n is endowed with a fat triangulation $\mathcal{T}$ such that $\inf_{\sigma \in \mathcal{T}} diam\, \sigma > 0$.*

*Mediterranean Journal of Mathematics, vol. 2, no. 2(2005), 215-229.

Since every PL manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) smoothing*, and every topological manifold of dimension $n \leq 3$ admits a PL structure†. We obtain from our results the following corollary:

Corollary 10 *Let M^n be an n-dimensional, $n \leq 4$ (resp. $n \leq 3$), PL (resp. topological) connected manifold with boundary, having a finite number of compact boundary components. Then any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

*cf. Munkres, Thurston

†cf. Moise, Thurston

By applying Alexander's Trick to Theorem 8, we obtain the following theorem of existence of quasimeromorphic mappings, which represents a generalization of Peltonen's theorem:

Theorem 11 *Let M^n be a connected, oriented C^1 Riemannian manifold without boundary or having a finite number of compact boundary components. Then there exists a non-constant quasimeromorphic mapping $f : M^n \rightarrow \widehat{\mathbb{R}^n}$.*

And thus, by Corollary 10 we obtain, in addition, the following corollary:

Corollary 12 *Let M^n be a connected, oriented C^1 n-dimensional manifold ($n \geq 2$), without boundary or having a boundary consisting of a finite number of compact boundary components.*
Then in each of the following cases there exists a non-constant quasimeromorphic mapping $f : M^n \to \widehat{\mathbb{R}^n}$:

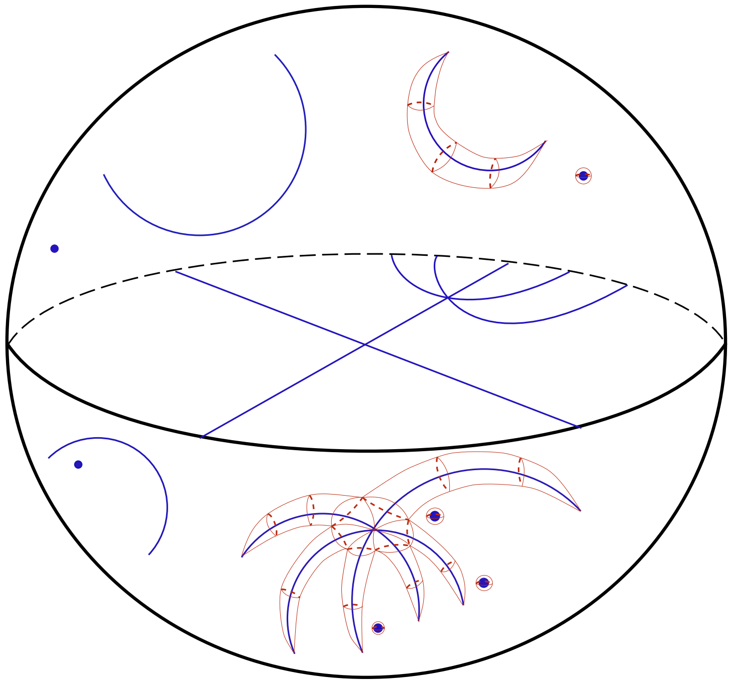
1. *M^n is a PL manifold and $n \leq 4$;*

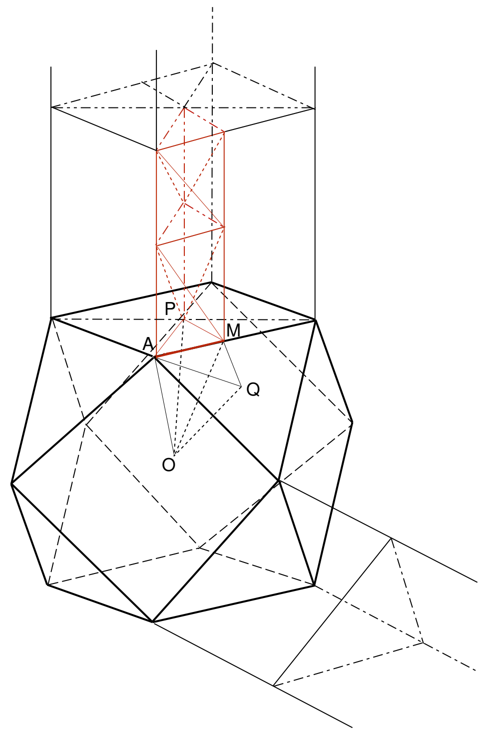
2. *M^n is a topological manifold and $n \leq 3$.*

The Existence of Automorphic Quasimeromorphic Mappings

Method of Proof

- Based upon the geometry of the elliptic transformations construct a fat triangulation $\mathcal{T}_1$ of N_e^*, where N_e^* is a certain closed neighbourhood of the singular set of $\mathbb{H}^n/G$.





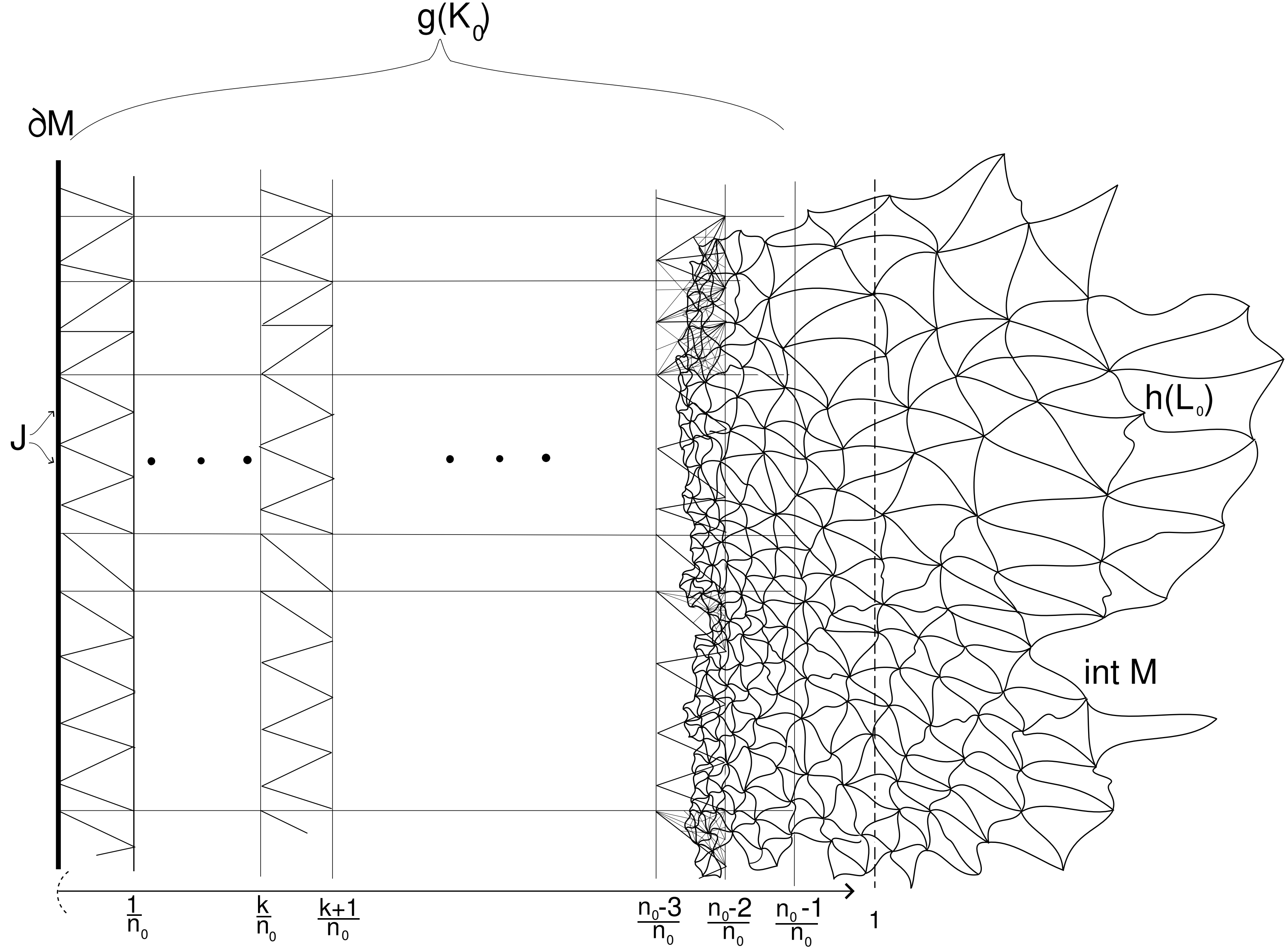
- Since $M_p = \Big(\mathbb{B}^n \setminus Fix(G)\Big)\,/\,G$ [*] is an orientable analytic manifold, we can apply Peltonen's result to gain a triangulation $\mathcal{T}_2$ of M_p.

- Therefore, if the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are chosen properly, each of them will induce a triangulation of $N_e^* \setminus N_e^{*\prime}$, for a certain $N_e^{*\prime} \subsetneq N_e^*$.

- "Mash" $\mathcal{T}_1$ and $\mathcal{T}_2$ (in $N_e^* \setminus N_e^{*\prime}$) i.e. ensure that the given triangulations intersect into a new triangulation $\mathcal{T}_0$ (Munkres technique). Modify $\mathcal{T}_0$ to receive a new fat triangulation $\mathcal{T}$ of $\mathbb{B}^n/G$ (Cheeger method[†]).

[*] $Fix(G) = \big\{x \in \mathbb{B}^n \,|\, \exists g \in G \setminus \{Id\},\, g(x) = x\big\}$

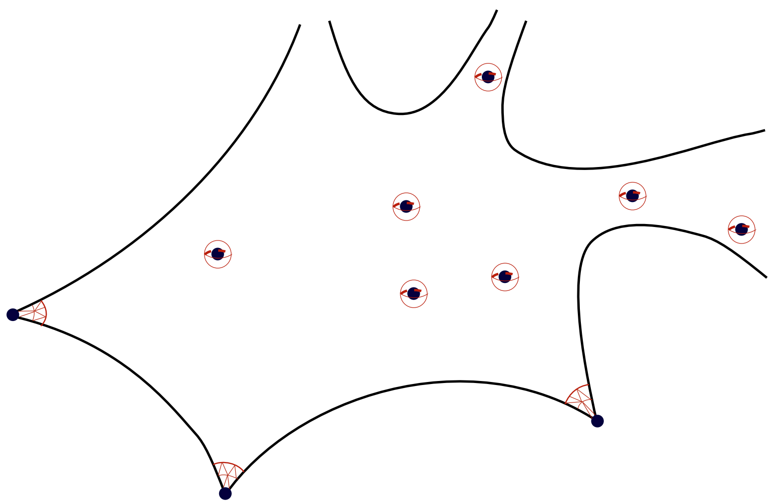
[†] Or use the one employed in Conform. Geom. Dyn. **10** (2006), 21-40.

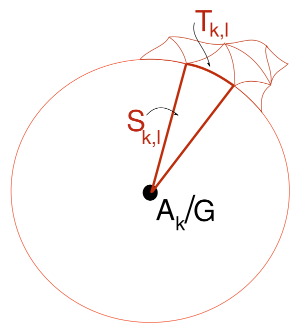

g(K0)
∂M
J
h(L0)
int M
1/n0
k/n0
k+1/n0
n0-3/n0
n0-2/n0
n0-1/n0
1

- Apply Alexander's trick to receive a quasimeromorphic mapping $f : \mathbb{B}^n/G \to \widehat{\mathbb{R}^n}$. The lift $\widetilde{f}$ of f to $\mathbb{B}^n$ represents the required G-automorphic quasimeromorphic mapping.

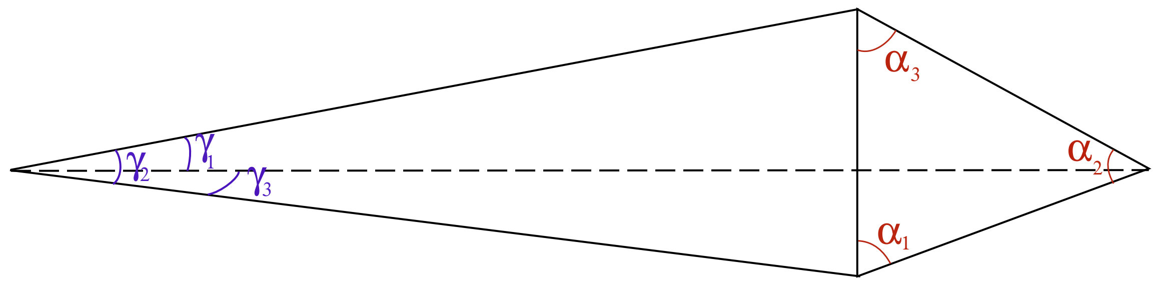
- If the orders of elliptics with degenerate fixed sets are **not** bounded from above than modification of this construction is needed:

- Excise from $\mathbb{B}^n/G$ ball neighourhoods B_k centered at A_k/G. Then $S_k = \partial B_k$ is an $(n-1)$-manifold that admits a fat triangulation extending that of $(\mathbb{B}^n \setminus \mathcal{S})/G$ (Cheeger).





- Simplices of the type of $S_{k,l} = J(A_k/G, T_{k,l})$



can be mapped quasiconformally on the upper half-space (on the standard simplex) (Caraman, Gehring, Väisalä).

We conclude with some Questions (and a few partial Answers) that arise naturally from our study(investigation):

Question $\mathbf{A}_0$ What is the largest class of manifolds that admit quasimeromorphic mappings?

Indeed, one can ask the following slightly more general

Question $\mathbf{A}_1$ What is the largest class of geometric objects that admit qm-mapings onto $\mathbb{S}^n$?

In particular, one can sharpen the question above in a natural sense, by extending the class of groups that admit qm-automorphic mappings:

Question A$_2$ (M. Kapovich) Do *quasiconformal groups* admit qm-automorphic mappings?

Recall that

Definition 13 *A discrete group G of homeomorphisms of $\mathbb{B}^n$ (or $\widehat{\mathbb{R}^n}$) is called* quasiconformal *iff there exists $1 \leq K < \infty$ such that g is K-quasiconformal , for any $g \in G$.*

Another question arises from the Alexander method applied:

Question B$_1$ Let $f_A : M^n \to \mathbb{S}^n$ be the qm-mapping constructed using the Alexander method. What is the minimal branched qm-mapping $f_0 : M^n \to \mathbb{S}^n$, such that $K(f_0) = K(f_A)$?

A related (yet stronger) problem is formulated in

Question B$_2$ (Martio) Given a manifold M^n, does there exist a qm-mapping $f_{min} : M^n \to \mathbb{S}^n$ with minimal dilatation?

This conducts us immediately to the following question:

Problem B$_3$ (Martio) Compute $K(f_{min})$.

Yet another problem stems from Theorem 11:

Problem C (Martio) Let M^n ba a manifold with boundary, as in Theorem 11, and let $f_{int\,M^n} : int\,M^n \to \mathbb{S}^n$ be the qm-mapping given by Peltonen's result.

(i) Can $f_{int\,M^n}$ be extended to a qm-mapping $\widetilde{f} = f_{M^n} : M^n \to \mathbb{S}^n$?

and if it can:

(ii) What is the relationship between $\widetilde{f}|_{\partial M^n}$ and the qm-mapping $f_{\partial M^n}$ constructed in Theorem 11. (In particular how do $K(\widetilde{f}|_{\partial M^n})$ and $K(f_{\partial M^n})$ relate?)

We conclude with the following Problem related to the technique(method) of proof of Theorem 7 in the 3-dimensional case:

Problem D Compute collars for elliptic axes in higher-dimensional discrete groups, by using the extensions of Jørgensen's inequality.*

*Hint: Use results of Friedland and Hersonsky; Martin; Waterman.

We have only some partial Answers to Question A_0 (and its relatives):

Answer A_{0_1}* Any Lipschitz manifold admits qm-mappings onto $\mathbb{S}^n$.

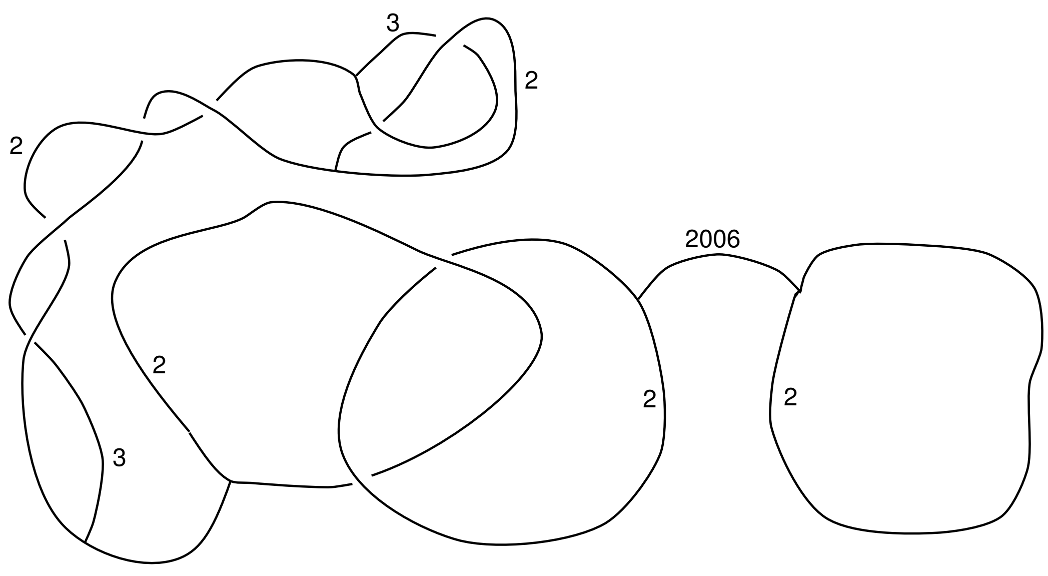
Since the boundary of any (PL) manifold is collared and since the fatness of the mashing of two fat triangulations depends solely upon the initial fatness and upon the dimension, we have the following generalization:

Answer A_{0_2} Let M^n be a (smooth) manifold with boundary, such that the boundary components admit fat triangulations of fatness $\geq \varphi_0$. Then M^n admits a global fat triangulation (hence admits qm-mappings).

*This was already conjectured by Cairns.

Also, we can give at least a partial answer to Question A_1:

Answer A_1 Any 3-dimensional orientable geometric orbifold* with tame singular locus and with isotropy groups of bounded orders (in particular for *Seifert fibred orbifolds*, that possess natural, canonical geometric neighbourhoods†).



*Given by a 1-complex in e.g. $\mathbb{S}^3$ with certain labellings.

†See Bonahon-Siebenmann.