The Existence of Quasimeromorphic Mappings

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The existence of Qusimeromorphic Mappings

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"... the uncountable time of eternity had come to an end."

Gabriel Garcia Marquez – *The Autumn of the Patriarch*

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Abstract

It is classical that every Riemann surface carries non-constant meromorphic functions, implying that every Fuchsian group G has non-constant G-automorphic (meromorphic) functions.

In higher dimensions $n \geq 3$ the only locally conformal mappings are restrictions of Möbius transformations, and since they are injective, an extension of the classical existence theorem requires to look at quasimeromorphic mappings.

Following partial results by Martio, Srebro and Tukia on the problem of existence or non-existence of non-constant quasimeromorphic G-automorphic mappings (G being a discrete Möbius group acting on the hyperbolic space $\mathbb{H}^n$) we now give a complete characterization of all discrete Möbius groups G acting on hyperbolic space $\mathbb{H}^n$, that admit non-constant G-automorphic quasimeromorphic mappings, for any $n \geq 2$.

Following results by Tukia and Peltonen on the existence of non-constant quasimeromorphic mappings on complete $\mathcal{C}^\infty$ Riemannian manifolds, we now prove the existence of such mappings on manifolds with boundary, of lower differentiability class. Since the proofs are based on the existence of fat triangulations, we extend a classical result of Munkres by showing that every $\mathcal{C}^1$ manifold M^n with boundary consisting of finitely many compact components has a fat triangulation. We also prove that any fat triangulation of ∂M^n can be extended to M^n.

Notation

$\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ - the one point compactification of $\mathbb{R}^n$
$\mathbb{S}^{n-1}$ - the unit $(n-1)$-sphere in $\mathbb{R}^n$
$\mathbb{H}^n$ - the hyperbolic n-space
ACL - absolutely continuous on lines
ACL^p - absolutely continuous on lines with partial derivatives in L^p_{loc}
J_f - Jacobian
$f'(x)$, $Df(x)$ - the formal derivative of f at x
$K_O, K_O(f)$ - the outer dilatation of f
$K_I, K_I(f)$ - the inner dilatation of f
$K, K(f)$ - the maximal dilatation of f
B_f - the branch set of f
$G \rtimes X$ - the group G acts on the space X
$\mathbb{B}^n$ - the unit ball in $\mathbb{R}^n$, the ball model of $\mathbb{H}^n$
$\mathbf{H}^n_+$ - the half-space model of $\mathbb{H}^n$
$Isom_+(\mathbb{H}^n)$ - the group of orientation-preserving isometries of $\mathbb{H}^n$
d_{eucl} - Euclidean distance
d_{hyp} - Hyperbolic distance
$A(f)$ - the axis of the elliptic transformation f
$\sigma < \tau$ - σ is a face of τ
K - complex
$|K|$ - the polyhedron of the complex K
$St(a, K)$ - the star of a in K
Q - orbifold
X_Q - the underlying space of the orbifold Q
$Diff(\mathbb{R}^n)$ - the group of diffeomeorphisms of $\mathbb{R}^n$
$f : X \xrightarrow{\sim} Y$ - f is a homeomorphism between X and Y
$int\, U$ - the interior of U
$cl\, U$ - the closure of U
Σ_Q - the singular locus of the orbifold Q
$i(x, f)$ - the local topological index of f at x
$\mathcal{N}(x)$ - the set of neighbourhoods of x
$\measuredangle(\tau, \sigma)$ - the dihedral angle of $\sigma < \tau$
$\sigma_1 \pitchfork_\delta \sigma_2$ - the simplices σ_1 and σ_2 are δ-transverse
$Vol_j(\sigma)$ - the Euclidean j-volume of σ

$diam\,\sigma$ - the Euclidean diameter of σ
$\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ - triangulations
N_f, N_i - geometric neighbourhoods
$O^+(n)$ - the special orthogonal group

Chapter 1

Introduction

1.1 The Main Problems

Quasiconformal (qc), quasiregular (qr) and quasimeromorphic (qm) mappings in $\mathbb{R}^n, n \geq 2$ represent natural generalizations of conformal, analytic, and meromorphic functions in $\mathbb{C}$, respectively. The theory of these mappings represents an extension of the geometric and metric function theory in the plane.

In dimension $n = 2$, the study of quasimeromorphic mappings reduces to the study of quasiconformal mappings, namely injective quasiregular mappings. This is a consequence of the fact that any qm mapping is of the form $f = g \circ h$, where h is quasiconformal and g is meromorphic. On the other hand, by Liouville's theorem, any non-constant (n.c.) 1-quasimeromorphic mapping, i.e. with dilatation equal to one, is the restriction of a Möbius transformation. Therefore a meaningful theory in the case $n \geq 3$ requires dilatation greater than one.

Moreover, since in general the classical methods of complex analysis can not be applied for the study of quasiregular and quasimeromorphic mappings in $\mathbb{R}^n$, the methods of proof usually are more direct and mainly of a geometric nature. The class of quasiregular mappings was introduced in 1966 by Yu. G. Reshetnyak. The study of quasiregular and quasimeromorphic mappings was continued and further enhanced starting from 1969 by O. Martio, S. Rickman and J. Väisälä. Their approach was more geometric than that the original one of Reshetnyak, whose methods were analytic in nature.

The notion of qm extends to mappings $f : M^n \rightarrow \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$, where M^n is an n-manifold, by using coordinate charts..

Let G be a discrete group of Möbius transformations on the hyperbolic space $\mathbb{H}^n$. Following Martio and Srebro ([MS1]) a qm mapping $f : \mathbb{H}^n \rightarrow \widehat{\mathbb{R}^n}$ is called *automorphic* iff $f \circ g = g$, for all $g \in G$.

Our study concerns mainly two problems:

1. Characterize all discrete Möbius groups acting on $\mathbb{H}^n$ which have n.c. qm mappings.

2. Determine the existence of n.c. qm mappings $f : M^n \to \widehat{\mathbb{R}^n}$; where M^n is an orientable n-manifold with or without boundary.

The two problems are classical and completely solved for $n = 2$. In fact, every Riemann surface carries non-constant meromorphic mappings and therefore every discrete Möbius group acting on $\mathbb{C}$ or on $\mathbb{B}^2$ has non-constant automorphic mappings (see [Fo], [K]).

The question whether n.c. qm mappings exist for any $n \geq 3$ was originally posed by Martio and Srebro in [MS1]; subsequently in [MS2] they proved the existence of the above mentioned mappings in the case of finite co-volume groups, i.e. groups such that $Vol_{hyp}(\mathbb{H}^n/G) < \infty$. Also, it was later proved by Tukia ([Tu]) that the existence of n.c. automorphic qm mappings is assured in the case when G acts torsionfree on $\mathbb{H}^n$. Moreover, since for torsionfree discrete Möbius groups G, $\mathbb{H}^n/G$ is a (analytic) manifold, the next natural question to ask is whether there exist n.c. qm mappings $f : M^n \to \widehat{\mathbb{R}^n}$; where M^n is an orientable n-manifold. An affirmative answer to this question is due to Peltonen ([Pe]) in the case of connected, orientable $\mathcal{C}^\infty$ Riemannian manifolds (see B.2.1 below).

In contrast with the above results it was proved by Srebro ([Sr]) that, for any $n \geq 3$, there exists a Kleinian group $G \rtimes \mathbb{H}^n$ with no non-constant, G-automorphic functions $f : \mathbb{H}^n \to \mathbb{R}^n$. More precisely, he showed that for any $n \geq 3$ there exists a Kleinian group G (i.e. a discrete Möbius group) containing elliptics of unbounded orders with non-degenerate fixed set, and that these groups do not have non-constant G-automorphic qm mappings.

To obtain a complete answer to the existence problem we consider the case when the orders of all elliptics with non-degenerate fixed set are bounded, and show that such groups do carry n.c. qm automorphic mappings, in any dimension $n \geq 3$. This result, in conjunction with Srebro's non-existence theorem, gives a complete characterization of those Kleinian group which admit non-constant G-automorphic quasimeromorphic mappings.

The classical methods employed in proving the existence in the case $n = 2$ do not apply in higher dimensions. Therefore, different methods are needed. Following other researchers, we shall employ the classical "Alexander trick" (see [Al]). According to the Alexander method, first one constructs a G-invariant chessboard triangulation (Euclidian or hyperbolic) of $\mathbb{H}^n$, i.e. a triangulation whose simplices satisfy the condition that every $(n-2)$-face is incident to an even number of n-simplices. Then one alternately maps in G-invariant manner the simplices of the triangulation onto the interior and the exterior of the standard simplex in $\mathbb{R}^n$ using *qc* maps. If the dilatations of the *qc* maps constructed above are uniformly bounded, then the resulting mapping will be quasimeromorphic and G-automorphic.

If the simplices are uniformly *fat* (that is they satisfied a uniform non-degeneracy condition – for the formal definition see Section A.3 below), then the restrictions of the mapping to the simplices can be made quasiconformal with uniformly bounded dilatations, yielding a quasimeromorphic mapping. (see [MS2], [Tu]).

Another natural direction of study stems from Tukia's and Peltonen's theorems: since they proved the existence of quasimeromorphic mappings for complete (analytic) hyperbolic manifolds and $\mathcal{C}^\infty$ complete Riemannian manifolds, respectively. We want to prove the existence of quasimeromorphic mappings for manifolds with boundary, and when the regularity condition is relaxed. To this end we extend a classical theorem of Munkres regarding the existence of triangulation of manifolds with boundary, to the case of fat triangulations (see Theorem 2.2.1).

The existence of triangulations for $\mathcal{C}^1$ manifolds without boundary has been known since the classical work of Whithead ([Wh], 1940). This result was extended in 1960 by Munkres ([Mun]) to include $\mathcal{C}^r$ manifolds with boundary, $1 \leq r \leq \infty$. To be more precise, he proved that any $\mathcal{C}^r$ triangulation of the boundary can be extended to a $\mathcal{C}^r$ triangulation of the whole manifold.

Earlier, in 1934-1935, Cairns ([Ca1], [Ca2]) proved the existence of triangulations for compact $\mathcal{C}^1$ manifolds and for compact manifolds with boundary having a finite number of compact boundary components. It should be noted that, although far better known and widely cited, Whitehead's work is rooted in Cairns' studies. Moreover Cairns' and Whitehead's studies complement each other. Moreover, the Cairn's method produced *fat* triangulations, while Munkres' method produced fat simplices only away from the boundary.

The interest in the existence of a fat triangulations was rekindled by the study of quasiregular and quasimeromorphic functions, since the existence of a fat triangulations is crucial, as we have noted above, to the proof of existence of quasiregular (quasimeromorphic) mappings (see [MS2], [Tu]). In 1992 Peltonen ([Pe]) proved the existence of fat triangulations for $\mathcal{C}^\infty$ Riemannian manifolds, using methods partially based upon another technique of Cairns (originally developed for triangulating manifolds of class $\mathcal{C}^r, r \geq 2$).

Recently, the existence of fat triangulations has been revived in the context of Combinatorial Geometry and its various applications. Mashing and fattening methods for 'mesh improvement' – albeit for finite Euclidean triangulations (mostly in $\mathbb{R}^2$ and $\mathbb{R}^3$) were developed – see, e.g. [E], [PA], [Rup], [BCER].

1.2 Preliminaries

1.2.1 Quasimeromorphic mappings

We bring below the basic definitions and results concerning quasiregular and quasimeromorphic mappings that are needed here. The main reference text we employ is [Ric2]. Additional relevant material can be found in [V].

Definition 1.2.1 *Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$. A mapping $f : D \rightarrow \mathbb{R}^m$ is called ACL (absolutely continuous on lines) iff:*
(i) f is continuous

(ii) for any n-interval $Q = \bar{Q} = \{a_i \leq x_i \leq b_i \,|\, i = 1, \ldots, n\}$ in D, f is absolutely continuous on almost every line segment in Q, parallel to the coordinate axes.

Definition 1.2.2 *$f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ACL^p iff its derivatives are locally L^p integrable, $p \geq 1$.*

If $D \subseteq \mathbb{R}^n$ is a domain and if $f : D \rightarrow \mathbb{R}^n$, then $f \in ACL^p$ iff f is continuous and belongs to the *Sobolev space* $W^1_{p,loc}$ (see [Ric2], pp. 5-11).

Definition 1.2.3 *Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$ and let $f : D \rightarrow \mathbb{R}^n$ be a continuous mapping. f is called*

1. *quasiregular iff (i) f is ACL^n*
   and
   (ii) there exists $K \geq 1$ such that

   $$|f'(x)|^n \leq K J_f(x) \text{ a.e.} \tag{1.2.1}$$

   where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup_{|h|=1} |f'(x)h|$, and where $J_f(x) = det f'(x)$.
   The smallest K that satisfies (1.2.1) is called the outer dilatation $K_O(f)$ of f.

2. *quasiconformal iff $f : D \rightarrow f(D)$ is a quasiregular homeomorphism.*

3. *quasimeromorphic iff $f : D \rightarrow \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is quasiregular, where the condition of quasiregularity at $f^{-1}(\infty)$ can be checked by conjugation with auxiliary Möbius transformations.*

Remark 1.2.4 *For other, equivalent definitions of quasiregularity, see [Car], [He], [Ric2], [V].*

One can extend the definitions above to oriented, connected $\mathcal{C}^\infty$ Riemannian manifolds as follows: let M^n, N^n be to oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \rightarrow N^n$ be a continuous function. One can define the formal derivative of f by using coordinate charts.

Definition 1.2.5 *Let M^n, N^n be oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \rightarrow N^n$ be a continuous function. f is called locally quasiregular iff for every $x \in M^n$, there exist coordinate charts (U_x, φ_x) and $(V_{f(x)}, \psi_{f(x)})$, such that $f(U_x) \subseteq V_{f(x)}$ and $g = \psi_{f(x)} \circ f \circ \varphi_x^{-1}$ is quasiregular.*

If f is locally quasiregular, then $T_x f : T_x(M^n) \rightarrow T_{f(x)}(N^n)$ exist for a.e. $x \in M^n$ (see [V], 26.4).

Definition 1.2.6 *Let M^n, N^n be to oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \to N^n$ be a continuous function. f is called quasiregular iff*
(i) f is locally quasiregular
and
(ii) there exists $K, 1 \leq K < \infty$, such that

$$|T_x f|^n \leq K J_f(x) \tag{1.2.2}$$

for a. e. $x \in M^n$.
A quasiregular homeomorphism is called a quasiconformal mapping.

Remark 1.2.7 *For more details regarding the definition of quasiconformal and quasiregular mappings on manifolds, see [Suo].*

Definition 1.2.8 *Let $f : M^n \to N^n$ be a quasiregular mapping. The set $B_f = \{x \in M^n \mid f \text{ is not a local homeomorphism at } x\}$ is called the branch set (or critical set) of f.*

Definition 1.2.9 *Let $f : D \to \mathbb{R}^n$ be an orientation preserving map. The local topological index of f at x is defined as:*

$$i(x, f) = \inf_{U \in \mathcal{N}(x)} \sup_y |f^{-1}(y) \cap U|$$

If $f : M^n \to N^n$ is quasiregular, then there exists $K \geq 1$ such that the following inequality holds a.e. in M^n:

$$J_f(x) \leq K' \inf_{|h|=1} |T_x f h|^n \tag{1.2.3}$$

By analogy with the outer dilatation we have the following definition:

Definition 1.2.10 *The smallest number K' that satisfies inequality (1.2.3) is the inner dilation $K_I(f)$ of f, and $K(f) = \max(K_O(f), K_I(f))$ is the maximal dilatation of f. If $K(f) < \infty$ we say that f is called K-qr.*

The dilations are $K(f), K_O(f)$ and $K_I(f)$ are simultaneously finite or infinite. Indeed, the following inequalities hold: $K_I(f) \leq K_O^{n-1}(f)$ and $K_O(f) \leq K_I^{n-1}(f)$.

We bring a few facts regarding the index of quasiregular mapping: First, note that if $f : M^n \to N^n$ be a quasiregular mapping, then $i(x, f) \geq 1$. Moreover: $x \in B_f$ iff $i(x, f) > 1$. Also, let $f : D \to \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$, $n \geq 3$, be a non-constant quasimeromorphic mapping. Then:

1. The local topological index cannot be uniformly too large on all the points of a non-degenerate continuum F. To be more precise, the following inequality holds:
$$\inf_{x \in F} i(x, f) < n^{n-1} K_I(f)\,.$$
(See e.g. [Ric2], III. 5.9.)

   However:

2. The local topological index can be arbitrarily large at an isolated point (see [Ric1], pp. 263-264).

Note that, by classical results of Reshetnyak, if $f : D \to \mathbb{R}^n$ is a non-constant quasiregular mapping, then:

1. f is open (i.e. it maps open sets in D onto open sets in), discrete (i.e. $f^{-1}(y)$ is a discrete set for any $y \in \mathbb{R}^n$) and orientation-preserving (see e.g. [Ric2], VI. 5.7.);

   and

2. f maps sets of zero measure onto sets of zero measure (see e.g. [Ric2], [V]).

We conclude this section with the following remark: If $D \subseteq \mathbb{R}^n$ is a domain and if $f : D \to \mathbb{R}^n$ is an L-bilipschitz map, then that is f is quasiconformal and $K(f) \leq L(f)^{2(n-1)}$ (see [V]).

1.2.2 Kleinian Groups and Elliptic Transformations

We bring the basic definitions and results concerning Kleinian groups which are needed here. For proofs see [Abi], [AP2], [Ms], [Th1].

We denote by $\mathbb{H}^n$ the hyperbolic n-space. We shall employ either one of the Poincaré conformal models: the unit ball model $\mathbb{B}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,\big|\, ||x|| < 1\}$, equipped with the metric $ds^2 = \frac{dx^2}{(1-r^2)^2}$, where dx denotes the Euclidean length element and r denotes the distance from the origin; or the half-space model $\mathbf{H}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,\big|\, x_n > 0\}$ equipped with the metric $ds^2 = \frac{dx^2}{(x_n)^2}$. In both models planes are generalized half-spheres orthogonal to $\partial\mathbb{B}^n$, respectively $\partial\mathbf{H}^n_+$. In particular, geodesic lines are generalized half-circles orthogonal to $\partial\mathbb{B}^n$, respectively $\partial\mathbf{H}^n_+$.

Let $Isom_+(\mathbb{H}^n)$ denote the group of orientation-preserving isometries of $\mathbb{H}^n$. If $f \in Isom_+(\mathbb{H}^n)$, then it extends to a homeomorphism of $\partial\mathbb{B}^n$. Thus, by Brouwer fixed-point theorem, f has a fixed point in the compact set $\mathbb{B}^n \cup \partial\mathbb{B}^n$. We classify hyperbolic isometries by their fixed points, as follows:

Definition 1.2.11 *A transformation $f \in Isom_+(\mathbb{H}^n)$, $f \neq Id$ is called:*

1. *elliptic iff f has a fixed point in $\mathbb{H}^n$;*

2. *parabolic iff f has no fixed points in $\mathbb{H}^n$ and has exactly one fixed point in $\partial\mathbb{H}^n$;*

3. *loxodromic iff f has no fixed points in $\mathbb{H}^n$ and exactly two fixed points in $\partial\mathbb{H}^n$.*

Definition 1.2.12 *Let G be a topological group, and let $H \subseteq G$. Then H is called discrete iff H, with the induced topology, is a discrete space.*

Note that $H \subseteq G$ is discrete iff the identity is an isolated element of H.

It is also well known that if $G < Isom_+(\mathbb{H}^n)$ is a discrete group of orientation-preserving isometries, then $|G| \leq \aleph_0$.

Recall that a group G of homeomorphisms acts *properly discontinuously* on a a locally compact topological space iff the following conditions hold for any $g \in G_x$: (a) the stabilizer of x, $G_x = \{g \in G \,|\, g(x) = x\}$ is finite; and (b) there exists a neighbourhood V_x of x, such that (b_1) $g(V_x) \cap V_x = \emptyset$, for any $g \in G \setminus G_x$; and (b_2) $g(V_x) \cap V_x = V_x$. While every discontinuous group is discrete, not every discrete group is properly discontinuous (see [Ms], [AP2]). However, the following holds: $G < Isom_+(\mathbb{H}^n)$ is properly discontinuous iff G acts discretely on $\partial\mathbb{H}^n$.

Definition 1.2.13 *A properly discontinuous group* $G < Isom_+(\mathbb{H}^n)$ *is called a Kleinian group.*

Let $G < Isom_+(\mathbb{H}^n)$ be a discrete group. If G contains no elliptic elements, then $\mathbb{H}^n/G$ is a complete hyperbolic manifold. If G does contain elliptic elements, then $\mathbb{H}^n/G$ is a complete hyperbolic orbifold. Recall that a smooth geometric n-orbifold Q is – loosely speaking – a connected Hausdorff space X_Q (*the underlying space*) locally modelled on $\mathbb{R}^n$, modulo a finite group $G < Diff(\mathbb{R}^n)$ – the group of diffeomorphisms of $\mathbb{R}^n$. The elements of G are called *folding maps.* More generally, let M^n be a smooth manifold, and let G be a group that acts properly discontinuous on M^n. Then M^n/G is a smooth orbifold.

Note however that, in general, $X_Q \not\approx Q$. Indeed, Q is not necessarily a manifold. (For example any manifold with boundary M has an orbifold structure, where the points on the boundary ∂M^n have neighbourhoods homeomorphic to $\mathbb{R}^n/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection in hyperplane.) However, any 2-dimensional orbifold is homeomorphic to a manifold. Moreover, any 3-dimensional orbifold whose folding maps are all orientation-preserving is homeomorphic to a manifold. In particular, if G is a Kleinian group acting with torsion on $\mathbb{H}^3$, then $X_Q = \mathbb{H}^3/G$ is topologically a manifold.

If $f \in G$ is an elliptic transformation, where $G < Isom(\mathbb{H}^n)$ is a discrete group, then there exists $m \geq 2$ such that $f^m = Id$, and the smallest m satisfying this condition is called the *order* of f. In the 3-dimensional case the *fixed point set* of f i.e. $Fix(f) = \{x \in \mathbb{H}^3 | f(x) = x\}$, is a hyperbolic line and will be denoted by $A(f)$ – the *axis of f.* In dimension $n \geq 4$ the fixed set (or *axis of f*) of an elliptic transformation is a k-dimensional hyperbolic plane, $0 \leq k \leq n-2$. An axis A is called *degenerate* iff $dim\, A = 0$. In dimensions higher than $n = 3$, different elliptics may have fixed sets of different dimensions.

If G is a discrete group, G is countable so we can write $G = \{g_j\}_{j \geq 1}$ and let $\{f_i\}_{i \geq 1} \subset G$ denote the set of elliptic elements of G. Therefore the set $\mathcal{A} = \{A_{f_i}\}$ is countable, hence the set $\mathcal{S} = \{C_j\}$ of connected components of the singular set $\mathcal{A}^* = \bigcup_i A_i$ is also countable. Moreover, by the discreteness of G, it follows that the set $\mathcal{A}$ – and hence $\mathcal{S}$ – have no accumulation points in $\mathbb{H}^n$.

Remark 1.2.14 *Given any finitely generated Kleinian group acting on* $\mathbb{H}^3$ *the number of conjugacy classes of elliptic elements is finite (see [FM]). However, this is not true for Kleinian groups acting upon* $\mathbb{H}^n$*,* $n \geq 4$ *; (for counterexamples, see [FM], [Po] and [H]).*

Remark 1.2.15 *Hamilton ([H], Theorem 4.1.) constructed examples of Kleinian groups* G *acting on* $\mathbb{H}^4$ *such that there exists an infinite sequence* $\{f_n\}_{n\in\mathbb{N}} \subset G$ *of elliptic transformations, with* $ord(f_n) \to \infty$ *and such that the fixed set of each* f_n *is degenerate.*

If the discrete group G is acting upon $\mathbb{H}^n$, then by the discreteness of G, there exists no accumulation point of the elliptic axes in $\mathbb{H}^n$. Moreover, if G contains no elliptics with intersecting axes, and if G contains no order two elliptics, then the distances between the axes are bounded from bellow (see [GM1]). Otherwise it is possible that $dist_{hyp}(A_i, A_j) \underset{i,j}{\longrightarrow} \infty$. However our proof of Theorem 2.1.1 holds in all cases.

Let $f \in Isom(\mathbb{H}^n)$ be an elliptic transformation, let $g \in Isom(\mathbb{H}^n)$, and let $x \in A(x)$, such that $f \circ g = f$. Then the order of g divides $i(x, f)$ (see [Sr]). It follows, by Remark 2.1.10 that the following holds:

Proposition 1.2.16 ([Sr], Proposition 1.2.) *Let* G *be a Kleinian group, acting upon* $\mathbb{H}^n, n \geq 3$*. If* G *has elliptic elements of arbitrarily large orders with non-degenerate fixed sets, then any* G*-invariant quasimeromorphic mappings is constant.*

Moreover we have the following theorem:

Theorem 1.2.17 ([Sr], Theorem 3.1.) *For any* $n \geq 2$*, there exits a Kleinian group* G*, acting upon* $\mathbb{H}^n$*, which contains elliptic elements* g_k*, with non-degenerate fixed sets, such that* $ord(g_k) = k, k \geq 2$*.*

1.2.3 Triangulations

We recall a few classical definitions and notations:

Definition 1.2.18 *Let* $a_0, \ldots, a_m \in \mathbb{R}^n$*.* $a_0, \ldots, a_m$ *are called independent iff the vectors* $v_i = a_0a_i\,, i = 1, ..., m$ *are linearly independent.*

The set $\sigma = a_0a_1 \ldots a_m = \{x = \alpha_i a_i \,|\, \alpha_i \geq 0, \sum \alpha_i = 1\}$ *is called the* m*-simplex spanned by* $a_0, \ldots, a_m$*. The points* $a_0, \ldots, a_m$ *are called the vertices of* σ*.*

The numbers α_i *are called the barycentric coordinates of* σ*. The point* $\tilde{\sigma} = \sum \frac{\alpha_i}{m+1}$ *is called the barycenter of* σ*.*

If $\{a_0, \ldots, a_k\} \subseteq \{a_0, \ldots, a_m\}$*, then* $\tau = a_0 \ldots a_k$ *is called a face of* σ*, and we write* $\tau < \sigma$*.*

Definition 1.2.19 *Let* $A, B \subset \mathbb{R}^n$*. We define the join* $A * B$ *of* A *and* B *as* $A * B = \{\alpha a + \beta b \,|\, a \in A, b \in B\,;\, \alpha, \beta \geq 0,\ \alpha + \beta = 1\}$*. If* $A = \{a\}$*, then* $A * B$ *is called the cone with vertex* a *and base* B*.*

Definition 1.2.20 *A collection K of simplices is called a simplicial complex if*

1. *If $\tau < \sigma$, then $\tau \in K$.*

2. *Let $\sigma_1, \sigma_2 \in K$ and let $\tau = \sigma_1 \cap \sigma_2$. Then $\tau < \sigma_1, \tau < \sigma_1$.*

3. *K is locally finite.*

$|K| = \bigcup_{\sigma \in K} \sigma$ *is called the underlying polyhedron (or polytope) of K.*

Definition 1.2.21 *A complex K' is called a subdivision of K iff*

1. *$K' \subset K$;*

2. *if $\tau \in K'$, then there exists $\sigma \in K$ such that $\tau \subseteq \sigma$.*

If K' is a subdivision of K we denote it by $K' \lhd K$.
Let K be a simplicial complex and let $L \subset K$. If L is a simplicial complex, then it is called a subcomplex of K.

Definition 1.2.22 *Let $a \in |K|$. Then*

$$St(a, K) = \bigcup_{\substack{a \in \sigma \\ \sigma \in K}} \sigma$$

is called the star of $a \in K$.

If $S \subset K$, then we define: $St(S, K) = \bigcup_{a \in S} St(a, K)$.

Definition 1.2.23 *Let $\sigma = a_0 a_1 \ldots a_m$ and let $f : \sigma \to \mathbb{R}^p$. The map f is called linear iff for any $x = \sum \alpha_i a_i \in \sigma$, it holds that $f(x) = \sum \alpha_i f(a_i)$.*
Let K, L be complexes, and let $f : |K| \to |L|$. Then f is called linear (relative to K and L) iff for any $\sigma \in K$, $\tau = f(\sigma) \in L$.
The map $f : K \to L$ is called piecewise linear (PL) iff there exists a subdivision K' of K such that $f : K' \to L$ is linear.
If (i) $f : K \to L$ is a homeomorphism of $|K|$ onto $|L|$, (ii) $f|_\sigma$ is linear and (iii) $\tau = f|_\sigma \in L$, for any $\sigma \in K$, then f is called a linear homeomorphism.

Definition 1.2.24 *A cell γ is a bounded subset of $\mathbb{R}^n$ defined by:*

$$\gamma = \{x \in \mathbb{R}^n \mid \sum_j \alpha_{ij} x_j \geq \beta_i;\ i = 1, \ldots, p\},$$

for some constants $\alpha_{i,j}$ and β_i.
The dimension m of γ is defined as $\min\{dim\,\Pi \mid \gamma \subset \Pi, \Pi$ *a hyperplane in* $\mathbb{R}^n\}$.

Let γ be an m-dimensional cell. The $(m-1)$-cells β_j of $\partial\gamma$ are called its $(m-1)$-faces, the $(m-2)$-faces of each β_j are called the $(m-2)$-faces of γ, etc. By convention $\emptyset$ and γ are also faces of γ.

A cell complex is defined in the same manner as a simplicial complex, more exactly, a cell complex K is a collection of cells that satisfy conditions 1.–3. of Definition 1.2.20.

Subcomplexes are also defined analogous to the simplicial case. In particular, the q skeleton K^q of K, $K^q = \{\gamma \,|\, \gamma \in K,\ dim\, \gamma \leq q\}$ is a subcomplex of K.

Lemma 1.2.25 *Let K be cell complex. Then K has a simplicial subdivision.*

Proof See [Mun], Lemma 7.8 and Appendix B.2.3.

□

We define the concept of embedding for complexes, but first we need some basic definitions:

Definition 1.2.26 *Let K be a simplicial complex.*

1. *$f : |K| \to M^n$ is $\mathcal{C}^r$ differentiable (relative to $|K|$) iff $f|_\sigma \in \mathcal{C}^r(\sigma)$, for any simplex $\sigma \in K$.*

2. *$f : |K| \to M^n$ is non-degenerate iff $rank(f|_\sigma) = dim(\sigma)$, for any simplex $\sigma \in K$.*

Definition 1.2.27 *Let σ be a simplex, and let $f : \sigma \to \mathbb{R}^n$, $f \in \mathcal{C}^r$. If $a \in \sigma$ we define $df_a : \sigma \to \mathbb{R}^n$ as follows: $df_a(x) = Df(a) \cdot (x - a)$, where $Df(a)$ denotes the formal derivative $Df(a) = (\partial f_i/\partial x^j)_{1\leq i,j\leq n}$, computed with respect to some orthogonal coordinate system contained in $\Pi(\sigma)$, where $\Pi(\sigma)$ is the hyperplane determined by σ. The map $df_a : \sigma \to \mathbb{R}^n$ does not depend upon the choice of this coordinate system.*

Note that $df_a|_{\sigma\cap\tau}$ is well defined, for any $\sigma, \tau \in \overline{St}(a, K)$. Therefore the map $df_a : \overline{St}(a, K) \to \mathbb{R}^n$ is well-defined and continuous, and it is called – analogous to the case of differentiable manifolds – the differential of f.

Remark 1.2.28 *In contrast to the differential case, the tangent space $T_{f(p)}(M^n)$ is a union of of polyhedral tangent cones, Therefore it possess no natural vector space structure (see [Th2], p. 196).*

Definition 1.2.29 *Let K be a simplicial complex, let M^n be a $\mathcal{C}^r$ submanifold of $\mathbb{R}^N$, and let $f : K \to M^n$ be a $\mathcal{C}^r$ map. Then f is called*

1. *an immersion iff $df_\sigma : \overline{St}(\sigma, K) \to \mathbb{R}^n$ is injective for each and every $\sigma \in K$;*

2. *an embedding iff it is an immersion and a homeomorphism on the image f(K);*

3. *a $\mathcal{C}^r$ triangulation iff it is an embedding such that $f(K) = M^n$.*

Remark 1.2.30 *If the class of the map f is not relevant, f will be called simply a triangulation.*

1.2.4 Fat Triangulations

We shall employ mainly Cheeger's definition ([CMS]) of fatness:

Definition 1.2.31 *Let $\tau \subset \mathbb{R}^n$; $0 \leq k \leq n$ be a k-dimensional simplex. The fatness φ of τ is defined as being:*

$$\varphi = \varphi(\tau) = \inf_{\substack{\sigma<\tau \\ dim\,\sigma = l}} \frac{Vol(\sigma)}{diam^l\,\sigma} \tag{1.2.4}$$

The infimum is taken over all the faces of τ, $\sigma < \tau$, and $Vol_{eucl}(\sigma)$ and $diam\,\sigma$ stand for the Euclidian l-volume and the diameter of σ respectively. (If $dim\,\sigma = 0$, then $Vol_{eucl}(\sigma) = 1$, by convention.)
A simplex τ is φ_0-fat, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$. A triangulation (of a submanifold of $\mathbb{R}^n$) $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is φ_0-fat if all its simplices are φ_0-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is fat if there exists $\varphi_0 > 0$ such that all its simplices are φ_0-fat.

Remark 1.2.32 *There exists a constant $c(k)$ that depends solely upon the dimension k of τ s.t.*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\substack{\sigma<\tau \\ dim\,\sigma = l}} \measuredangle(\tau, \sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{1.2.5}$$

and

$$\varphi(\tau) \leq \frac{Vol(\sigma)}{diam^l\,\sigma} \leq c(k) \cdot \varphi(\tau)\,; \tag{1.2.6}$$

where $\measuredangle(\tau, \sigma)$ denotes the (internal) dihedral angle of $\sigma < \tau$.

We give a formal definition of the notion of (*internal*) *dihedral angle* (see also [Som] IV. 2, IX. 15), but first we need a few preliminary definitions:

Definition 1.2.33 *A simplicial cone $C^k \subset \mathbb{R}^k \subset \mathbb{R}^n$, is defined as: $C^k = \bigcap_{j=1}^{k} H_j$, where H_j are open half spaces in general position, such that $0 \in H_j$, $j = 1, \ldots, k$.*

$L^{k-1} = C^k \bigcap \mathbb{S}^{n-1}$ is called a spherical simplex.

Definition 1.2.34 *Consider the simplices $\sigma^k < \tau^m$, and let $p \in \sigma^k$. Define the normal cone: $C^{\perp}(\sigma^k, \tau^m) = \{\overrightarrow{px} \,|\, x \in \tau^m,\ \overrightarrow{px} \perp \sigma^k\}$, where $\overrightarrow{px}$ denotes the ray through x and base-point p.*

The spherical simplex $L(\sigma^k, \tau^m)$ associated to $C^{\perp}(\sigma^k, \tau^m)$ is called the link of σ^k in τ^m.

Remark 1.2.35 *$C^{\perp}(\sigma^k, \tau^m)$ does not depend upon the choice of p.*

Definition 1.2.36 *Denote by $\measuredangle(\tau^k, \sigma^m)$ the normalized volume of $L(\sigma^k, \tau^m)$, where the normalization is such that the volume of $\mathbb{S}^{n-1}$ equals 1, for any $n \geq 2$.*

Although briefer, Peltonen's definition of fatness is less convenient for actual computations:

Definition 1.2.37 *A k-simplex* $\tau \subset \mathbb{R}^n$ *(or* $\mathbb{H}^n$*);* $2 \leq k \leq n$ *is* φ*-fat if there exists* $\varphi \geq 0$ *such that the ratio* $\frac{r}{R} \geq \varphi$*; where* r *denotes the radius of the inscribed sphere of* τ *(*inradius*) and* R *denotes the radius of the circumscribed sphere of* τ *(*circumradius*).*
A triangulation of a submanifold of $\mathbb{R}^n$ *(or* $\mathbb{H}^n$*)* $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ *is* φ*-fat if all its simplices are* φ*-fat. A triangulation* $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ *is fat if there exists* $\varphi \geq 0$ *such that all its simplices are* φ*-fat;* $\forall i \in \mathbf{I}$.

Remark 1.2.38 *For other equivalent definitions of fatness, see [Ca1], [Ca2], [Mun].*

The reminder of this work is structured as follows: In Chapter 3 we present our main results, their proofs being sketched in Chapter 4. The full proofs are presented in Appendices A, and B, which represent our papers [S2], [S3], respectively.

Chapter 2

Results

In this chapter we list our main results. We present here a sketch of the proofs and methods employed therein. The details are given in Appendices A and B, which form our papers [S2] and [S3], respectively.

2.1 The Existence of Automorphic Quasimeromorphic Mappings

The following existence theorem , that represents a generalization of previous results of Tukia ([Tu]) and Martio and Srebro ([MS2]), is the main result in this topic. While we present bellow an outline of the proof, the full details are given in Appendix A, which forms the paper [S2].

Theorem 2.1.1 *Let G be a Kleinian group with torsion acting upon $\mathbb{H}^n, n \geq 3$. If the elliptic elements (i.e. torsion elements) of G with non-degenerate fixed set have uniformly bounded orders, then there exists a non-constant G-automorphic quasimeromorphic mapping $f : \mathbb{H}^n \to \widehat{\mathbb{R}^n}$.*

The theorem above, together with Srebro's non-existence theorem, gives the following complete characterization of the discrete groups which carry non-constant automorphic mappings:

Theorem 2.1.2 *Let G be a Kleinian group acting on $\mathbb{B}^n$. Then G admits non-constant automorphic qm-mappings iff:*

1. *$n = 2$;*
2. *$n \geq 3$, and the orders of the elliptic elements of G having non-degenerate fixed sets are bounded.*

Note that, by applying Remark 1.2.14, it follows that we get the following corollary:

Corollary 2.1.3 *Let G be a finitely generated Kleinian group with torsion acting upon $\mathbb{H}^3$. Then there exists a non constant G-automorphic quasimeromorphic mapping $f : \mathbb{H}^3 \to \widehat{\mathbb{R}^3}$.*

The idea of the proof of Theorem 2.1.1 is, in a nutshell, as follows: Based upon the geometry of the elliptic transformations construct a fat triangulation $\mathcal{T}_1$ of $N_e^* = (\bar{N}_e \cap \mathbb{B}^n)/G$. Since $M_p = (\mathbb{B}^n/\,G) \setminus N_e^*$ is an analytic manifold, we can apply Peltonen's result to gain a triangulation $\mathcal{T}_2$ of M_p. Therefore the set $T_e^* = T_e/G$ will be endowed with two triangulations: the restrictions of $\mathcal{T}_1$ and $\mathcal{T}_2$.

'Mash' $\mathcal{T}_1$ and $\mathcal{T}_2$ i.e. ensure – by applying infinitesimal moves of the vertices so that the two triangulation will intersect in general position with respect to each other, and by performing suitable subdivisions – that the given triangulations intersect into a new triangulation $\mathcal{T}_0$ (see [Mun], Theorem 10.4).

Modify $\mathcal{T}_0$ to receive a new fat triangulation $\mathcal{T}$ of $\mathbb{B}^n/G$.

In the presence of degenerate components $A_k = A(f_k)$ of the singular sets, where the transformations f_k have arbitrarily large orders, a modification of this construction is needed – see Section A.

Apply Alexander's trick to receive a quasimeromorphic mapping $f : \mathbb{B}^n/G \to \widehat{\mathbb{R}^n}$. The lift $\widetilde{f}$ of f to $\mathbb{B}^n$ represents the required G-automorphic quasimeromorphic mapping.

We employ a method for fattening triangulations developed in [CMS]. While for full details we refer the interested reader to [CMS], we bring in Appendix B.2.3 a succinct presentation of the main steps of the fattening process. The essential tool of this method is the following result, which represents a slight modification of [CMS], Lemma 6.3.:

Proposition 2.1.4 *Let $\mathcal{T}_1, \mathcal{T}_2$ be two fat triangulations of open sets $U_1, U_2 \subset M^n$, $U_1 \cap U_2 \neq \emptyset$ having common fatness $\geq \varphi_0$. Then there exist fat triangulations $\mathcal{T}_1', \mathcal{T}_2'$ and there exist open sets $U \subset U_1 \cap U_2 \subset V$, such that*

1. $(\mathcal{T}_1' \cap \mathcal{T}_2')\big|_{U_i \setminus V} = \mathcal{T}_i\,,\ i = 1, 2\,;$

2. $(\mathcal{T}_1' \cap \mathcal{T}_2')\big|_{U} = \mathcal{T};$

where

3. *$\mathcal{T}$ is a fat triangulation of U.*

A different proof for the case $n = 3$ is given in [S1]. In this proof we develop and use a technique for mashing distinct fat triangulations while preserving fatness, technique that employs mainly elementary tools. This technique can be adapted to higher dimensions and it is also relevant in Computational Geometry and Mathematical Biology (see [S4]).

2.2 The Existence of Fat Triangulations and the Existence of Quasimeromorphic Mappings on Manifolds

The main results we have proved in this topic are listed below. An outline of the proof is given in the next chapter and the full proof is given Appendix B, which forms the paper [S3]. The following theorem represents a generalization of [Mun], Theorem 10.6.

Theorem 2.2.1 *Let M^n be an n-dimensional $\mathcal{C}^\infty$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then any uniformly fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Remark 2.2.2 *We prove that the theorem above also holds when the compactness condition of the boundary components is replaced by the condition that ∂M^n is endowed with a fat triangulation $\mathcal{T}$ such that $\inf\limits_{\sigma \in \mathcal{T}} diam\, \sigma > 0$.*

Since every PL manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) smoothing (see [Mun1], [Mun], [Th2]), and every topological manifold of dimension $n \leq 3$ admits a PL structure (cf. [Moi], [Th2]), we obtain from our results the following corollary:

Corollary 2.2.3 *Let M^n be an n-dimensional, $n \leq 4$ (resp. $n \leq 3$), PL (resp. topological) connected manifold with boundary, having a finite number of compact boundary components. Then any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

By applying Alexander's Trick to Theorem 2.2.1, we obtain the following theorem about the existence of quasimeromorphic mappings. This theorem represents a generalization of Peltonen's Theorem (see [Pe], II.3.).

Theorem 2.2.4 *Let M^n be a connected, oriented C^1 Riemannian manifold without boundary or having a finite number of compact boundary components. Then there exists a non-constant quasimeromorphic mapping $f : M^n \rightarrow \widehat{\mathbb{R}^n}$.*

And thus, by Corollary 2.2.3 we obtain, in addition, the following corollary:

Corollary 2.2.5 *Let M^n be a connected, oriented C^1 n-dimensional manifold ($n \geq 2$), without boundary or having a boundary consisting of a finite number of compact boundary components.*
Then in each of the following cases there exists a non-constant quasimeromorphic mapping $f : M^n \rightarrow \widehat{\mathbb{R}^n}$:

1. *M^n is a PL manifold and $n \leq 4$;*

2. M^n is a topological manifold and $n \leq 3$.

In the proof of Theorem 2.2.4 we again use Alexander's Trick, Peltonen's result and Cheeger's fattening method.

We conclude by noticing that the arguments of the proofs of Theorems 2.1.1 and Theorem 2.2.1 extend to include any orientable (see [Dr], p. 46) geometric orbifold with tame singular locus (at least in dimension 3) and with isotropy groups (see e.g. [Th2]) with bounded orders. Moreover, since the constructions applies also in the case when the groups Γ_i contain orientation-reversing elements, one can extend Theorem 2.1.1 to prove the existence of G-automorphic quasimeromorphic mappings, for any group $G < Isom(\mathbb{H}^n)$. In this case one employs (following Reshetnyak) a somewhat different definition of quasiregularity, with $J_f(x)$ being replaced by $|J_f(x)|$. This allows for orientation-reversing quasiregular (and quasimeromorphic) mappings (see [Vu] 10.8, 10.9).

Appendix A

The Existence of Quasimeromorphic Mapping

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ABSTRACT. Let G be a Kleinian group G acting on $\mathbb{B}^n$, $n \geq 2$. We show that if the orders of the elliptic elements in G which have non-degenerate fixed set are bounded, then G carries non-constant G-automorphic quasimeromorphic mappings. This together with an earlier non-existence theorem by Srebro gives a complete characterization of Kleinian groups that admit non-constant quasimeromorphic automorphic mappings.

A.1 Introduction

Definition A.1.1 *Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$ and let $f : D \to \mathbb{R}^n$ be a continuous mapping. f is called*

1. *quasiregular iff (i) f belongs to $W^{1,n}_{loc}(D)$ and*
   (ii) there exists $K \geq 1$ such that:

$$|f'(x)|^n \leq K J_f(x) \ a.e. \tag{A.1.1}$$

   where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup_{|h|=1} |f'(x)h|$, and where $J_f(x) = det f'(x)$.

2. *quasiconformal iff $f : D \to f(D)$ is a quasiregular homeomorphism.*

3. *quasimeromorphic iff $f : D \to \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is quasiregular, where the condition of quasiregularity at $f^{-1}(\infty)$ can be checked by conjugation with auxiliary Möbius transformations.*

The smallest number K that satisfies (B.3.1) is called the outer dilatation of f.

One can extend the definitions above to oriented, connected $\mathcal{C}^\infty$ Riemannian manifolds as follows:

Definition A.1.2 *Let M^n, N^n be oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \rightarrow N^n$ be a continuous function. f is called locally quasiregular iff for every $x \in M^n$, there exist coordinate charts (U_x, φ_x) and $(V_{f(x)}, \psi_{f(x)})$, such that $f(U_x) \subseteq V_{f(x)}$ and $g = \psi_{f(x)} \circ f \circ \varphi_x^{-1}$ is quasiregular.*

If f is locally quasiregular, then $T_x f : T_x(M^n) \rightarrow T_{f(x)} N^n$ exist for a.e. $x \in M^n$.

Definition A.1.3 *Let M^n, N^n be oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \rightarrow N^n$ be a continuous function. f is called quasiregular iff*
(i) f is locally quasiregular
and
(ii) there exists $K, 1 \leq K < \infty$, such that

$$|T_x f|^n \leq K J_f(x) \tag{A.1.2}$$

for a. e. $x \in M^n$.

Recall that a group G of homeomorphisms acts *properly discontinuously* on a locally compact topological space X iff the following conditions hold for any $g \in G_x\,, x \in X$: (a) the stabilizer $G_x = \{g \in G \,|\, g(x) = x\}$ of x is finite; and (b) there exists a neighbourhood V_x of x, such that (b_1) $g(V_x) \cap V_x = \emptyset$, for any $g \in G \setminus G_x$; and (b_2) $g(V_x) \cap V_x = V_x$.

Definition A.1.4 *A discontinuous group of orientation-preserving isometries of $\mathbb{B}^n$ is called a Kleinian group.*

It is well known that a discontinuous group is discrete (see [Ms]).

Definition A.1.5 *Let $f : \mathbb{B}^n \rightarrow \widehat{\mathbb{R}^n}$, and let G be a Kleinian group acting upon $\mathbb{B}^n$. The function f is called G-automorphic iff:*

$$f(g(x)) = f(x)\,; \quad \text{for any } x \in \mathbb{B}^n \text{ and for all } g \in G\,; \tag{A.1.3}$$

Recall the definition of elliptic transformations:

Definition A.1.6 *A Möbius transformation $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$, $f \neq Id$ is called elliptic iff f has a fixed point in $\mathbb{B}^n$.*

The existence of non-constant automorphic meromorphic functions in dimension $n = 2$ represents a classical result which follows from the existence of meromorphic functions on Riemann surfaces (see [Fo], [K]).

The question whether quasimeromorphic mappings (or qm-mappings, in short) exist in any dimension $n \geq 3$ was originally posed by Martio and Srebro in [MS1]; subsequently in [MS2] they proved the existence of the fore-mentioned mappings in the case of co-finite groups i.e. groups such that $Vol_{hyp}(\mathbb{B}^n/G) < \infty$ (the important case of geometrically finite groups being thus included). Also, it was later proved by Tukia ([Tu]) that the existence of non-constant quasimeromorphic mappings is assured in the case when G acts torsionless upon $\mathbb{B}^n$. Moreover, since for torsionless Kleinian groups G, $\mathbb{B}^n/G$ is a (analytic) manifold, the next natural question to ask is whether there exist non-constant qm-mappings $f : M^n \to \widehat{\mathbb{R}^n}$; where M^n is an orientable $n-$manifold. A partial affirmative answer to this question is due to Peltonen (see [Pe]); to be more precise she proved the existence of qm-mappings in the case when M^n is a complete, connected, orientable $\mathcal{C}^\infty$-Riemannian manifold.

Our main result is the following theorem:

Theorem A.1.7 *Let G be a Kleinian group G acting on $\mathbb{B}^n$, $n \geq 2$. If the orders of the elliptic elements of G which have non-degenerate fixed set are bounded, then G admits non-constant G-automorphic quasimeromorphic mappings.*

In contrast with the above results it was proved by Srebro ([Sr]) that, if G is a Kleinian group acting on $\mathbb{B}^n$, $n \geq 3$, containing elliptic elements with non-degenerate fixed set, of arbitrarily large orders, then G does not admit non-constant G-automorphic qm-mappings; and showed that such groups exist in all dimensions $n \geq 3$.

This non existence result, together with Theorem A.1.7 gives a complete characterization of those Kleinian groups which admit G-automorphic quasimeromorphic mappings. Namely:

Theorem A.1.8 *Let G be a Kleinian group acting on $\mathbb{B}^n$. Then G admits non-constant automorphic qm-mappings iff:*

1. *$n = 2$;*

   or

2. *$n \geq 3$, and the orders of the elliptic elements of G having non-degenerate fixed sets are uniformly bounded.*

Remark A.1.9 *Given any finitely generated Kleinian group acting on $\mathbb{B}^3$ the number of conjugacy classes of elliptic elements is finite (see [FM]). However, this is not true for Kleinian groups acting upon $\mathbb{B}^n$, $n \geq 4$; (for counterexamples, see [FM], [Po] and [H]).*

Remark A.1.10 *Hamilton ([H], Theorem 4.1.) constructed examples of Kleinian groups G acting on $\mathbb{B}^4$ such that there exists an infinite sequence $\{f_n\}_{n\in\mathbb{N}} \subset G$ of elliptic transformations, with $ord(f_n) \to \infty$ and such that the fixed set of each f_n is degenerate. (For the relevant definitions, see Section A.2 below.) (Here $ord(f_n)$ denotes the order of f_n.)*

Note that by Remark A.1.9 we have the following corollary:

Corollary A.1.11 *Let G be a finitely generated Kleinian group acting upon $\mathbb{B}^3$. Then there exists a non constant G-automorphic qm-mapping $f : \mathbb{B}^3 \rightarrow \widehat{\mathbb{R}^3}$.*

The classical methods employed in proving the existence in the case $n = 2$ do not apply in higher dimensions – indeed, for $n \geq 4$, $\mathbb{B}^n/G$ is not even a manifold, but an orbifold. Therefore, different methods are needed. Following other researchers, we shall employ the classical "Alexander trick" (see [Al]).

A uniform bound for the dilatations can be attained (see [MS2], [Tu]) if the considered triangulation is *fat*, i.e. such that each of its individual simplices may be mapped onto a standard n-simplex, by a L-bilipschitz map, followed by a homothety, for a fixed L. (For a precise definition of fatness see Section A.3 below.)

The idea of the proof of Theorem A.1.7 is, in a nutshell, as follows: Based upon the geometry of the elliptic transformations construct a fat triangulation $\mathcal{T}_1$ of N_e^*, where N_e^* is a certain closed neighbourhood of the singular set of $\mathbb{B}^n/G$. Since $M_p = \left(\mathbb{B}^n \setminus Fix(G)\right)/G$, $Fix(G) = \left\{x \in \mathbb{B}^n \,|\, \exists g \in G \setminus \{Id\},\, g(x) = x\right\}$ is an orientable analytic manifold, we can apply Peltonen's result to gain a triangulation $\mathcal{T}_2$ of M_p. Therefore, if the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are chosen properly, each of them will induce a triangulation of $N_e^* \setminus N_e^{*\prime}$, for a certain $N_e^{*\prime} \subsetneq N_e^*$ (see Section A.2).

'Mash' $\mathcal{T}_1$ and $\mathcal{T}_2$ (in $N_e^* \setminus N_e^{*\prime}$) i.e. ensure that the given triangulations intersect into a new triangulation $\mathcal{T}_0$ (see [Mun], Theorem 10.4). Modify $\mathcal{T}_0$ to receive a new fat triangulation $\mathcal{T}$ of $\mathbb{B}^n/G$.

In the presence of degenerate components $A_k = A(f_k)$ of the fixed set of G, where the transformations f_k may have arbitrarily large orders, a modification of this construction is needed – see Section A.4.

Apply Alexander's trick to receive a quasimeromorphic mapping $f : \mathbb{B}^n/G \rightarrow \widehat{\mathbb{R}^n}$. The lift $\widetilde{f}$ of f to $\mathbb{B}^n$ represents the required G-automorphic quasimeromorphic mapping.

In [S3] we showed how to build $\mathcal{T}_1$ using a generalization of a theorem of Munkres ([Mun], 10.6) on extending the triangulation of the boundary of a manifold (with boundary) to the whole manifold. Munkres' technique also provided us with the basic method of mashing the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$. In this paper we present a more direct, geometric method of triangulating N_e^* and mashing the two triangulations. We already employed this simpler method in [S1], where we proved Theorem A.1.7 in the case $n = 3$. The original technique used in [S1] for fattening the intersection of $\mathcal{T}_1$ and $\mathcal{T}_2$ is, however, restricted to dimension 3. Therefore here we make appeal to the method employed in [S3], which is essentially the one developed in [CMS].

This paper is organized as follows: in Section 2 we show how to triangulate the closed neighbourhood N_e^* of the singular set of $\mathbb{B}^n/G$. Section 3 is dedicated to the main task of mashing the triangulations and fattening the resulting common triangulation. In Section 4 we show how to apply the main result in the construction of a G-automorphic quasimeromorphic mapping from $\mathbb{B}^n$ to $\widehat{\mathbb{R}^n}$.

A.2 Geometric Neigbourhoods

If G is a discrete Möbius group and if $f \in G$, $f \neq Id$ is an elliptic transformation, then there exists $m \geq 2$ such that $f^m = Id$. The smallest m satisfying this condition is called the *order* of f, and it is denoted by $ord(f)$. In the 3-dimensional case the *fixed point set* of f i.e. $Fix(f) = \{x \in \mathbb{B}^3 | f(x) = x\}$, is a hyperbolic line and will be denoted by $A(f)$ – the *axis of f*. In dimension $n \geq 4$ the fixed set (or *axis of f*) of an elliptic transformation is a k-dimensional hyperbolic plane, $0 \leq k \leq n-2$. An axis A is called *degenerate* iff $dim\,A = 0$. In dimensions higher than $n = 3$, different elliptics may have fixed sets of different dimensions.

If G is a discrete group, G is countable and so is the set of elliptics and the set of connected components of $Fix(G)$, which we denote by $\{f_i\}_{i\geq 1}$ and $\{C_j\}$, respectively.

Moreover, by the discreteness of G, the sets $\mathcal{A} = \{A_i\}_{i\geq 0}$ – and hence $\mathcal{S} = \{C_j\}$ – have no accumulation points in $\mathbb{B}^n$.

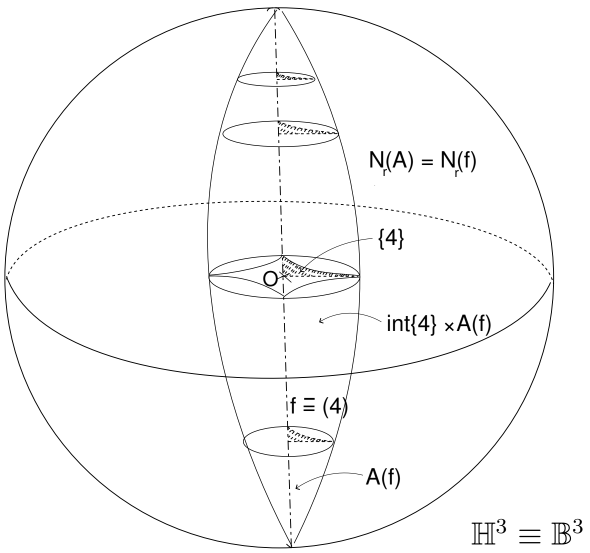

Figure A.1: Geometric neighbourhood for $n = 3$ and $m = 4$. Here $\{4\}$ denotes the regular (hyperbolic) polygon with 4 sides.

Hence we can choose disjoint, G-invariant neighbourhoods N_j and N'_j of C_j, $N'_j \subsetneq N_j$. Indeed, first choose a neighbourhood N_1 of C_1, such that $\bar{N}_1 \cap \bigcup_{j\geq 2} C_j = \emptyset$; then recursively build a neighbourhood N_k of C_k, such that $N_k \subset \mathbb{B}^n \setminus (N_1 \cup \cdots \cup N_{k-1})$ and $\bar{N}_k \cap \bigcup_{j>k} C_j = \emptyset$, for all $k \geq 2$. Denote $N_e = \bigcup_{j\in\mathbb{N}} N_j$, $N'_e = \bigcup_{j\in\mathbb{N}} N'_j$. Define $N^*_e = (\bar{N}_e \cap \mathbb{B}^n)/G$, $N^{*\prime}_e = (\bar{N}'_e \cap \mathbb{B}^n)/G$.

To produce the desired closed neighbourhood N^*_e of the singular set of $\mathbb{B}^n/G$ and its triangulation $\mathcal{T}_1$, we first consider the case where $C_i = A(f)$, for some $f \in G$, and then construct a standard neighbourhood $N_f = N\big(A(f)\big)$ of the axis of each elliptic element of G such that $N_f \simeq A(f) \times I^{n-k}$, where $A(f) = \mathbb{S}^k$ and where I^{n-k} denotes the unit $(n-k)$-dimensional interval. The construction of N_f proceeds as follows: By [Cox], Theorem $11 \cdot 23$. the fundamental region for the local action of the stabilizer group of the axis of f, $G_f = G_{A(f)} = \{g \in G \,|\, g(x) = x\}$ is a simplex or a product a simplices. Let $\mathcal{S}_f$ be the fundamental region (see Figure A.2).

Then we can define the generalized prism (or *simplotope* – see [Som], VII. 25.) $\mathcal{S}_f^{\perp}$, defined by translating $\mathcal{S}_f$ in a direction perpendicular to $\mathcal{S}_f$, where the translation length is $dist_{hyp}\big(\mathcal{S}_f, A(f)\big)$. It naturally decomposes into simplices (see [Som], VII. 25., [Mun], Lemma 9.4). We have thus constructed an f-invariant triangulation of a prismatic neigbourhood N_f of $A(f)$. We can reduce the mesh of this triangulation as much as required, while controlling its fatness by dividing $\mathcal{S}_f$ into similar simplices and partitioning N_f into a finite number of radial strata of equal width ϱ. In the special case when the minimal distance between axes $\delta = \min\{dist_{hyp}\big(A(f), A(g)\big) \,|\, g \text{ elliptic},\ g \neq f\}$ is attained we can chose $\varrho = \delta/\kappa_0$, for some integer κ_0, and further partition it into 'slabs' of equal hight h. (In particular one can use this approach in the case when G acts on $\mathbb{B}^3$ and it contains no order two elliptics, since in this particular case, according to a result of Gehring and Martin [GM1], the minimum exists and is strictly positive.) Henceforth we shall call the neighbourhood thus produced, together with its fat triangulation, a *geometric neighbourhood*.

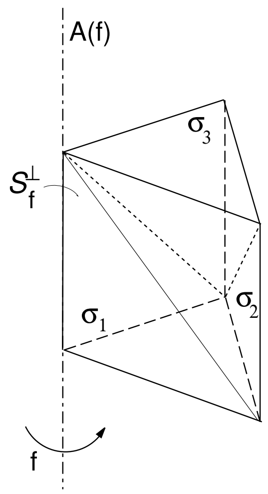

Figure A.2: Canonical decomposition into simplices of $\mathcal{S}_f^{\perp}$, for $n = 3$

Since the stabilizer $Stab(A_{1,\dots,k})$ of the intersection of axes $A_{1,\dots,k} = A_{i_1} \cap \cdots \cap A_{i_k}$ is a finite subgroup of $O^+(n)$, and since in any dimension there exist only a finite number of such groups of orders $\leq M_0$, for any $M_0 \in \mathbb{N}$ (see [Cox], Chap. 11.), the angles between the axes of transformations of orders $\leq m_0$ admit a bound $\alpha =$

$\alpha(m_0, n)$. Therefore, the intersection $N(A_{1,\dots,k}) = N_{f_1} \cap \dots \cap N_{f_k}$ of the geometric neighbourhoods of several axes is also endowed with a natural fat triangulation, invariant under the group $G = < G_{f_1}, \dots, G_{f_k} >$. (In the particular case $n = 3$ one can choose as a geometric neighbourhood of A a regular or a semi-regular polyhedron together with its interior (see Figure 3 below).

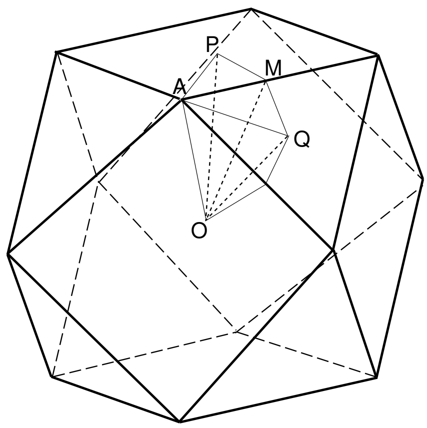

Figure A.3: A Euclidean semi-regular polyhedron and two of its fundamental tetrahedra ($n = 3$)

If $q \in \mathbb{B}^n$, $dim\, q = 0$, is a degenerate element of the singular locus, we replace the tubular neighbourhood considered above by $\mathcal{P}_q \cup int\, \mathcal{P}_q$, where $\mathcal{P}_q$ is a regular polytope invariant under the stabilizer G_q of q in G, together with its canonical simplicial subdivision (see [Cox], $7 \cdot 6.$). Indeed, every finite group generated by reflections is the symmetry group of a regular polyhedron $\mathcal{P}$ (see [Cox], p. 209) Moreover, the rotation group of $\mathcal{P}$ has order $nl/2$, where l is the number of faces of $\mathcal{P}$ (see [Cox], pp. 227-231).

Remark A.2.1 *As noted above, if* G *is a Kleinian group acting with torsion on* $\mathbb{B}^n$*, then* $M_p = \left(\mathbb{B}^n \setminus Fix(G)\right) / G$ *is a complete orientable manifold. Moreover, since the isotropy groups of any point in* $Q_G = \mathbb{B}^n / G$ *are subgroups of* $O^+(n)$*, it follows that* $\mathbb{B}^n / G$ *is complete orientable orbifold (see [Dr], p. 46). The singular locus* $\Sigma_{Q_G} = Fix(G)/G$ *of* Q_G *contains all the non-manifold points of* Q_G*, yet the two sets are not equal. Indeed, in dimension* $n = 2$ *(*$n = 3$*) any orbifold (orientable orbifold) is homeomorphic to a manifold. The local structure of* Σ_{Q_G} *at a point* $x_Q \in Q_G$ *is determined by the stabilizer in* G *of its preimage in* $\mathbb{B}^n$*, i.e. by the finite subgroups of* $O^+(n)$*. (For instance, in dimension* $n = 3$ *only two infinite families and three more special cases of branching points (of* $Fix(G)$ *and thus of* Σ_{Q_G}*) can occur – see [Th1], 5.6.) However, the global structure of* Σ_{Q_G} *can be very complicated (see [Th1], 5.6.).*

A.3 Mashing and Fattening Triangulations

We present the main steps of the Munkres ([Mun], Chap. 10) and Chegeer ([CMS], 432-440) techniques, and we indicate how to adapt them to our particular setting. First let us establish some definitions and notations:

Definition A.3.1 *Let M^n be a PL-manifold. Two triangulations $\mathcal{T}_1, \mathcal{T}_2$ of M^n intersect transversally iff for any $p \in M^n$, there exist neighbourhoods U_1, U_2, U_3 of p in $|\mathcal{T}_1|, |\mathcal{T}_2|$ and M^n, respectively, such that the triple (U_1, U_2, U_3) is PL-homeomorphic to a neighbourhood of 0 in $(\mathbb{R}^n \times 0, 0 \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$.*

To ensure the fatness of the common triangulation we need to make appeal to a stronger notion of transversality, namely:

Definition A.3.2 *Let $\sigma_i \in K$, $\dim \sigma_i = k_i$, $i = 1, 2$; such that $diam\, \sigma_1 \leq diam\, \sigma_2$. Denote by $[\sigma_i]$ the affine subspace of $\mathbb{R}^N$ generated by σ_i, and let $<\sigma_i>$ denote the subspace parallel to $[\sigma_i]$, such that $0 \in <\sigma_i> \subset \mathbb{R}^N$; $i = 1, 2$. We say that σ_1, σ_2 are δ-*transverse *iff*
(i) $dim([\sigma_1] \cap [\sigma_2]) = \max(0, k_1 + k_2 - n)$;
(ii) $0 < \delta < \measuredangle([\sigma_1], [\sigma_2])$, where $\measuredangle([\sigma_1], [\sigma_2]) = \measuredangle(<\sigma_1>, <\sigma_2>)$, and where $\measuredangle(<\sigma_1>, <\sigma_2>) = \min\limits_{(e_1, e_2)} \arccos(e_1, e_2)$, $e_i \in (<\sigma_1> \cap <\sigma_1>)^{\perp} \cap <\sigma_i>$, $||e_i|| = 1$, $i = 1, 2$; where (e_1, e_2) denotes the standard inner product on $\mathbb{R}^n$;
and if $\sigma_3 \subsetneq \sigma_1$, $\sigma_4 \subsetneq \sigma_2$, such that $dim\, \sigma_3 + dim\, \sigma_4 < n = dim\, K$, then
(iii) $dist(\sigma_3, \sigma_4) > \delta \cdot d_1$, where $d_1 = diam\, \sigma_1$.
In this case we write: $\sigma_1 \pitchfork_\delta \sigma_2$.

Definition A.3.3 *Let $\tau \subset \mathbb{R}^n$; $0 \leq k \leq n$ be a k-dimensional simplex. The fatness φ of τ is defined as being:*

$$\varphi = \varphi(\tau) = \inf_{\substack{\sigma < \tau \\ dim\, \sigma = l}} \frac{Vol(\sigma)}{diam^l\, \sigma} \tag{A.3.1}$$

The infimum is taken over all the faces of τ, $\sigma < \tau$, and $Vol_{eucl}(\sigma)$ and $diam\, \sigma$ stand for the Euclidian l-volume and the diameter of σ respectively. (If $dim\, \sigma = 0$, then $Vol_{eucl}(\sigma) = 1$, by convention.)

A simplex τ is φ_0-fat, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$. A triangulation (of a submanifold of $\mathbb{R}^n$) $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is φ_0-fat if all its simplices are φ_0-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is fat if there exists $\varphi_0 > 0$ such that all its simplices are φ_0-fat.

Remark A.3.4 *There exists a constant $c(k)$ that depends solely upon the dimension k of τ s.t.*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\substack{\sigma < \tau \\ dim\, \sigma = l}} \measuredangle(\tau, \sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{A.3.2}$$

and

$$\varphi(\tau) \leq \frac{Vol(\sigma)}{diam^l\,\sigma} \leq c(k)\cdot\varphi(\tau)\,; \tag{A.3.3}$$

where $\measuredangle(\tau,\sigma)$ denotes the (internal) dihedral angle of $\sigma < \tau$. (For a formal definition, see [CMS], pp. 411-412, [Som].)

Remark A.3.5 *The definition above is the one introduced in [CMS]. For equivalent definitions of fatness, see [Ca1], [Ca2], [Mun], [Pe], [Tu].*

The first step is that of mashing the triangulations $\mathcal{T}_1, \mathcal{T}_2$:

We approximate the triangulation $\mathcal{T}_2$ of M_p by a locally finite Euclidian triangulation, by means of the secant map (see [Mun], p. 90). Also, the hyperbolic simplices of $\mathcal{T}_1$ can be approximated arbitrarily well by Euclidean simplices, by considering $diam\,\sigma,\ \sigma \in \mathcal{T}_1$ small enough (see [Tu]). Therefore the mashing and fattening of triangulations reduces to that of Euclidean ones.

Next we ensure that the given triangulations intersect into a new triangulation $\mathcal{T}_0$. This is first done locally by modifying these local triangulations coordinate chart by chart, so they will be PL-compatible wherever they overlap. More precisely, we first apply infinitesimal moves of the vertices so that the two triangulations will intersect transversally. Next we perform suitable barycentric subdivisions of the closed, convex polyhedral cells $\bar{\gamma} = \bar{\sigma}_1 \cap \bar{\sigma}_2$, $\sigma_i \in \mathcal{T}_i,\ i = 1,2$; in the following manner: suppose each cell $\beta \subset \partial\gamma$ already has a subdivision into simplices $\beta_i\,,\ i = 1,\ldots,p\,$; choose an interior point $p_\gamma \in int\,\gamma$, construct the joins $J(p_\gamma, \beta_i)$, $i = 1,\ldots,p\,$; and consider all their simplices. (see [Mun], 10.2 - 10.3).

To extend the local triangulations to a global triangulation $\mathcal{T}_0$, we work in $\mathbb{R}^n$, by using the coordinate charts and maps. Here again we have to approximate the given triangulation by a PL-map, such that the given triangulation and the one we produce will be PL-compatible (see [Mun], Theorem 10.4). The existence of the common triangulation $\mathcal{T}_0$ follows immediately (see [Mun], Theorem 10.5).

We next present the main steps of the fattening process (for details see [CMS]):

One begins by triangulating and fattening the intersection of two individual simplices belonging to the two given triangulations, respectively. First one shows that if two individual simplices are fat and if they intersect δ-transversally, one can choose the points p_γ such that the barycentric subdivision $\bar{\gamma}^*$ will be composed of fat simplices. (See [CMS], Lemma 7.1.)

Next one shows that given two fat Euclidian triangulations that intersect δ-transversally, then it is possible to infinitesimally move any given point of one of the triangulations such that the resulting intersection will be δ^*-transversal, where δ^* depends only on δ, the common fatness of the given triangulations, and on the displacement length (see [CMS], Lemma 7.3.).

By repeatedly applying this results to the simplices of dimensions $0\,,\ldots,n\,$, of the intersection of two fat triangulations, one can now prove the main fattening result:

Proposition A.3.6 *([CMS], Lemma 6.3.) Let $\mathcal{T}_1, \mathcal{T}_2$ be two fat triangulations of open sets $U_1, U_2 \subset \mathbb{R}^n$, $B_r(0) \subseteq U_1 \cap U_2$, having common fatness $\geq \varphi_0$ and such that $d_1 = \inf_{\sigma_1 \in \mathcal{T}_1} diam\, \sigma_1 \leq d_2 = \inf_{\sigma_2 \in \mathcal{T}_2} diam\, \sigma_2$. Then there exist φ_0^*-fat triangulations $\mathcal{T}_1', \mathcal{T}_2'$, $\varphi_0^* = \varphi_0^*(\varphi_0)$, of open sets $V_1, V_2 \subseteq B_r(0)$, such that*

1. $\mathcal{T}_i'\big|_{B_{r-8d_2}(0)} = \mathcal{T}_i\big|_{B_{r-8d_2}(0)}$, $i = 1, 2$;
2. *$\mathcal{T}_1'$ and $\mathcal{T}_2'$ agree near their common boundary.*

Moreover:

3. $\inf_{\sigma_1' \in \mathcal{T}_1'} diam\, \sigma_1' \leq 3d_1/2, \quad \inf_{\sigma_2' \in \mathcal{T}_2'} diam\, \sigma_2' \leq d_2$.

We apply Proposition A.3.6 above to our particular context in the following manner: Let $\mathcal{T}_1$, $\mathcal{T}_2$ be the triangulations of $N_e^* \setminus N_e^{*\prime}$ constructed above. To gain a globally fat triangulation from the mashing of $\mathcal{T}_1$ and $\mathcal{T}_2$, we start by partitioning $N_e^* \setminus N_e^{*\prime}$ into (almost) cubes Q. If the diameters of the sets Q are small enough we can apply Proposition A.3.6, for Q instead $B_r(0)$. Extend $\mathcal{T}_0$ by $\mathcal{T}_2$ on the face included in ∂N_e and by $\mathcal{T}_1$ on the other faces, to receive the desired triangulation $\mathcal{T}$. (Further fattening of the triangulations induced on the lower dimensional faces may be necessary. However, by the locally finiteness of the triangulation, the number of steps required for fattening the lower dimensional intersections is finite and depends solely upon the dimension n.) This gives the required globally fat triangulation of $\mathbb{B}^n/G$.

A.4 The Existence of Quasimeromorphic Mappings

We first prove the following lemma:

Lemma A.4.1 *([MS1], [Pe]) Let $M^n \subset \mathbb{R}^N$ be an orientable n-manifold, let $\mathcal{T}$ be a chessboard fat triangulation of M^n, let $\sigma \in \mathcal{T}$, $\sigma = (p_0, \ldots, p_n)$ and let $\tau_0 = (p_{0,1}, \ldots, p_{0,n})$ denote the equilateral n-simplex inscribed in the unit sphere $\mathbb{S}^{n-1}$. Then there exists a orientation-preserving homeomorphism $h = h_\sigma : |\sigma| \to \widehat{\mathbb{R}^n}$ s.t.*

1. *$h(|\sigma|) = |\tau_0|$, if σ is positively oriented*
   and
   $h(|\sigma|) = \widehat{\mathbb{R}^n} \setminus |\tau_0|$, otherwise.
2. *$h(p_i) = p_{0,i}$, $i = 0, \ldots, n$.*
3. *$h|_{\partial|\sigma|}$ is a PL-homeomorphism.*
4. *$h|_{int|\sigma|}$ is quasiconformal.*

Proof If $det(p_0, \dots, p_n) > 0$, then the PL-mapping h defined by condition *2* above also satisfies conditions *1*, *3* and *4*. If $det(p_0, \dots, p_n) < 0$, we define h as follows: $h = \varphi^{-1} \circ J \circ \varphi \circ h_0$, where φ is the *radial linear stretching* $\varphi : \tau_0 \to \mathbb{R}^n$, J denotes the reflection in the unit sphere $\mathbb{S}^{n-1}$ and $h_0 : |\sigma| \to |\tau_0|$ is the orientation-reversing PL-mapping defined by condition *2*. Recall that φ is onto and bilipschitz (see [MS2]). Moreover, by a result of Gehring and Väisalä, φ is also quasicomformal (see [V]). We can extend φ to $\widehat{\mathbb{R}^n}$ by defining $\varphi(\infty) = \infty$. It follows that h indeed represents the required PL-homeomorphism.

□

The existence theorem of quasimeromorphic mappings now follows immediately:

Proof of Theorem A.1.7 Let $\mathcal{T}$ be the φ_0^*-fat chessboard triangulation of $\mathbb{B}^n/G$ constructed in Section A.3 above. Let $f : \mathbb{B}^n/G \to \widehat{\mathbb{R}^n}$ be defined by: $f|_{|\sigma|} = h_\sigma$, where h is the homeomorphism constructed in the lemma above. Then f is a local homeomorphism on the $(n-1)$-skeleton of $\widetilde{\mathcal{T}}$ too, while its branching set B_f is the $(n-2)$-skeleton of $\widetilde{\mathcal{T}}$. By its construction f is quasiregular and its (outer) dilatation depends only φ_0^* and on the dimension n (see [Tu], Lemma E.). The lift $\widetilde{f}$ of f to $\mathbb{B}^n$ represents the required G-automorphic quasimeromorphic mapping.

In the case of degenerate components A_k of the fixed set $\mathcal{S}$, the proof is essentially the same as in the classical case of Riemann surfaces (see, e.g. [Fo], pp. 233-238). More precisely, we proceed as follows: We excise from $\mathbb{B}^n$ disjoint ball neighbourhoods B_k of A_k. Let $S_k = \partial B_k$. Then each of the quotients S_k/G admits a fat triangulation $\mathcal{T}_k$. The manifold $\big(\mathbb{B}^n \setminus \bigcup_{k \geq 1} B_k\big)/G$ admits a fat triangulation that extends the fat triangulation of S_k (see [CMS], p. 444 and [S3], Theorem 2.9.). We build the simplices P_k with vertex A_k/G and base T_{kl}, where T_{kl} are simplices belonging to $\mathcal{T}_k$. Then each of the simplices P_k can be quasiconformally mapped onto a half-space, with bounded dilatation which depends only on n and not on the angles at the vertices A_i, even if the orders of the transformations f_k are not bounded from above (see [Car], Theorem 3.6.10. and Theorem 3.6.13.).

□

Acknowledgment

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Appendix B

Note on a Theorem of Munkres

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ABSTRACT. We prove that given a $\mathcal{C}^\infty$ Riemannian manifold with boundary, having a finite number of compact boundary components, any fat triangulation of the boundary can be extended to the whole manifold. We also show that this result extends to $\mathcal{C}^1$ manifolds and to embedded PL manifolds of dimensions $2, 3$ and 4. We employ these results to prove that manifolds of the types above admit quasimeromorphic mappings onto $\widehat{\mathbb{R}^n}$. As an application we prove the existence of G-automorphic quasimeromorphic mappings, where G is a Kleinian group acting on $\mathbb{H}^n$.

B.1 Introduction

The existence of triangulations for $\mathcal{C}^1$ manifolds without boundary has been known since the classical work of Whithead ([Wh], 1940).

This result was extended in 1960 by Munkres ([Mun]) to include $\mathcal{C}^r$ manifolds with boundary, $1 \leq r \leq \infty$. To be more precise, he proved that any $\mathcal{C}^r$ triangulation of the boundary can be extended to a $\mathcal{C}^r$ triangulation of the whole manifold.

Earlier, in 1934-1935, Cairns ([Ca1], [Ca2]) proved the existence of triangulations for compact $\mathcal{C}^1$ manifolds and for compact manifolds with boundary having a finite number of compact boundary components. It should be noted that, although far better known and widely cited, Whitehead's work is rooted in Cairns' studies, to whom it gives due credit in the very opening phrase: "`This paper is supplementary to S.S. Cairns' work on the triangulation ... of manifolds of class` $\mathcal{C}^1$".

Moreover, the resulting triangulations were *fat* (that is they satisfied a uniform non-degeneracy condition – for the formal definition see 1.2. below), while Munkres' method produced fat simplices only away from the boundary (see Section 2.2).

Unfortunately, it seems that little interest existed during the following decades, for studying generalizations of the results above ([Fe] representing a notable exception).

The interest in the existence of a fat triangulations was rekindled by the study of quasiregular and quasimeromorphic functions, since the existence of a fat triangulations is crucial in the proof of existence of quasiregular (quasimeromorphic) mappings (see [MS2], [Tu]) and in 1992 Peltonen ([Pe]) proved the existence of fat triangulations for $\mathcal{C}^\infty$ Riemannian manifolds, using methods partially based upon another technique of Cairns (originally developed for triangulating manifolds of class $\geq \mathcal{C}^2$).

In this paper we extend Munkres' theorem to the case of fat triangulations of manifolds with or without boundary and we show how to apply this main result in order to prove the existence of quasimeromorphic mappings. Our main result is the following theorem:

Theorem B.1.1 *Let M^n be an n-dimensional $\mathcal{C}^\infty$ Riemannian manifold with boundary, having a finite number of boundary components. Then any uniformly fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Here fat triangulations are defined as follows:

Definition B.1.2 *Let $\tau \subset \mathbb{R}^n$; $0 \leq k \leq n$ be a k-dimensional simplex. The fatness φ of τ is defined as being:*

$$\varphi = \varphi(\tau) = \inf_{\substack{\sigma<\tau \\ dim\,\sigma = j}} \frac{Vol_j(\sigma)}{diam^j\,\sigma} \tag{B.1.1}$$

The infimum is taken over all the faces of τ, $\sigma < \tau$, and $Vol_j(\sigma)$ and $diam\,\sigma$ stand for the Euclidian j-volume and the diameter of σ respectively. (If $dim\,\sigma = 0$, then $Vol_j(\sigma) = 1$, by convention.)
A simplex τ is φ_0-fat, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$. A triangulation (of a submanifold of $\mathbb{R}^n$) $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is φ_0-fat if all its simplices are φ_0-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is fat if there exists $\varphi_0 \geq 0$ such that all its simplices are φ_0-fat.

Remark B.1.3 *There exists a constant $c(k)$ that depends solely upon the dimension k of τ such that*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\sigma<\tau} \measuredangle(\tau,\sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{B.1.2}$$

and

$$\varphi(\tau) \leq \frac{Vol_j(\sigma)}{diam^j\,\sigma} \leq c(k) \cdot \varphi(\tau)\,; \tag{B.1.3}$$

where $\measuredangle(\tau,\sigma)$ denotes the (internal) dihedral angle of $\sigma < \tau$. (For a formal definition, see [CMS], pp. 411-412, [Som].)

Remark B.1.4 *The definition above is the one introduced in [CMS]. For equivalent definitions of fatness, see [Ca1], [Ca2], [Mun], [Pe], [Tu].*

The idea of the proof of Theorem 1.1. is first to build two fat triangulations: $\mathcal{T}_1$ of a product neighbourhood N of ∂M^n in M^n and $\mathcal{T}_2$ of $int\, M^n$, and then to "mash" the two triangulations into a new triangulation $\mathcal{T}$, while retaining their fatness.

While the mashing procedure of the two triangulations is basically that developed in the original proof of Munkres' theorem, the triangulation of $\mathcal{T}_1$ was modified, in order to ensure the fatness of the simplices of $\mathcal{T}_1$. The existence of the second triangulation is assured by Peltonen's result.

Thus our main efforts should be dedicated to the task of fattening the newly obtained triangulation into a new fat triangulation. However, such a technique was already developed in [CMS], and we employ it here. (For a more direct approach in dimensions 2 and 3 see [S1]. Also, for the treatment of the same problem in the context of Computational Geometry, see [E].)

Once a fat triangulation of an orientable manifold M^n is provided, the construction of the required quasimeromorphic mapping is canonical (see [Al], [Pe], [MS2], [Tu]) and is based upon the so called "Alexander Trick", which we present here succinctly: one starts by constructing a suitable triangulation of M^n. Since M^n is orientable, a consistent orientation of all the simplices of the triangulation (i.e. such that two given n-simplices having a $(n-1)$-dimensional face in common will have opposite orientations) can be chosen. Then one quasiconformally maps the simplices of the triangulation into $\widehat{\mathbb{R}^n}$ in a chessboard manner: the positively oriented ones onto the interior of the standard simplex in $\mathbb{R}^n$ and the negatively oriented ones onto its exterior. To ensure the existence of such a chessboard triangulation, we may have to perform a barycentric type of subdivision, thus rendering a triangulation whose simplices satisfy the condition that every $(n-2)$-face is incident to an even number of n-simplices. If the dilatations of the quasiconformal maps constructed above are uniformly bounded – which condition is fulfilled if the simplices of the triangulation are of uniform fatness – then the resulting map will be quasimeromorphic.

This paper is organized as follows: in Section 2 we bring the proof of Theorem 1.1. and we present the main techniques we employ: Peltonen's method of triangulating $int\, M^n$, the adaptation to our context of the Proof of Munkres' theorem on the extension of the triangulation of ∂M^n to a product neighbourhood N of ∂M^n in M^n, and the fattening method of the resulting common triangulation. In Section 3 we show how to apply the main result in the construction of quasimeromorphic mappings from M^n to $\widehat{\mathbb{R}^n}$. In Section 4 we propose some generalizations. Finally, in Section 5 we bring an application of our main results to the proof of existence of G-automorphic quasimeromorphic mappings, where G is a Kleinian group acting on $\mathbb{H}^n$.

B.2 Extending $\mathcal{T}_1$ to $int\, M^n$

B.2.1 Peltonen's Technique

Peltonen's method is an extension of one due to Cairns, developed in order to triangulate $\mathcal{C}^2$-compact manifolds ([Ca3]). It is based on the subdivision of the given manifold into a closed cell complex generated by a Dirichlet (Voronoy) type partition whose vertices are the points of a maximal set that satisfy a certain density condition. We give below a sketch of the Peltonen's method and refer the interested reader to [Pe] for the full details.

The construction devised by Peltonen consists of two parts:

Part 1 This part proceeds in two steps:

Step A Build an exhaustation $\{E_i\}$ of M^n, generated by the pair (U_i, η_i), where:

1. U_i is the relatively compact set $E_i \setminus \bar{E}_{i-1}$ and
2. η_i is a number that controls the fatness of the simplices of the triangulation of E_i, that will be constructed in Part 2, such that it will not differ too much on adjacent simplices, i.e.:
   (i) The sequence $(\eta_i)_{i\geq 1}$ descends to 0;
   (ii) $2\eta_i \geq \eta_{i-1}$.

Step B

1. Produce a maximal set A, $|A| \leq \aleph_0$, such that $A \cap U_i$ satisfies:
   (i) a density condition, and
   (ii) a "gluing" condition (for U_i, U_{i+1}).
2. Prove that the Dirichlet complex $\{\bar{\gamma}_i\}$ defined by the sets A_i is a cell complex and every cell has a finite number of faces (so it can be triangulated in a standard manner).

Part 2 Consider first the dual complex Γ and prove that it is a Euclidian simplicial complex with a "good" density, then project Γ on M^n (using the normal map). Finally, prove that the resulting complex can be triangulated by fat simplices.

Remark B.2.1 *In the course of Peltonen's construction M^n is presumed to be isometrically embedded in some $\mathbb{R}^{N_1}$, where the existence of N_1 is guaranteed by Nash's theorem (see [Pe], [Spi]).*

B.2.2 The Extension of $\mathcal{T}_1$ to $int\, M^n$

We first establish some notations and definitions:

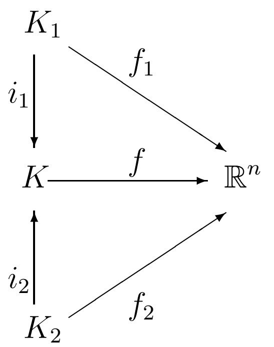
Let K denote a simplicial complex, and let $K' < K$ denote a subcomplex of K.

Definition B.2.2 *Let* $f_i : K_i \xrightarrow{\sim} \mathbb{R}^n,\ i = 1, 2$ *be such that* $f(|K_i|)$ *is closed.*
We say that $(K_1, f_1), (K_2, f_2)$ intersect in a subcomplex *iff:*
(i) $f_i^{-1}\big(f_1(|K_1|) \cap f_2(|K_2|)\big) = |L_i|$ *; where* $L_i < K_i$ *,* $i = 1, 2.$
and
(ii) $f_2^{-1} \circ f_1 : L_1 \to L_2$ *is a linear isomorphism, i.e. (a)* $f : |L_1| \xrightarrow{\sim} |L_2|$ *and (b)* $f|_\sigma$ *is linear for any simplex* $\sigma \in L_1$*.*

Definition B.2.3 *Let* $L < K$*.* L *is called* full *iff* $\sigma \cap L$ *either is a face of* σ *or else it is empty; for any simplex* $\sigma \in K$*.*

Remark B.2.4 *If* $L < K$ *is full, and* $\sigma \in K$*, then* $\partial\sigma \cap L \neq \partial\sigma$*.*

If $(K_1, f_1), (K_2, f_2)$ intersect in a full subcomplex, then there exist a complex K and a homeomorphism $f : K \to \mathbb{R}^n$ such that the following diagram is commutative:



Here i_1, i_2 are linear isomorphisms. The pair (K, f) is unique up to isomorphism.

Definition B.2.5 *Let* $(K_1, f_1), (K_2, f_2)$ *and* (K, f) *be as above. Then* (K, f) *is called the* union *of* (K_1, f_1) *and* (K_2, f_2)*.*

Definition B.2.6 *Let* $f : K \to \mathbb{R}^n$ *be a* $\mathcal{C}^r$ *map, and let* $\delta : K \to \mathbb{R}_+^*$ *be a continuous function. Then* $g : |K| \to \mathbb{R}^n$ *is called a* δ*-approximation to* f *iff:*
(i) There exists a subdivision K' *of* K *such that* $g \in \mathcal{C}^r(K', \mathbb{R}^n)$ *;*
(ii) $d_{eucl}\big(f(x), g(x)\big) < \delta(x)$ *, for any* $x \in |K|$ *;*
(iii) $d_{eucl}\big(df_a(x), dg_a(x)\big) \leq \delta(a) \cdot d_{eucl}(x, a)$ *, for any* $a \in |K|$ *and for all* $x \in \overline{St}(a, K')$*.*

Definition B.2.7 *Let* K' *be a subdivision of* K*,* $U = \overset{\circ}{U}$*, and let* $f \in \mathcal{C}^r(K, \mathbb{R}^n)$*,* $g \in \mathcal{C}^r(K', \mathbb{R}^n)$*.* g *is called a* δ*-approximation of* f *(on* U*) iff conditions (ii) and (iii) of Definition 2.6. hold for any* $a \in U$*.*

Definition B.2.8 *Let K' be a subdivision of K and let $f \in \mathcal{C}^r(K,\mathbb{R}^n)$, $g \in \mathcal{C}^r(K',\mathbb{R}^n)$ be non-degenerate mappings (i.e. $rank(f|_\sigma) = rank(g|_\sigma) = dim\,\sigma$, for any $\sigma \in K$) and let $U = \overset{\circ}{U} \subset |K|$. The mapping g is called an α-approximation (of f on U) iff:*

$$\measuredangle\big(df_a(x), dg_a(x)\big) \leq \alpha\,;\ \text{for any } a \in U, \text{ and any } x \in \overline{St}(a,K'),\ a \neq x. \qquad \text{(B.2.1)}$$

We now bring the extension of Munkres' theorem ([Mun], 10.6). While we will initially apply it for $\mathcal{C}^\infty$ manifolds, we give the proof for the general case of $\mathcal{C}^r$ manifolds, $1 \leq r \leq \infty$. We modify the original construction so that the triangulation of a certain neighbourhood of ∂M^n will be fat.

Theorem B.2.9 *Let M^n be a $\mathcal{C}^r$ Riemannian manifold with boundary, having a finite number of compact boundary components. Then any fat $\mathcal{C}^r$-triangulation of ∂M^n can be extended to a $\mathcal{C}^r$-triangulation $\mathcal{T}$ of M^n, $1 \leq r \leq \infty$, the restriction of which to a product neighbourhood $\bar{K}_0 = \partial M^n \times I_0$ of ∂M^n in M^n is fat.*

Proof We shall assume that M^n is isometrically embedded in some $\mathbb{R}^N$. To ensure the existence of such an embedding, one first has to consider the double $D(M)$ of the given manifold with respect to its boundary while smoothing without perturbing the original metric, and then apply Nash's theorem for the open manifold $D(M)$.

Let $f : J \to \partial M^n$ be a $\varphi_{\partial M}$-fat $\mathcal{C}^r$ triangulation, for some $\varphi_{\partial M}$. We construct a triangulation of $|J| \times [0,1)$ in the following way:
If J is isometrically embedded in $\mathbb{R}^{N-1}$, we consider (in $\mathbb{R}^N$) the cells of type:

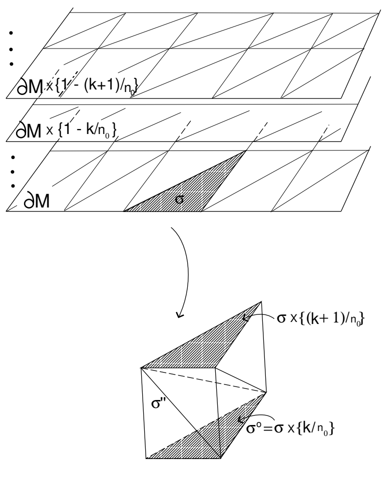


Figure B.1: Triangulation of a cell

$$\sigma' = \sigma \times \Big[\frac{k}{n_0}\,,\ \frac{k+1}{n_0}\Big],\ k = 0,\ldots,n_0-2\,; \qquad \text{(B.2.2)}$$

$$\sigma'' = \sigma \times \Big[\frac{n_0-1}{n_0}\,,\ 1\Big)\,; \qquad \text{(B.2.3)}$$

and

$$\sigma^\circ = \sigma \times \left\{ \frac{k}{n_0} \right\} ; \text{ for any } \sigma \in J \,. \tag{B.2.4}$$

Let K denote the resulting cell complex: $|K| = |J| \times [0,1)$. Let K be isometrically embedded in M. The cells of the complex above may be divided into simplices without subdividing the cells of type (2.4). (See Fig. 1 for the case $N = 2$.)

For reasons that will become clear in the next section, we choose $n_0 = n_0(\varphi_{\partial M}, d_0)$, $d_0 = \min\limits_{\sigma \in \partial M} diam\, \sigma$, such that the fatness of any simplex $\sigma \in K$ is $\geq \varphi_0$, for some $\varphi_0 = \varphi_0(\varphi_{\partial M}, \varphi_{int\, M})$ and such that $diam\, \sigma \leq diam\, \tau$, for any $\sigma \in K_0$, $\tau \in L_0$, where K_0, L_0 are defined as follows:

Let K_0 be the subcomplex of K such that $|K_0| = |J| \times \left[0\,, \frac{k_1}{n_0}\right]$, $k_1 = [\frac{5n_0}{6}]$, (where $[\frac{5n_0}{6}]$ denotes the integer part of $\frac{5n_0}{6}$) and let $\psi : \partial M^n \times [0,1) \to M^n$ be a product neighbourhood of ∂M^n (in M^n). Then, if g makes the following diagram commutative:

$$\begin{array}{ccc} J \times [0,1) & \xrightarrow{\ f \times id\ } & \partial M^n \times [0,1) \\ & {\scriptstyle g} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & M^n & \end{array}$$

then $g|_{K_0}$ is a $\mathcal{C}^r$ embedding such that:

(i) $g(K_0) = \overline{g(K_0)}$ (in M^n)
and
(ii) $\psi\big(\partial M^n \times [0, \frac{k_2}{n_0})\big) \subset int\, g(K_0)$, $k_2 = [\frac{4n_0}{5}]$.

Remark B.2.10 *To determine the integer n_0 with the required properties, further subdivisions may be necessary – their number depending upon the respective η_i-s given by Peltonen's construction.*

Now, if $h : L \to M^n$ is a $\mathcal{C}^r$ triangulation of $int\, M^n$, then, by further (eventual) subdivision, we may suppose (see [Mun]) that: $\sigma' \cap \psi\big(\partial\, M^n \times [0, \frac{k_3}{n_0}]\big) = \emptyset$, $k_3 = [\frac{3n_0}{4}]$; for all $\sigma' \in L$; $\sigma' \cap \psi\big(\partial\, M^n \times \{\frac{k_2}{n_0}\}\big) \neq \emptyset$.

Let L_0 be the complex given by:

$$\left\{ \begin{array}{l} L_0^i = \big\{\sigma \in L \,\big|\, h(\sigma) \cup \big(M^n \setminus \psi\big(\partial\, M^n \times [0, \frac{k_2}{n_0})\big)\big) \neq \emptyset \big\} \,; \\ L_0^f = \big\{ \text{faces of } \sigma \,\big|\, \sigma \in L_0^i \big\} \,; \\ L_0 = L_0^i \cap L_0^f \,. \end{array} \right. \tag{B.2.5}$$

Then, by [Mun], Theorem 10.4, (see also Fig. 2) there exists $g' : K_0' \to M^n$, $h' : L_0' \to M^n$; where g' is a δ-approximation of g and h is a δ-approximation of h, such

that:
(i) $g'(K_0') \cap h'(L_0')$ is full
and
(ii) The union of (K_0', g') and (L_0', h') is an embedding.

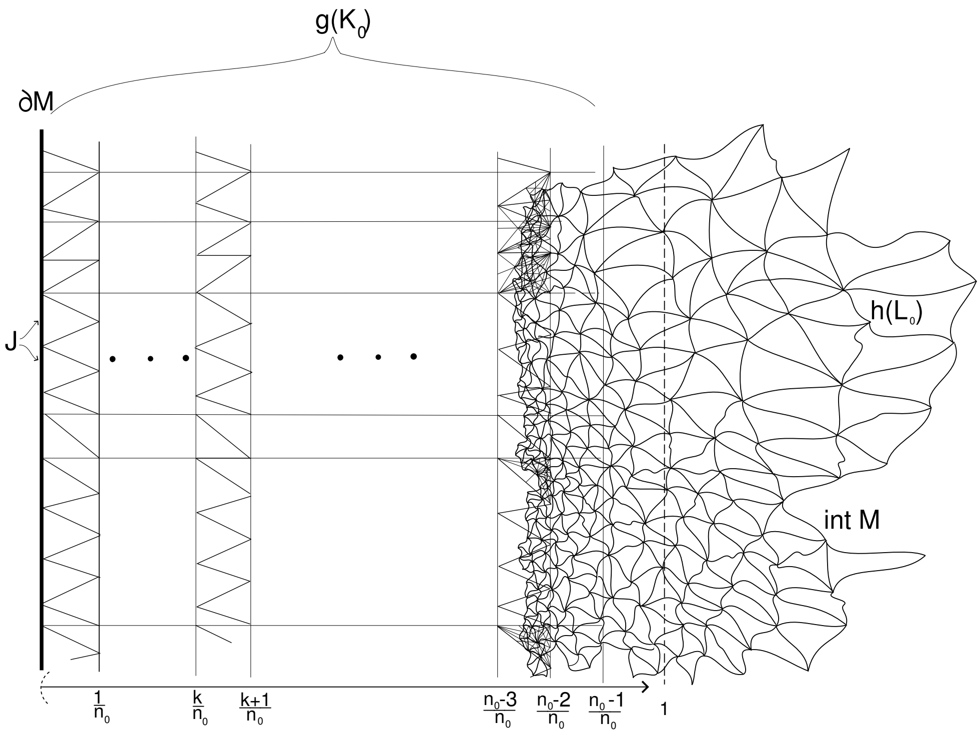


Figure B.2: Mashing the triangulations of $\partial\, M^n$ and $int\, M^n$

Also, by applying again [Mun] Theorem 10.4, we may suppose that
(a) $K_0'\big|_{|J|\times[0,\,\frac{k_4}{n_0}]} \equiv K_0\big|_{|J|\times[0,\,\frac{k_4}{n_0}]}$
(b) $g'\big|_{|J|\times[0,\,\frac{k_4}{n_0}]} \equiv g\big|_{|J|\times[0,\,\frac{k_4}{n_0}]}$; $k_4 = [\frac{n_0}{2}]$

Then $(K_0', g') \cup (L_0', h')$ will be the sought for triangulation, but only if the following condition also holds:

$$g'\big(|K_0|\big) \cup h'\big(|L_0|\big) = M^n\,. \tag{B.2.6}$$

But this condition also takes hold in our case, by virtue of a more general result about topological manifolds (see [Mun], pp. 36-38, 105).

□

Remark B.2.11 *The compactness condition in the theorem above is not essential to the construction employed in the proof, more precisely to the ability to choose an integer n_0 with the required properties: it can be replaced by the condition that ∂M^n*

is endowed with a fat triangulation such that $\inf_{\sigma \in \partial M} diam\, \sigma > 0$*. Thus it is possible to extended Theorem 2.9. to manifolds whose boundary is endowed with a uniformly fat triangulation and that has a finite number of components. (See Section 5 for an example of such a manifold.)*

B.2.3 Fattening Triangulations

We bring here a succinct presentation of the main steps of the fattening process, for full details see [CMS].

First let us establish some definitions and notations:

Definition B.2.12 *Let* $\sigma_i \in K$*,* $dim\, \sigma_i = k_i$*,* $i = 1, 2$*; such that* $diam\, \sigma_1 \leq diam\, \sigma_2$*. Denote by* $[\sigma_i]$ *the affine subspace of* $\mathbb{R}^N$ *generated by* σ_i*, and let* $<\!\sigma_i\!> \parallel [\sigma_i]$*,*
$0 \in <\!\sigma_i\!> \subset \mathbb{R}^N$*;* $i = 1, 2$*. We say that* σ_1, σ_2 *are* δ-transverse *iff*
(i) $dim([\sigma_1] \cap [\sigma_2]) = \max(0, k_1 + k_2 - n)$*;*
(ii) $0 < \delta < \measuredangle([\sigma_1], [\sigma_2])$*, where* $\measuredangle([\sigma_1], [\sigma_2]) = \measuredangle(<\!\sigma_1\!>, <\!\sigma_2\!>)$*, and where*
$\measuredangle(<\!\sigma_1\!>, <\!\sigma_2\!>) = \min_{(e_1, e_2)} \arccos\,(e_1, e_2)$*,* $e_i \in (<\!\sigma_1\!> \cap <\!\sigma_1\!>)^{\perp} \cap <\!\sigma_i\!>$ *,*
$||e_i|| = 1$ *,* $i = 1, 2$*; where* (e_1, e_2) *denotes the standard inner product on* $\mathbb{R}^N$*;*
and if $\sigma_3 \subsetneq \sigma_1$*,* $\sigma_4 \subsetneq \sigma_2$*, such that* $dim\, \sigma_3 + dim\, \sigma_4 < n = dim\, K$*, then*
(iii) $dist(\sigma_3, \sigma_4) > \delta \cdot d_1$ *, where* $d_1 = diam\, \sigma_1$ *.*
In this case we write: $\sigma_1 \pitchfork_\delta \sigma_2$*.*

One begins by triangulating and fattening the intersection of two individual simplices belonging to the two given triangulations, respectively. Given two closed simplices $\bar{\sigma}_1, \bar{\sigma}_2$, their intersection (if not empty) is a closed, convex polyhedral cell: $\bar{\gamma} = \bar{\sigma}_1 \cap \bar{\sigma}_2$. If $dim\, \gamma = 0$ or $dim\, \gamma = 1$, then γ is already a simplex. If $dim\, \gamma \geq 2$, one canonically triangulates $\bar{\gamma}$ by using the *barycentric subdivision* $\bar{\gamma}^*$ of $\bar{\gamma}$, defined inductively upon the dimension of the cells of γ in the following manner: suppose each cell $\beta \subset \partial\gamma$ already has a subdivision into simplices β_i , $i = 1, \ldots, p$; choose an interior point $p_\gamma \in int\, \gamma$, construct the joins $J(p_\gamma, \beta_i)$, $i = 1, \ldots, p$; and consider all their simplices.

One first shows that if the simplices are fat and if they intersect δ-transversally, then one can choose the points such that the barycentric subdivision $\bar{\gamma}^*$ will be composed of fat simplices. (See [CMS], Lemma 7.1.) The proof of this assertion is based upon the following two facts:

1. The following sets are compact:
   $S_1 = \{\sigma_1 \,|\, diam\, \sigma_1 = 1 \,, \varphi(\sigma_1) \geq \varphi_0\}$, $S_2 = \{\sigma_2 \,|\, diam\, \sigma_2 = 2(1 + \delta) \,, \varphi(\sigma_2) \geq \varphi_0\}$, $S(\phi_0, \delta) \subset S_1 \cap S_2$, $S(\phi_0, \delta) = \{(\sigma_1, \sigma_2) \,|\, \exists v_0 \in \sigma_1 \,, \forall \sigma_1 \,;$ and $\bar{\sigma}_1 \cap \bar{\sigma}_2 \neq \emptyset\}$.

2. There exists a constant $c(\varphi)$ such that $\mathcal{S} = \mathcal{S}'$, where
   $\mathcal{S} = \{\sigma_1 \cap \sigma_2 \,|\, diam\, \sigma_2 \leq d_2\}$, $\mathcal{S}' = \{\sigma_1 \cap \sigma_2 \,|\, diam\, c(\varphi)(1 + \delta)d_1\}$,
   i.e. the sets of all possible intersections remains unchanged under controlled dilations of one of the families of simplices.

Next one shows that given two fat Euclidian triangulations that intersect δ-transversally, then one can infinitesimally move any given point of one of the triangulations such that the resulting intersection will be δ^*-transversal, where δ^* depends only on δ, the common fatness of the given triangulations, and on the displacement length. (See [CMS], Lemma 7.3.) By repeatedly applying this results to the simplices of dimensions $0, \ldots, n$, of the intersection of two fat triangulations, one can now prove the main result of this section, namely:

Proposition B.2.13 ([CMS], Lemma 6.3.) *Let $\mathcal{T}_1, \mathcal{T}_2$ be two fat triangulations of open sets $U_1, U_2 \subset M^n$, $U_1 \cap U_2 \neq \emptyset$ having common fatness $\geq \varphi_0$. Then there exist fat triangulations $\mathcal{T}_1', \mathcal{T}_2'$ and there exist open sets $U \subset U_1 \cap U_2 \subset V$, such that*

1. $(\mathcal{T}_1' \cap \mathcal{T}_2')\big|_{U_i \setminus V} = \mathcal{T}_i\,,\ i = 1, 2\,;$
2. $(\mathcal{T}_1' \cap \mathcal{T}_2')\big|_{U} = \mathcal{T};$

where

3. *$\mathcal{T}$ is a fat triangulation of U.*

Now let $\mathcal{T}_1$, $\mathcal{T}_2$ be the triangulations of $\partial M^n \times [0, 1)$ and *int* M^n, respectively, as in the proof of Theorem 2.9.. Then the local fat triangulation obtained in Proposition 2.13 extends globally to a fat triangulation of $\mathcal{T}_1 \cap \mathcal{T}_2$, by applying Lemma 10.2 and Theorem 10.4 of [Mun]. This concludes the proof of Theorem 1.1.

□

B.3 The Existence of Quasimeromorphic Mappings

B.3.1 Quasimeromorphic Mappings

Definition B.3.1 *Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$, and let $f : D \to \mathbb{R}^m$.*
f is called ACL (absolutely continuous on lines) iff:
(i) f is continuous
and
(ii) for any n-interval $Q = \bar{Q} = \{a_i \leq x_i \leq b_i \,|\, i = 1, \ldots, n\}$, f is absolutely continuous on almost every line segment in Q, parallel to the coordinate axes.

Lemma B.3.2 ([V], 26.4) *If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is ACL, then f admits partial derivatives almost everywhere.*

The result above justifies the following definition:

Definition B.3.3 *$f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is ACL^p iff its derivatives are locally L^p integrable, $p \geq 1$.*

Definition B.3.4 *Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$ and let $f : D \to \mathbb{R}^n$ be a continuous mapping. f is called*

1. *quasiregular iff (i) f is ACL^n and*
   (ii) there exists $K \geq 1$ such that:

$$|f'(x)|^n \leq K J_f(x) \text{ a.e.} \tag{B.3.1}$$

   where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup_{|h| = 1} |f'(x)h|$, and where $J_f(x) = det f'(x)$.

2. *quasiconformal iff $f : D \to f(D)$ is a quasiregular homeomorphism.*

3. *quasimeromorphic iff $f : D \to \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is quasiregular, where the condition of quasiregularity at $f^{-1}(\infty)$ can be checked by conjugation with auxiliary Möbius transformations.*

The smallest number K that satisfies (B.3.1) is called the outer dilatation of f.

One can extend the definitions above to oriented, connected $\mathcal{C}^\infty$ Riemannian manifolds as follows: let M^n, N^n be to oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \to N^n$ be a continuous function. One can define the formal derivative of f by using coordinate charts. The ACL^p (ACL^n) property can be defined directly, by using the fact that if $U = int\, U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ then $f \in ACL^p$ iff f is continuous and belongs to the *Sobolev space* $W^1_{p,loc}$ (see [?], pp. 5-11).

Definition B.3.5 *Let M^n, N^n be oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \to N^n$ be a continuous function. f is called locally quasiregular iff for every $x \in M^n$, there exist coordinate charts (U_x, φ_x) and $(V_{f(x)}, \psi_{f(x)})$, such that $f(U_x) \subseteq V_{f(x)}$ and $g = \psi_{f(x)} \circ f \circ \varphi_x^{-1}$ is quasiregular.*

If f is locally quasiregular, then, by Lemma 3.2. , $T_x f : T_x(M^n) \to T_{f(x)} N^n$ exist for a.e. $x \in M^n$.

Definition B.3.6 *Let M^n, N^n be to oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \to N^n$ be a continuous function. f is called quasiregular iff*
(i) f is locally quasiregular
and
(ii) there exists $K, 1 \leq K < \infty$, such that

$$|T_x f|^n \leq K J_f(x) \tag{B.3.2}$$

for a. e. $x \in M^n$.

B.3.2 Alexander's Trick

The technical ingredient in Alexander's trick is the following lemma:

Lemma B.3.7 *([MS1], [Pe]) Let $\mathcal{T}$ be a fat triangulation of $M^n \subset \mathbb{R}^N$, and let $\tau, \sigma \in \mathcal{T}$, $\tau = (p_0, \ldots, p_n)$, $\sigma = (q_0, \ldots, q_n)$; and denote $|\tau| = \tau \cup int\, \tau$.*
Then there exists a orientation-preserving homeomorphism $h = h_\tau : |\tau| \to \widehat{\mathbb{R}^n}$ such that:

1. *$h(|\tau|) = |\sigma|$, if $det(p_0, \ldots, p_n) > 0$*
   and
   $h(|\tau|) = \widehat{\mathbb{R}^n} \setminus |\sigma|$, if $det(p_0, \ldots, p_n) < 0$.
2. *$h(p_i) = q_i$, $i = 0, \ldots, n$.*
3. *$h|_{\partial|\sigma|}$ is a PL homeomorphism.*
4. *$h|_{int|\sigma|}$ is quasiconformal.*

Proof Let $\tau_0 = (p_{0,0}, \ldots, p_{0,n})$ denote the equilateral n-simplex inscribed in the unit sphere $\mathbb{S}^{n-1}$. The *radial linear stretching* $\varphi : int\,|\tau| \to \bar{B}^n$ is onto and bi-lipschitz (see [MS2]). Moreover, by a result of Gehring and Väisalä, φ is also quasicomformal (see [V]). We can extend φ to $\widehat{\mathbb{R}^n}$ by defining $\varphi(\infty) = \infty$. Let J denote the reflection in the unit sphere $\mathbb{S}^{n-1}$ and let $h_0 : |\sigma| \to |\tau|$ denote the orientation-reversing PL mapping defined by: $h_0(p_i) = q_i$, $i = 0, \ldots, n$. Then $h = \varphi^{-1} \circ J \circ \varphi \circ h_0$ is the required mapping.

□

The existence theorem of quasimeromorphic mappings now follows immediately:

Theorem B.3.8 *Let M^n be a connected, oriented C^∞ Riemannian manifold without boundary or having a finite number of compact boundary components. Then there exists a non-constant quasimeromorphic mapping $f : M^n \to \widehat{\mathbb{R}^n}$.*

Proof Let $\mathcal{T}$ be the fat triangulation provided by Theorem 2.9., if $\partial M^n \neq \emptyset$, or by Peltonen's theorem, otherwise. Furthermore, by performing a barycentric type subdivision before starting the fattening process of the triangulation given by Theorem 2.9., ensure that all the simplices of the triangulation satisfy the condition that every $(n-2)$-face is be incident to an even number of n-simplices. Let $f : M^n \to \widehat{\mathbb{R}^n}$ be defined by: $f|_{|\sigma|} = h_\sigma$, where h is the homeomorphism constructed in the lemma above. Then f is a local homeomorphism on the $(n-1)$-skeleton of $\mathcal{T}$ too, while its branching set B_f is the $(n-2)$-skeleton of $\mathcal{T}$. By its construction f is quasiregular. Moreover, given the uniform fatness of the triangulation $\mathcal{T}$, the dilatation of f depends only on the dimension n.

□

B.4 Smoothings

We succinctly present some immediate generalizations of Theorems 1.1. and 2.9. In this Section we consider only submanifolds of a Euclidian space $\mathbb{R}^N$, and the metric considered is that induced by the ambient space.

Theorem 1.1. was restricted to $\mathcal{C}^\infty$ manifolds because the triangulation $\mathcal{T}_2$ of $int\,M^n$ was obtained by applying Peltonen's theorem; so our overall argument is valid only for $\mathcal{C}^\infty$ manifolds. But the class of any n-manifold may be elevated up to $\mathcal{C}^\infty$ (see [Mun], Theorems 4.8 and 5.13), so we can apply the methods of [Pe] on the smoothed $\mathcal{C}^\infty$ manifold, and then project the fat triangulation received to the original structure. Since in the smoothing process we employed only δ-approximations that are, by [Mun], Lemma 8.7. , α-approximations too, we will obtain a fat triangulation, as desired. We can thus formulate the following corollary:

Corollary B.4.1 *Let M^n be an n-dimensional $\mathcal{C}^r$, $1 \leq r \leq \infty$ manifold with boundary, having a finite number of compact boundary components. Then any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Moreover, every PL manifold of dimension $n \leq 4$ admits a (unique, for $n \leq 3$) smoothing (see [Mun1], [Mun], [Th2]), and every topological manifold of dimension $n \leq 3$ admits a PL structure (cf. [Moi], [Th2]). Therefore we can start with a PL manifold (or even just a topological one in dimensions 2 and 3) and smooth it, thus receiving

Corollary B.4.2 *Let M^n be an n-dimensional, $n \leq 4$ (resp. $n \leq 3$), PL (resp. topological) manifold with boundary, having a finite number of compact boundary components. Then any fat triangulation of ∂M^n can be extended to a fat triangulation of M^n.*

Using again Alexander's Trick renders the following result:

Corollary B.4.3 *Let M^n be a connected, oriented n-dimensional manifold ($n \geq 2$), without boundary or having a finite number of compact boundary components. Then in the following cases there exists a non-constant quasimeromorphic mapping $f : M^n \rightarrow \widehat{\mathbb{R}^n}$:*

1. *M^n is of class $\mathcal{C}^r$, $1 \leq r \leq \infty$, $n \geq 2$;*
2. *M^n is a PL manifold and $n \leq 4$;*
3. *M^n is a topological manifold and $n \leq 3$.*

Remark B.4.4 *The dilatation may increase only when we linearize the tangent cone at cone points. Fortunately, the nature of linearization process is such that, when the cone angles are bounded from below, then the dilatations will be bounded from above (see, e.g. [Th2]).*

B.5 Kleinian Groups

Since the construction of fat triangulations was motivated mainly by the study of G-automorphic quasimeromorphic mappings with respect to a Kleinian group G, i.e. a discontinuous group of orientation preserving isometries of $\mathbb{H}^n$, it is natural to employ Theorems 1.1. and 2.9. to prove the following result, that represents a generalization of a result of Tukia ([Tu]):

Theorem B.5.1 *Let G be a Kleinian group with torsion acting upon $\mathbb{H}^n$, $n \geq 3$. If the elliptic elements (i.e. torsion elements) of G have uniformly bounded orders, then there exists a non constant G-automorphic quasimeromorphic mapping $f : \mathbb{H}^n \rightarrow \widehat{\mathbb{R}^n}$, i.e. such that*

$$f(g(x)) = f(x)\,, \text{ for any } x \in \mathbb{H}^n \text{ and for all } g \in G\,. \tag{B.5.1}$$

While for full details we refer the reader to [S2] and – for a different fattening method (albeit in dimension 3 only), to [S1] – we bring here the following sketch of proof:

Proof By Lemma 3.7. it suffices to produce a fat G-invariant triangulation of $\mathbb{H}^n$. The *singular locus* $\mathcal{L}$ of $\mathbb{H}^n/G$ is the image, under the natural projection $\pi : \mathbb{H}^n \rightarrow \mathbb{H}^n/G$, of the union $\mathcal{A} = \bigcup_{i\,\in\,\mathbb{N}} A_{f_i}$ of the elliptic axes of G. ($\mathcal{A} = \{A_{f_i}\}_i$ is a countable set, by the discreteness of G.) For each elliptic axes A_{f_i} it is possible to choose a *collar* N_i and triangulate it in an f_i-invariant manner. Denote by $\mathcal{T}_i$ the f_i-invariant triangulation of N_i. Put $\mathcal{N} = \bigcup_{i\,\in\,\mathbb{N}} N_i$. Then $M_e = (\mathbb{H}^n \setminus \mathcal{N})/\,G$ is a manifold with boundary. Then $\partial M_e = \bigcup_{i\,\in\,\mathbb{N}} \partial N_i$ has the triangulation induced by that of $\mathcal{N}$ and, since the orders of the elliptic elements are bounded from above, the induced triangulation will be fat. By slightly modifying the proof of Theorem 1.1., one can show that this triangulation can be extended to a fat triangulation $\mathcal{T}$ of $\mathbb{H}^n/G$. Then $\pi^{-1}(\mathcal{T}) \cup \bigcup_{i\,\in\,\mathbb{N}} \mathcal{T}_i$ will represent the desired fat G-invariant triangulation. □

Remark B.5.2 *The existence of quasimeromorphic automeromorphic mappings in the case $n = 2$ represents a classical result closely connected to the existence of meromorphic mappings on Riemann surfaces.*

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Bibliography

[Abi] Abikoff, W.: *Kleinian Groups*, Lecture Notes, given at "The Technion – Israel Institute of Technology", Haifa, Israel, 1996-1997.

[AH] Abikoff, W. and Hass, A.H.: *Nondiscrete groups of hyperbolic motions*, Bull. London Math. Soc. **22**(3), 1990, 233-238.

[Al] Alexander, J.W.: *Note on Riemmann spaces*, Bull. Amer. Math. Soc. **26**, 1920, 370-372.

[AP1] Apanasov, B.N.: *Klein Groups in Space*, Sib. Math. J. **16**, 1975, 679-684.

[AP2] Apanasov, B.N.: *Discrete groups in space and uniformization problems*, Kluwer Academic Publishers, Dordrecht, 1991.

[B]

[BAC] Brânzei, D., Aniţa, S. and Cocea, C.: *Planul si spaţiul euclidian*, Editura Academiei R.S.R., Bucureşti, 1986.

[Bea] Beardon, A. F.: *The Geometry of Discrete Groups*, Springer Verlag, GTM **91**, N.Y., 1982.

[BCER] Bern, M., Chew, P. Eppstein, P.C.D. and Ruppert, J.: *Dihedral Bounds for Mesh Generation in High Dimensions*, 6th ACM-SIAM Symp. Discrete Algorithms, San Francisco, 1995, 189-196.

[BM] Bowditch, B.H. and Mess, G.: *A 4-Dimensional Kleinian Group*, Transaction of the Amer. Math. Soc., **344**(1), 1994, 390-405.

[BoS] Bonahon, F. and Siebenmann, L.C.: *The classification of Seifert fibred 3-orbifolds* , in: *Low-dimensional topology"(Chelwood Gate, 1982)*, 19-85, London Math. Soc. Lecture Notes **95**, Cambridge University Press, 1985.

[BrM] Brooks, R. and Matelski, J.P.: *Collars in Kleinian groups*, Duke Math.J., **49**(1), 1982, 163-182.

[Ca1] Cairns, S.S.: *On the triangulation of regular loci*, Ann. of Math. **35**, 1934, 579-587.

[Ca2] Cairns, S.S.: *Polyhedral approximation to regular loci*, Ann. of Math. **37**, 1936, 409-419.

[Ca3] Cairns, S.S.: *A simple triangulation method for smooth manifolds*, Bull. Amer. Math. Soc. **67**, 1961, 380-390.

[Car] Caraman, P.: *n-Dimensional Quasiconformal (QCf) Mappings*, Editura Academiei Române, Bucharest, Abacus Press, Tunbridge Wells Haessner Publishing, Inc., Newfoundland, New Jersey, 1974.

[CMS] Cheeger, J., Müller, W., and Schrader, R.: *On the Curvature of Piecewise Flat Spaces*, Comm. Math. Phys., **92**, 1984, 405-454.

[Cox] Coxeter, H. S. M.: *Regular Polyopes*, Second Edition, MacMillan, London, 1963.

[Dr] Druschel, K.S.: *The Cobordism of Oriented Three Dimensional Orbifolds*, Pacific Journal of Mathematics, **193**(1), 2000, 45-55.

[DM] Derevin, D.A. and Mednikh, A.D.: *Geometric proprieties of discrete groups acting with fixed points in Lobachevsky space*, Soviet Math. Dokl., **37**(3), 1988, 614-617.

[E] Edelsbrunner, H.: *Geometry and Topology for Mesh Generation*, Cambridge University Press, Cambridge, 2001.

[Ep^{++}] Epstein, D.B.A. et al. (editor): *Word Processing in groups*, Jones and Bartlett, Boston, MA, 1992.

[Fe] Ferrarotti, M.: *Triangulatione analitica per varietà*, Bolletino U.M.I. (5) **17**-A , 1980, 79-84.

[Fo] Ford, L.R.: Automorphic functions. - Chelsea, New York, N.Y., 1951.

[FH] Friedland, S. and Hersonsky, S.: *Jorgensen's inequality for discrete groups in normed algebras*, Duke Mathematical Journal, **69**(3), 1993, 593–614.

[FM] Feighn, M. and Mess, G.: *Conjugacy classes of finite subgroups of Kleinian groups*, Amer. J. of Math., **113**, 1991, 179-188.

[GM1] Gehring, F.W., Martin G.J., *Commutators, collars and the geometry of Möbius groups*, J. Anal. Math. **63**, 1994, 174-219.

[GM2] Gehring, F.W., Martin, G.J., *On the Margulis constant for Kleinian groups, I*, Ann. Acad. Sci. Fenn., **21**, 1996, 439-462.

[GMMR] Gehring, F.W., Maclachlan, C., Martin, G.J., and Reed, A.W.: *Arithmecity, Discreteness and Volume*, Trans. Amer. Math. Soc. **349**, 1997, 3611-3643.

[GV] Gehring, W. F. and Väisala, J. *The coefficients of quasiconformality*, Acta Math. **114**, 1965, 1-70.

[Gro] Gromov, M. : *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, Boston, Mass., 1999.

[H] Hamilton, E. : *Geometrical finiteness for hyperbolic orbifolds*, Topology, **37**(3), 1998, 635-657.

[He] Heinonen, J. : *Analysis on metric spaces*, Lecture Notes, Helsinki, May 1979.

[Ho] Holopainen, I. : *Quasiregular mappings and the p-Laplace operator*, Contemporary Mathematics **338**, 2003, 219-239.

[Hu] Hudson, J.F. : *Piecewise Linear Topology*, Math. Lect. Notes Series, Benjamin, N.Y., 1969.

[J] Jørgensen, T. : *On discrete groups of Möbius transformations*, Amer. Journ. of Math. **98**(3), 1976, 739-749.

[JM] Jørgensen, T. and Marden, A. : *Algebraic and Geometric Convergence in Kleinian Groups*, Math. Scand. **66**, 1990, 42-72

[K] Kra, I. : Automorphic Forms and Kleinian Groups. - W.A. Benjamin, Reading, Mass., 1972.

[Ku] Kuiper, N. H. : *A Short History of Triangulation and Related Matters*, in *History of topology* (I. M. James ed.), Elsevier, Amsterdam , 1999.

[M] Martin, G.J. : *On discrete Möbius groups in all dimensions: A generalization of Jørgensen's inequality*, Acta Math., **163**, 1989, 253-289.

[Med] Mednikh, A.D. : *Automorphism groups of the three-dimensional hyperbolic manifolds*, Soviet Math. Dokl., **32**(3), 1985, 633-636.

[Moi] Moise, E. E. : *Geometric Topology in Dimensions 2 and 3*, Springer-Verlag, New-York, 1977.

[Mor] Morgan, J.W. : *On Thurston's Uniformization Theorem for Three-Dimensional Manifolds*, in "The Smith Conjecture", (Morgan, J.W. and Bass, H. ed.), Academic Press, N.Y., 1984, 37-126.

[MS1] Martio, O. and Srebro, U. : *Automorphic quasimeromorphic mappings in* R^n, Acta Math. **195**, 1975, 221-247.

[MS2] Martio, O. and Srebro, U. : *On the existence of automorphic quasimeromorphic mappings in* R^n, Ann. Acad. Sci. Fenn., Series I Math., **3**, 1977, 123-130.

[Ms] Maskit, B.: *Kleinian Groups*, Springer Verlag, GDM 287, N.Y., 1987.

[My] Mayer, V.: *Uniformly Quasiregular Mappings of Lattès Type*, Conformal Geometry and Dynamics **1**, 1997, 104-11.

[Mun1] Munkres J. R.: *Obstructions to the smoothening of piecewise-differentiable homeomorphisms*, Annals of Math., **72**(2), 1960, 521-554.

[Mun] Munkres, J. R.: *Elementary Differential Topology*, (rev. ed.) Princeton University Press, Princeton, N.J., 1966.

[Pe] Peltonen, K.: *On the existence of quasiregular mappings*, Ann. Acad. Sci. Fenn., Series I Math., Dissertationes, 1992.

[Po] Potyagailo, L.: *Finitely generated Kleinian groups in 3-space and 3-manifolds of infinite homotopy type*, Trans. of Amer. Math. Soc., **344**(1), 1994, 57-77.

[PA] Pach, J. and Agarwal. P.K.: *Combinatorial Geometry*, Wiley-Interscience, 1995.

[Rat] Ratcliffe, J.G.: *Foundations of Hyperbolic Manifolds*, GTM 194, Springer Verlag, New York, 1994.

[Ric1] Rickman, S.: *Quasiregular Mappings*, in Proc. Romanian-Finnish Seminar on Teichmüler spaces and quasiconformal mappings, Braşov, 1963, 261-271.

[Ric2] Rickman, S.: *Quasiregular Mappings*, Springer-Verlag, Berlin - Heidelberg - New-York, 1993.

[Rup] Ruppert, J.: *A Delaunay Refinement Algorithm for Quality 2-Dimensional Mesh Generation*, Journal of Algorithms **18**(3), 1995, 548-585.

[RS] Rourke, C. and Sanderson, B.: *Introduction to Piecewise-linear Topology*, Ergebnesse der Mathematik **69** (1972), Springer-Verlag, Berlin - Heidelberg - New-York.

[S1] Saucan, E.: *The Existence of Quasimeromorphic Mappings in Dimension 3*, Conform. Geom. Dyn., to appear.

[S2] Saucan, E.: *The Existence of Quasimeromorphic Mappings*, to appear in Ann. Acad. Sci. Fenn., Series I A, Math. **31**, 2006.

[S3] Saucan, E.: *Note on a theorem of Munkres*, Mediterr. j. math., **2**(2), 2005, 215 - 229.

[S4] Saucan, E.: in preparation.

[SA] Saucan, E. and Apleboim, E.: *Quasiconformal Fold Elimination for Seaming and Tomography*, in preparation.

[Som] Sommerville, D.M.Y.: *An Introduction to the Geometry of N Dimensions*, Dover Publications, N.Y., 1958.

[Spi] Spivak, M.,: *A Comprehensive Introduction to Differential Geometry, volume V*, Publish or Perish, Boston, MA, 1970-1975.

[Sr] Srebro, U.,: *Non-existence of Automorphic Quasimeromorphic Mappings*, Analysis and topology, World Sci. Publishing, River Edge, NJ, 1998.

[Sr1] Srebro, U.,: *Quasiregular Mappings*, in Advances in Function Theory, Lecture Notes in Mathematics **505**, 148-163, Springer-Verlag, Berlin, 1976.

[Sr2] Srebro, U.,: *Topological proprieties of quasiregular mappings*, in Quasiconformal Space Mappings – A colection of surveys 1960-1990 (M. Vuorinen ed.), LNM **1508**, 104-118, Springer-Verlag, Berlin, 1991.

[Suo] Suominen, K.,: *Quasiconformal* , Ann. Acad. Sci. Fenn., Series I Math., **393**, 1966, 5-39.

[Th1] Thurston, W.: *The Geometry and Topology of* 3*-Manifolds*, preliminary version, 1990.

[Th2] Thurston, W.: *Three-Dimensional Geometry and Topology, vol.1, (Edited by S. Levy)*, Princeton University Press, Princeton, N.J. 1997.

[Tu] Tukia, P.: *Automorphic Quasimeromorphic Mappings for Torsionless Hyperbolic Groups*, Ann. Acad. Sci. Fenn., **10**, 1985, 545-560.

[V] Väisalä, J.: *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. **229**, Springer-Verlag, Berlin - Heidelberg - New-York, 1971.

[Vu] Vuorinen, M.: *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math. **1319**, Springer-Verlag, Berlin - Heidelberg - New-York, 1988.

[Wa] Waterman,, P.L.: *Möbius Transformations in Several Dimensions*, Advances in Mathematics, **101**, 1993, 87-113.

[Wh] Whitehead, J.H.C.: *On* $\mathcal{C}^1$*-complexes*, Ann. of Math., **41**, 1940, 809-824.