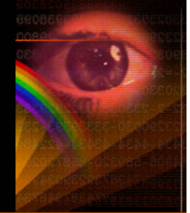

Metric Curvature and Applications

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We consider a number of applications of metric curvature to a variety of problems. Amongst them:

[A] The problem of better approximating surfaces by triangular meshes. We suggest to view the approximating triangulations (graphs) as finite metric spaces and the target smooth surface as their Haussdorff-Gromov limit.

[B] Employing metric differential geometry for the analysis of weighted graphs/networks. In particular, we employ Haantjes curvature as a tool in communication networks and DNA microarray analysis.

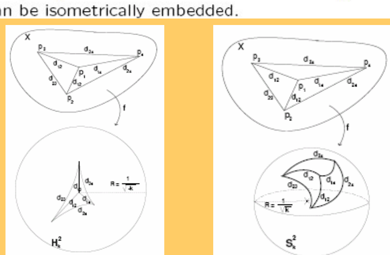
Embedding Curvature

Our goal: to define an intrinsic metric curvature for surfaces.

We do this by comparing quadruples on the given metric space, to those in a gauge surface.

Definition Let (M,d) be a metric space, and let $Q = \{p_1, ..., p_4\} \subset M$, together with the mutual distances: $d_{ij} = d_{ji} = d(p_i,p_j)$; $1 \leq i,j \leq 4$. The set Q together with the set of distances $\{d_{ij}\}_{1\leq i,j\leq 4}$ is called a *metric quadruple*.

Definition The *embedding curvature* $\kappa(Q)$ of the metric quadruple Q is defined be the curvature κ of S_κ into which Q can be isometrically embedded.



The Embedding Curvature at a point is defined by passing to the limit:

Definition Let (M,d) be a metric space, and let $p \in M$ be an accumulation point. Then p is said to have *Wald curvature* $\kappa_W(p)$ iff

(i) No neighbourhood of p is linear;

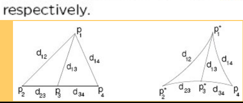
(ii) For any $\varepsilon > 0$, exists $\delta > 0$ s.t. (a) $Q = \{p_1, ..., p_4\} \subset M$, and (b) $d(p,p_i) < \delta\,(i = 1,...,4) \Longrightarrow |\kappa(Q) - \kappa_W(p)| < \varepsilon$.

An approximate formula (Robinson):

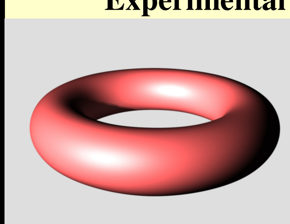
Given the semi-dependent metric quadruple $Q = Q(p_1,p_2,p_3,p_4)$, of distances $d_{ij} = dist(p_i,p_j)$, $i = 1,...,4$, the embedding curvature $\kappa(Q)$ is well approximated by:

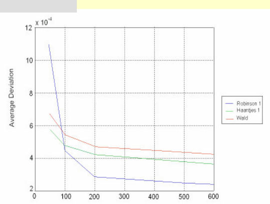
$$K(Q) = \frac{6(\cos\angle_0 2+\cos\angle_0 2')}{d_{24}(d_{12}\sin^2(\angle_0 2)+d_{23}\sin^2(\angle_0 2'))};$$

where: $\angle_0 2 = \angle(p_1p_2p_4)$, $\angle_0 2' = \angle(p_3p_2p_4)$ represent the angles of the Euclidean triangles of sides d_{12}, d_{14}, d_{24} and d_{23}, d_{24}, d_{34}, respectively.



Experimental Results

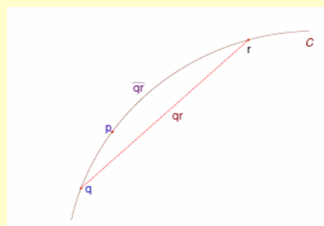




Haantjes Curvature

For Rectifiable Metric Spaces

Definition Let (M,d) be a metric space, let $c: I = [0,1] \xrightarrow{\sim} M$ be a homeomorphism, and let $p,q,r \in c(I)$, $q,r \neq p$. Denote by $\widehat{qr}$ the arc of $c(I)$ between q and r, and by qr segment from q to r.



Then c has *Haantjes Curvature* $\kappa_H(p)$ at the point p iff:

$$\kappa_H^2(p) = 24 \lim_{q,r\rightarrow p} \frac{l(\widehat{qr})-d(q,r)}{\left(l(\widehat{qr})\right)^3}$$

where "$l(\widehat{qr})$" denotes the length

Possible Application: As approximation of *sectional curvature* for triangulated surface reconstruction (see figure below).

For Weighted Graphs

Definition Let $G = (V,G,d)$ be a metric graph, and let $v \in V$. Let $\pi = v_1vv_2$ be a path through v. First we define the *curvature of triangles* with vertex v as being:

$$\kappa'_H(\triangle v_1vv_2) = \begin{cases} \sqrt{24\frac{|d(v_1,v)+d(v,v_2)-d(v_1,v_2)|}{\left(d(v_1,v)+d(v,v_2)\right)^3}} & e=(v_1,v_2)\in E; \\ 0 & e=(v_1,v_2)\notin E. \end{cases}$$

Then the *modified* Haantjes curvature $\kappa'_{H,\pi} = \kappa'_H(v)$ of π at v is defined to be the arithmetic mean of the curvatures of all the triangles with apex v:

$$\kappa'_H(p) = \frac{\sum_{v_i\sim v,\, v_{ij}\sim v,\, v_{ij}\neq v_i}\kappa'_H(\triangle v_ivv_{ij})}{|\{\triangle v_ivv_{ij}\,|\,v_i\sim v, v_{ij}\sim v, v_{ij}\neq v_i\}|}$$

This represents a generalization of *Combinatorial Curvature*

Definition Let G be a (connected) graph and let v be a vertex of G, s.t. $\rho(v) \geq 2$, where $\rho(v)$ denotes the degree of v i.e. $\rho(v) = \left|\{u\,|\,u\sim v\}\right|$. The combinatorial curvature of G at v is defined as:

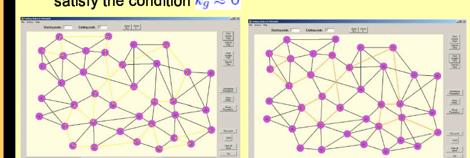
$$curv(v) \triangleq \frac{|\{\triangle vv_iv_{ij}\,|\,v_i\sim v, v_{ij}\sim v, v_{ij}\neq v\}|}{\rho(v)(\rho(v)-1)/2};$$

that is, it represents the ratio between the actual number of triangles and the maximum number of possible triangles with apex at v.

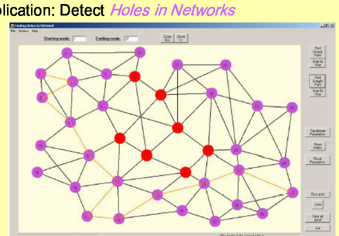
Applications

Quasi-Geodesics in Networks

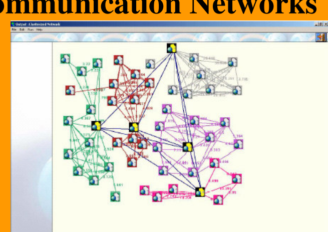
Use Haantjes curvature as *geodesic curvature* k_g in metric spaces, and define ***H-quasi-geodesics*** as the curves that satisfy the condition $k_g \approx 0$



Application: Detect *Holes in Networks*



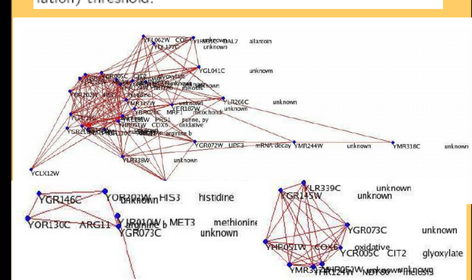
Communication Networks



Clustering

To perform clustering, one selects a curvature threshold $T_{curv} \in [0,1]^*$ and selects a subgraph $H_{T_{curv}} \subsetneq G$ by removing all nodes of curvature $< T_{curv}$ together with their adjacent edges.

DNA microarray data taken from http://rana.lbl.gov/EisenData.html is made into a graph by a method of correlation based "edging". Namely, one computes the correlation between different DNA microarrays and sets an edge between them according to a (correlation) threshold.



Here the metric is induced by the gene length, as they were shown to be relevant for the functioning of genes. More precisely we have employed – for the special case of gene length as weights – the following metric:

Definition Let (G,E,μ) be a connected vertex weighted graph. Define (for all $v \sim u$):

$$d(v,w) = \begin{cases} \frac{|\mu(v)|+|\mu(w)|}{|\mu(v)\mu(w)|} & v\neq w,\ \mu(v),\mu(w)\neq 0; \\ 1 & v\neq w,\ \mu(v)=0 \text{ or } \mu(w)=0; \\ 0 & v=w. \end{cases}$$

This project was supported by the Ollendorff Minerva Center http://visl.technion.ac.il Emil Saucan is a Viterbi Postdoctoral fellow