REMARKS ON THE THE EXISTENCE OF QUASIMEROMORPHIC MAPPINGS

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For Professors Dov Aharonov and Uri Srebro, on their retirement.

1. Introduction

The purpose of this note is to discuss and partially extend the following results on the existence of quasimeromorphic mappings:

Theorem 1.1 ([Sa06a])**.** *Let G be a Kleinian group with torsion acting upon $\mathbb{H}^n, n \geq 3$. If the elliptic elements of G have uniformly bounded orders, then there exists a non constant G-automorphic quasimeromorphic mapping $f : \mathbb{H}^n \rightarrow \widehat{\mathbb{R}^n}$.*

Recall that a Kleinian group is a discontinuous (hence discrete) group of orientation-preserving isometries of hyperbolic n-space $\mathbb{H}^n$, and that elliptic transformations are defined as follows:

Definition 1.2. An orientation-preserving isometry $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$, $f \neq Id$ is called *elliptic* iff f has a fixed point in $\mathbb{H}^n$.

If G is a discrete Möbius group and if $f \in G$, $f \neq Id$ is an elliptic transformation, then there exists $m \geq 2$ such that $f^m = Id$. The smallest m satisfying this condition is called the *order* of f, and it is denoted by $ord(f)$. The *fixed point set* (or *axis of f*) of an elliptic transformation, i.e. $Fix(f) = \{x \in \mathbb{H}^n | f(x) = x\}$ is a k-dimensional hyperbolic plane, $0 \leq k \leq n-2$. An axis A is called *degenerate* iff $dim\, A = 0$.

The existence result above, together with an earlier non-existence result of Srebro ([Sr98]) gives a complete characterization of those Kleinian groups which admit G-automorphic quasimeromorphic mappings. Namely:

Theorem 1.3 ([Sa06a])**.** *Let G be a Kleinian group acting on $\mathbb{H}^n$. Then G admits non-constant automorphic qm-mappings iff:*

(1) $n = 2$*;*

or

(2) $n \geq 3$*, and the orders of the elliptic elements of G having non-degenerate fixed sets are uniformly bounded.*

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Definition 1.4. Let M^n, N^n be oriented, Riemannian n-manifolds.

(1) $f : M^n \to N^n$ is called *quasiregular* (*qr*) iff
   (a) f is locally Lipschitz (and thus differentiable a.e.);
   and
   (b) $0 < |f'(x)|^n \leq KJ_f(x)$, for any $x \in M^n$;
   where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup\limits_{|h|=1} |f'(x)h|$,
   and where $J_f(x) = det f'(x)$;
(2) *quasimeromorphic* (*qm*) iff $f : D \to \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is quasiregular, where the condition of quasiregularity at $f^{-1}(\infty)$ is checked by conjugation with auxiliary Möbius transformations.

The smallest number K that satisfies condition (b) above is called the *outer dilatation* of f.

Theorem 1.5 ([Sa05]). *Let M^n be a connected, oriented n-dimensional manifold ($n \geq 2$), without boundary or having a finite number of compact boundary components. Then, in the following cases, there exists a non-constant quasimeromorphic mapping $f : M^n \to \widehat{\mathbb{R}^n}$:*

(1) *M^n is of class $\mathcal{C}^r$, $1 \leq r \leq \infty$, $n \geq 2$;*
(2) *M^n is a PL manifold and $n \leq 4$;*
(3) *M^n is a topological manifold and $n \leq 3$.*

By Selberg's Lemma ([Se60]), any finitely generated Möbius group contains a torsion-free subgroup of finite index. In particular, the orders of the elliptic elements of a finitely generated Kleinian group are uniformly bounded. Therefore, we obtain the following corollary:

Corollary 1.6. *Let G be a finitely generated Kleinian group acting upon $\mathbb{H}^n$. Then there exists a non-constant G-automorphic qm-mapping $f : \mathbb{H}^n \to \widehat{\mathbb{R}^n}$.*

The basic method of proof, both for Theorem 1.1 and of Theorem 1.5, is to produce a chess-board *fat* triangulation of $\mathbb{H}^n/G$, respectively of M^n, and to apply the well-known *Alexander method* (see [Al20], [Sa06a], [Sa05]). In both cases, we employ an existence result of fat triangulations on open, $\mathcal{C}^\infty$ Riemannian manifolds, due to Peltonen ([Pe92]) (for details, see [Sa06a], [Sa05]). Here, fat triangulations are defined as follows:

Definition 1.7. A k-simplex $\tau \subset \mathbb{R}^n$ (or $\mathbb{H}^n$); $2 \leq k \leq n$ is *f-fat* if there exists $f \geq 0$ such that the ratio $r/R \geq f$; where r, R denote the radius of the inscribed sphere, respectively the radius of the circumscribed sphere of τ. A triangulation of a submanifold of $\mathbb{R}^n$ (or $\mathbb{H}^n$) $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is *f-fat* if all its simplices are f-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is *fat* if there exists $f \geq 0$ such that all its simplices σ_i are *f-fat.*

Remark 1.8. Fat triangulations are precisely those for which the individual simplices considered in the Alexander method may each be mapped onto a standard n-simplex, by a L-bilipschitz map, followed by a homotety, with a fixed L.

2. Open Problems

We bring below a list of some questions (and a few partial answers) that arise naturally in conjunction with the results above and their proofs:

Question 1. *What is the largest class of manifolds that admit quasimeromorphic mappings?*

Indeed, one can ask the following slightly more general

Question 2. *What is the largest class of geometric objects that admit qm-mapings?*

Of course, in this generic context, one has first to provide a proper, generalized definition of *qr*-mappings. While diverse generalizations of the notion of quasiregularity have been considered, perhaps the best (and simplest) generalization is the one based upon the *linear dilatation*:

Definition 2.1. Let $(X,d),(Y,\rho)$ be metric spaces, and let $f:X\to Y$ be a continuous function. f is called quasiregular iff there exists K_0 such that, for all $x\in X$

$$H(f,x)=\limsup_{r\to 0}\frac{\sup\{\rho(f(x),f(y))\,|\,d(x,y)=r\}}{\inf\{\rho(f(x),f(y))\,|\,d(x,y)=r\}}\leq K_0<\infty\,.$$

$H(f,x)$ is called the *linear dilatation* of f (at x).

Moreover, one can sharpen Question 2 above, in a natural sense, by extending the class of groups that admit *qm*-automorphic mappings:

Question 3. (M. Kapovich) *Do* quasiconformal groups *admit qm-automorphic mappings?*

Recall that *quasiconformal groups* are defined as follows:

Definition 2.2. A discrete group G of homeomorphisms of $\mathbb{B}^n$ (or $\widehat{\mathbb{R}^n}$) is called *quasiconformal* iff there exists $1\leq K<\infty$ such that g is K-quasiconformal , for any $g\in G$.

Another question arises from the Alexander method employed:

Question 4. *Let $f_A:M^n\to\mathbb{S}^n$ be the qm-mapping constructed using the Alexander method. What is the minimal branched qm-mapping $f_0:M^n\to\mathbb{S}^n$, such that $K(f_0)=K(f_A)$?*

A related (yet stronger) problem is formulated in

Question 5. (Martio) *Given a manifold M^n, does there exist a qm-mapping $f_{min}:M^n\to\mathbb{S}^n$ attaining the minimal dilatation?*

This conducts us immediately to the following problem:

Problem 1. (Martio) *Compute $K(f_{min})$.*

Yet another question stems from Theorem 1.5:

Problem 2. (Martio) *Let M^n be a manifold with boundary, as in Theorem 1.5, and let $f_{int\,M^n} : int\,M^n \to \mathbb{S}^n$ be the qm-mapping given by Peltonen's result.*

(i) Can $f_{int\,M^n}$ be extended to a qm-mapping $\widetilde{f} = f_{M^n} : M^n \to \mathbb{S}^n$?

Moreover, if such an extension $\widetilde{f}$ exists,

(ii) What is the relationship between $\widetilde{f}|_{\partial M^n}$ and the qm-mapping $f_{\partial M^n}$ constructed in Theorem 1.5. (In particular how do $K(\widetilde{f}|_{\partial M^n})$ and $K(f_{\partial M^n})$ relate?)

We have only some partial answers to Question 1 (and the questions related to it). First let us note that the proof of Theorem 1.5 extends to include the case of Lipschtz manifolds, thus we have:

Answer 1. *Any Lipschitz manifold admits qm-mappings.*

Remark 2.3. This fact was already conjectured by Cairns ([Ca61]).

The fat triangulation for a manifold with boundary is obtained by "mashing" the triangulation of ∂M^n and of $int\,M^n$ into a new fat triangulation (see [Sa06b]). Since the boundary of any (PL) manifold is collared and since the fatness of the mashing of two fat triangulations depends solely upon the initial fatness and upon the dimension (see [CMS84], Lemma 6.3), we have the following generalization:

Answer 2. *Let M^n be a (smooth) manifold with boundary, such that the boundary components admit fat triangulations of fatness $\geq \varphi_0$. Then there exist a global fat triangulation of M^n (hence M^n admits qm-mappings).*

However, since the lower bound φ_0 for the fatness of the simplices of $int\,M^n$ assured by Peltonen's result is achieved via the specific construction of [Pe92], the following question arises naturally:

Question 6. *Does φ_0 represent the best lower bound?*

The answer to the question above seems to be negative, since Peltonen's construction depends upon the Nash isometric embedding technique of M^n into R^N, for some N large enough.

More important, the *mesh* of the triangulation we construct (thence the branching of the resulting *qm*-mapping) is a function of the *curvature radii*, therefore an *extrinsic* constraint, hence again strongly dependent upon the embedding in higher dimensional Euclidean space. This fact immediately generates the following question:

Question 7. *Does the Nash embedding technique impose any restrictions upon the curvature radii?*

The answer to this question is (at least partially) positive, since, in the Nash embedding method, the curvature of the embedding is controlled. Moreover, in the smoothing part of the Nash technique, a *star finite* partition of the embedding, obtained using curvature radii of an intermediate embedding, is considered (see [Na56], [An02]). Therefore, we should reformulate Question 7 above in the sharper form below:

Question 8. *What are the restrictions imposed upon the curvature radii by the Nash embedding technique?*

However, curvature-based considerations are applicable only if the manifold is smooth enough (i.e. at least of class $\mathcal{C}^{2,\alpha}$ for non-embedded manifolds). For $\mathcal{C}^1$ and, even more so, for PL manifolds, where the classical notion of curvature is not defined, one has therefore to consider either *generalized principal curvatures* ([Za05]) or even more general, metric curvatures ([SAZ06]). (For another approach to the problem of partitioning a PL manifold into Dirichlet (Voronoi) cells see [ILTC01].)

For smooth surfaces embedded in $\mathbb{R}^3$, partially intrinsic conditions for the existence of fat triangulation with mesh bounded from below are readily obtained, e.g. in the case of surfaces with pinched Gauss and mean curvature (see [SAZ06], Corollary 4.4). Of course, one would like to find such curvatures conditions (perhaps coupled with fitting topological constraints) in any dimension. However, in dimension greater or equal to three, even the problem of deciding which curvature (sectional, Ricci, scalar) is the relevant one, represents a problem that we defer for further study.

Also, we can give at least a partial answer to Question 2:

Answer 3. *Any* 3*-dimensional orientable geometric orbifold with tame singular locus and with isotropy groups of bounded orders admits qm-mappings. This holds, in particular, for Seifert fibred orbifolds, that possess natural, canonical geometric neighbourhoods (see* [BoS82]*).*

Indeed, any such orbifold Q is given by a 1-complex (in e.g. $\mathbb{S}^3$) with certain labellings (see [Th90]). Thus we can apply for the proof of Theorem 1.1, the technique developed in [Sa05], [Sa06b]: Construct a geometric fat triangulation $\mathcal{T}_1$ of N^*, where N^* is a certain closed neighbourhood of the singular set Σ_Q of Q, and mesh $\mathcal{T}_1$ with the triangulation of $\mathbb{S}^3 \setminus N^*$, given by Peltonen's technique, into a new fat triangulation.

We conclude with the following problem related to the method of proof of Theorem 1.1 in the 3-dimensional case:

Problem 3. *Compute collars for elliptic axes in higher-dimensional discrete groups, by using the extensions of Jørgensen's inequality.*

Hint: Use results of Friedland and Hersonsky ([FH93]); Martin ([Ma89]); Waterman ([Wa93].

Appendix

We address here the related problem of the existence of quasiconformal immersions of hyperbolic manifolds in $\mathbb{S}^n$, namely the following negative result of Martio and Srebro ([MS99], [Sr91]) that:

Proposition 2.4. *(*[MS99]*) Let M^n, $n \geq 3$ be a hyperbolic manifold. Then, if M^n has arbitrarily short geodesics, it can not be immersed quasiconformally in $\mathbb{S}^n$.*

This theorem does not hold in dimension $n = 2$ (see [Sr98], p. 115). Therefore, one is naturally conducted to formulate the following problem:

Problem 4 (Srebro, [Sr91])**.** *Let M^n , $n \geq 3$ be a hyperbolic manifold such that there exists a low bound λ_0 for the lengths of closed geodesics in M^n. Can M^n be immersed quasiconformally in $\mathbb{S}^n$?*

We show that the answer to this problem is a positive one: Indeed, since M^n, as a hyperbolic manifold, is a *space form* of constant *scalar curvature* $K \equiv -1$, the *injectivity radius* $i(M^n) = \inf_{x \in M^n} Inj(x)$, where $i(x) = \sup\{r \,|\, exp_x|_{\mathbb{B}^n(x,r)}$ is a diffeomorphism$\}$, is given by $i(M^n) = \frac{1}{2}\lambda_0$ (see [Be03]). (Here exp_x denotes the exponential map.) The desired result follows now easily from the properties of the exponential map (see, e.g. [CE]).

Since $M^n = \mathbb{H}^n/G$, where G is a discontinuous group ([Th90]), and since the only elements in G with *translation length* $l(g) = \inf_{x \in \mathbb{H}^n} d_{hyp}(x, g(x)) > 0$, are the loxodromics, we can formulate a sharper version of a theorem of Martio and Srebro ([MS99], [Sr91]):

Theorem 2.5. *Let G be a Kleinian group acting on $\mathbb{H}^n, n \geq 3$ and let $f : \mathbb{H}^n \to \widehat{\mathbb{R}^n}$ be a G-automorphic qm-map. Then f is locally injective iff G is purely loxodromic and $l(G) = \inf_{g \in G} l(g) > 0$.*

(Here d_{hyp} denotes the hyperbolic metric on $\mathbb{H}^n$.)

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